A Simple Demonstration of the Frequency Dependences of Elastic Moduli and the Damping of Stress Waves at Seismic Frequencies

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Abstract

Recognition of the frequency dispersion of seismic waves first arose from the study of the shapes of explosively generated pulses. Mathematically, the relevant dispersion relation has been derived with satisfying rigour, but the implied frequency dependences of elastic moduli have been difficult to demonstrate in a simple and convincing way. It is shown here that a torsional pendulum suffices to display the first order frequency dependence of the rigidity modulus for a hard plastic which is strongly dissipative, as represented by an anelastic Q factor of order 20. A similar direct demonstration for rocks is beset by the difficulty of reducing the strain amplitude to 10^{-6} or less, to bring the anelasticity into the linear range.

Introduction

Consider a sharp (δ function) displacement to be imposed at some point in an extensive elastic medium, so that a displacement pulse spreads out from it. We suppose that the medium is lossy and that high frequency elastic waves are attenuated more rapidly with distance than lower frequencies, as is always observed. The initial pulse may be Fourier analysed and its spectrum is white, that is waves of all frequencies have the same power per unit frequency interval. All the wave crests coincide at the initiating δ function but cancel elsewhere. However the selective attenuation of high frequencies means that the pulse becomes progressively less sharp as it propagates. If we assume that waves of all frequencies travel at the same speed, then all the superimposed crests stay together and the pulse spreads out maintaining a symmetrical shape as the high frequency components that are responsible for its sharpness disappear. But this means that half of the pulse arrives at a remote point before the time delay implied by the wave speed, and, even worse, there is actually no distinct pulse onset but the pulse begins to arrive before it has been initiated. This violation of the principle of causality compels abandonment of the assumption of wave speed independent of frequency. The speeds of the Fourier component waves must be so adjusted that their superposition gives complete mutual cancellation before the admissible arrival time.

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The general nature of this problem and its importance to explosion seismology appear to have been recognized first by N.H. Ricker in a series of papers beginning in 1940 but his approach was clouded by an erroneous mathematical starting point and an unwillingness to recognize that it led to results in conflict with observation. When this work appeared in book form (Ricker 1978) its scientific basis was fairly completely discredited (Kjartansson 1979; Gladwin 1980). More successful theoretical approaches that used particular assumptions about the frequency dependence of attenuation to make the problem mathematically tractable were by Kolsky (1956), Futterman (1962), Strick (1967, 1970, 1971) and Azimi *et al.* (1968). A completely general and mathematically rigorous treatment has now been given by Brennan & Smylie (1981, see also Brennan 1980).

The only intuitively obvious special case, attenuation with distance independent of frequency, requires no velocity dispersion because all Fourier components are equally attenuated and a pulse propagates with diminishing energy but unchanging shape. This case is not realized in any known medium, even approximately. Much of the literature on rock anelasticity indicates that attenuation is proportional to frequency, f (e.g. Knopoff 1964; Gordon & Davis 1968), although availability of better data in the last couple of years or so indicates that if a power law is an appropriate approximation, attenuation in the earth and in some geological materials varies as $f^{0.7}$ to $f^{0.8}$ rather than $f^{1.0}$ (e.g. Anderson & Minster 1979). These variations are usually represented in terms of the quality factor, Q, which, for $Q \gg 1$, is conveniently defined by the fractional loss of energy per cycle of a wave

$$-\Delta E/E = 2\pi/Q \qquad (1)$$

Then the energy of a wave of angular frequency ω and speed v decays with distance r as $\exp(-\omega r/vQ)$. The special case of attenuation proportional to frequency means very nearly constant Q (since variation of v with ω is slight). This case was shown by Kolsky (1956) to lead to velocity dispersion given to first order in Q^{-1} by

$$\frac{1}{v(\omega)} = \frac{1}{v(\omega_r)}\left[1 - \frac{1}{\pi Q}\ln\left|\frac{\omega}{\omega_r}\right|\right] \qquad (2)$$

For Q increasing slightly with frequency, as $\omega^{0.2}$ or $\omega^{0.3}$, theory suggests that the logarithmic dependence of velocity on frequency is still followed but with a correspondingly reduced variation (Brennan 1980). Conversely, if Q decreases slightly with frequency, as for the plastic which gave the data in Fig. 1, there is a correspondingly enhanced velocity dispersion. This has not, to our knowledge, previously been demonstrated experimentally.

For this theory to be applicable the anelastic hysteresis must be a linear phenomenon. This means that the loss of energy per cycle of a wave is proportional to its energy, that is, by equation (1), Q is independent of wave amplitude (or strain amplitude). It should be noted that even when anelasticity is linear, by this definition, the elasticity is non-linear (i.e. Hooke's law is not obeyed) because the stress-strain curve must form a loop whose area represents the energy loss per cycle. A perfectly Hookeian solid would be loss-less. The condition that anelasticity should be linear is that the amplitude of an applied strain cycle must be smaller than some limiting value that depends upon the nature of the material.

Direct observation of linear anelastic hysteresis in rocks has been difficult to achieve. Gordon & Davis (1968) plotted stress-strain loops at strain amplitudes of 10^{-4}, which they believed to be small enough for Q to be independent of strain amplitude, and found the loops to have cusped ends.

This is a demonstration of non-linearity, in which case consideration of the independent propagation and subsequent recombination of Fourier component waves, as in the linear theory that leads to equation (2), is invalid. McKavanagh & Stacey (1974) developed a method for observing loops at strain amplitudes down to 10^{-5}, again obtaining cusped loops, but the suspicion remained that linear anelastic behaviour would be observed at sufficiently small strain amplitudes (Stacey *et al.* 1975). The suspicion was confirmed by Brennan & Stacey (1977) and Brennan (1981) who found that the loops became elliptical and therefore anelasticity linear in rocks only for strain amplitudes below about 10^{-6}. Since seismic waves have strain amplitudes well below this limit except very close to the source, it is linear anelasticity that is important in seismology, although virtually all laboratory observations have been in the non-linear range. The only linear experiment at seismic frequencies is that of Brennan (1981) but at much higher frequencies observations of damping and frequency dependent elasticity by resonant oscillation of bars in the linear amplitude range were reported by Winkler *et al.* (1979).

Inferences of dispersion of body waves from seismic pulse shapes (Wuenschel 1965; see also McDonal *et al.* 1958) necessarily beg the question of linearity because they appeal to Fourier analyses of the pulses, which assume linearity. Dispersion could not, in principle, be distinguished from the effect of pulse shaping by non-linear attenuation. Measurements at a series of single frequencies are required. Validity of the dispersion principle is easier to confirm for certain plastic materials, for which the linear anelastic range appears to extend to strain amplitudes of 10^{-4} or so.

A simple experiment

We have measured the elastic moduli of plastics in the form of rods 6 to 8 mm in diameter and about 1 m long, using a torsional pendulum. The horizontal torsion bar, which is suspended by the vertical specimen, is a light aluminium tube about 1 m long with two 3 kg brass masses that could be moved to any positions on the tube. The period range of the torsional oscillations is adjustable over a range of 6 to 1 by moving the masses to adjust the moment of inertia of the torsion bar. This range suffices to demonstrate clearly the frequency dependence of elasticity for low Q materials such as perspex, which we have found to be particularly suitable.

The frequency of oscillation is given by

$$\omega = 2\pi f = \sqrt{\frac{\alpha}{I}} \tag{3}$$

where, for a specimen of radius r, length l and rigidity μ,

$$\alpha = \frac{\pi}{2}\frac{r^4 \mu}{l} \tag{4}$$

and the moment of inertia of the torsion bar is

$$I = I_o + 2I_m + 2MX^2 \tag{5}$$

I_o being the value for the aluminium tube alone, I_m for each of the brass masses about its own centre and M, X are the values of the masses and their displacement from the centre of the system.

Q is determined from the damping of the oscillations:

$$\frac{-\pi}{Q} = \frac{d \ln A(n)}{dn} \tag{6}$$

which represents the variation of amplitude A with oscillation number, n. It is important to plot ln A *vs* n and to consider only the section of the graph (at small amplitudes) which gives constant Q. Q independent of amplitude demonstrates that the anelastic damping is in the linear range.

Variation of I by movement of the brass masses allows any value of ω within the range of the apparatus to be selected. In such experiments it is usually assumed that the torsional 'constant' α is in fact constant, but, as this experiment shows, α is a function of ω and the dependence is determined by a series of measurements of I and ω. The rigidity of the plastic, μ, is related to α only by dimensions of the sample (equation (4)) and so has the same frequency dependence. The frequency dependences of μ and Q for a sample of perspex are plotted in Fig. 1.

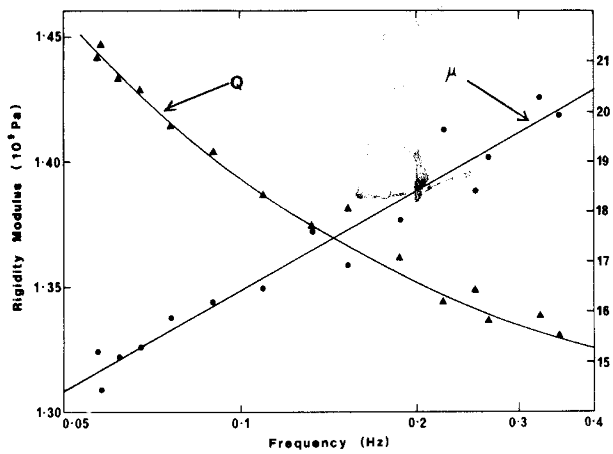

FIGURE 1

Variations with frequency of the rigidity modulus, μ, and anelastic damping parameter, Q, for a sample of perspex in a torsional cycle of low strain amplitude ($< 10^{-4}$).

Conclusion

By choosing a suitable material (perspex) with a Q of about 20 a 7% variation in rigidity modulus is found over a 6 to 1 frequency range. Over this range Q varies between 15.5 and 21, so that the simple relationship for constant Q that follows from equation (2), viz.

$$\frac{d \ln \mu}{d \ln f} = \frac{2}{\pi Q} \qquad (7)$$

requires modification. We can readily see in a qualitative way what is the effect of frequency dependent Q since equation (7) applies to constant Q but for $Q \propto f$, d ln μ/d ln f = 0 (the non-dispersive case). For the case of Q decreasing with f, as in our observations, d ln μ/d ln f is stronger than by equation (7). Taking the nearest fitting logarithmic relationship to the Q(f) curve to be d ln Q/d ln f = – 0.16, we suggest that for this material

$$\frac{d \ln \mu}{d \ln f} \approx 1.16 \cdot \frac{2}{\pi Q} \qquad (8)$$

This gives an average Q inferred from the elasticity data of 17.6 compared with the observed range 15.5 to 21. Thus we find detailed agreement between observation and theory.

We have contemplated an equivalent experiment using rocks, but by restricting the strain amplitude to 10^{-6} or less the oscillations become observable only by a special technique such as the capacitance micrometry method of Brennan & Stacey (1977) and Brennan (1981), in which the simplicity and directness of the present experiment are lost. Nevertheless the principle is clear and can be applied to any value of Q and any frequency range. If the frequency dependence of elasticity is scaled to a representative rock Q of 100, the elasticity varies by 1.5% (and seismic velocity by 0.75%) per decade change in frequency. Many physicists and geophysicists have been conditioned by elementary training not to think in terms of necessary intrinsic frequency dependences of elastic moduli, but even these apparently quite small effects are significant in some seismological applications (Futterman, 1962, Dziewonski & Anderson, 1981).

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