3D MT Data Modelling using Multi-order Hexahedral Vector Finite Element Method, including Anisotropy and Complex Geometry

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SUMMARY

We will present the progress made on the development of a computational algorithm to model 3D Magnetotelluric data using Vector Finite Element Method (VFEM). The differential equations to be solved are the decoupled Helmholtz equations for the secondary electric field, or the secondary magnetic field, with a symmetric conductivity tensor. These equations are modified to include anisotropic earth and complex geometry (such as surface topography, and subsurface interfaces). The primary field is the solution of an air domain, homogeneous half-space or layered earth.

This study will compare the application of two boundary conditions, the Generalize Perfect Matched Layers method (GPML) versus Dirichlet boundaries. Dirichlet boundary conditions are applied on the tangential fields, assuming that the boundaries lie far away from the inhomogeneous model. The GPML scheme defines an artificial boundary zone that absorbs the propagating and evanescent electromagnetic fields, to remove boundary effects (Fang, 1996).

In this algorithm, high order edge elements are defined based on covariant projections for hexahedral elements (Crowley, et al., 1988). The vector basis functions are defined for the 12 edges (linear) element, 24 edges (quadratic) element, and 48 edges (cubic) element. By this definition, the vector basis will have zero divergence in the case of rectangular elements and relatively small divergence in the case of distorted elements. They are defined to study their numerical accuracy and speed, and to see if the divergence correction is automatically satisfied.

Key words: 3D forward modelling, vector finite element, MT method, anisotropy

INTRODUCTION

In geophysics, EM modelling can be seen as the generation of synthetic EM data from a given geo-EM model of conductivity (σ), and magnetic permeability (μ). In the case of MT modelling, Maxwell's PDEs are solved with a propagating plane wave as a source field, and taking into consideration the quasi-static condition. Forward modelling is an important step for the inversion of MT data, which obtains successive geo-EM models that yields to synthetic MT data fitting the measured one. Consequently, the inversion needs a fast, accurate and reliable MT modelling solution (Avdeev, 2005; Börner, 2010).

Sometimes, the measured data can't be properly inverted with 1D or 2D models, so it is necessary to obtain reliable 3D models in order to interpret these data. The main numerical techniques applied to 3D MT modelling are Finite Differences (Mackie, et al., 1993; Weiss and Newman, 2002, 2003; Haber and Heldmann, 2007), Integral Equations (Wannamaker, 1991; Zhdanov, et al., 2006) and Finite Element (Farquharson and Miensopust, 2011; Mitsuhata and Uchida, 2004; Nam et al., 2007; Shi, et al., 2004) methods. The main problem of 3D EM modelling is the computational memory and time required to obtain the solutions. Therefore, the main target of these numerical implementations is to reduce the computational load and obtain accurate solutions including anisotropy and complex model geometry (Avdeev, 2005).

3D VFEM has been applied for MT modelling, with vector basis functions of linear order on a rectilinear mesh (Farquharson and Miensopust, 2011; Mitsuhata and Uchida, 2004; Shi et al., 2004) or on hexahedral elements (Nam et al., 2007). In this project, a VFEM algorithm is being developed with multi-order vector basis functions. These functions are based on covariant projections for hexahedral elements (Crowley et al., 1988), and are defined for Linear (12 edges, 8 nodes), Quadratic (24 edges, 20 nodes), and Cubic (48 edges, 27 nodes) hexahedral elements. The effectiveness of Dirichlet boundary conditions and Generalize Perfect Matched Layers method (Fang, 1996) will be compared in this study. An analysis of convergence of this model with analytical solutions (homogeneous or layered earth) will be carried out, in order to validate the program. Also, information of sampling densities, extension zones and air layer height that yields to faster and accurate models will be obtained.

VECTOR FINITE ELEMENT METHOD

For MT modelling, the decoupled Helmholtz equations (1) are obtained with the quasi-static condition, and in terms of the secondary field formulation (e.g. $\vec{E} = \vec{E}^p + \vec{E}^s$). The primary field $\vec{E}^p$ is the solution of a plane wave propagating through an air domain, a homogeneous earth or a layered earth. This formulation takes into consideration the anisotropy, by defining **σ** and **μ** as symmetric tensors.

$$\begin{aligned}
&\nabla \times \left(\boldsymbol{\mu}^{-1} \cdot \nabla \times \vec{E}^{\,s}\right) + i\omega\boldsymbol{\sigma} \cdot \vec{E}^{\,s} = \vec{s}^{\,E} \\
&\vec{s}^{\,E} = -i\omega\left[\delta\boldsymbol{\sigma} \cdot \vec{E}^{\,p} + \nabla \times \left(\boldsymbol{\mu}^{-1}\delta\boldsymbol{\mu} \cdot \vec{H}^{\,p}\right)\right] \\
&\nabla \times \left(\boldsymbol{\sigma}^{-1} \cdot \nabla \times \vec{H}^{\,s}\right) + i\omega\boldsymbol{\mu} \cdot \vec{H}^{\,s} = \vec{s}^{\,H} \\
&\vec{s}^{\,H} = -i\omega\delta\boldsymbol{\mu} \cdot \vec{H}^{\,p} + \nabla \times \left(\boldsymbol{\sigma}^{-1}\delta\boldsymbol{\sigma} \cdot \vec{E}^{\,p}\right)
\end{aligned} \tag{1}$$

In eq. (1), $\delta\boldsymbol{\sigma} = \boldsymbol{\sigma}^{s} - \boldsymbol{\sigma}^{p}$ is the difference between the geo-EM model of the primary fields and that of the unknown secondary field. Usually, one of these equations is solved and the other field is obtained directly from the numerical application of Maxwell's equations. These equations are to be solved for the TE and TM modes, to obtain the impedance of the subsurface, and be able to apply it in some inversion routine.

Applying the Galerkin method to the governing equations (1), is possible to obtain the integrals to be numerically solved with a basis vector function $\vec{\mathrm{w}}$, eq. (2).

$$\begin{aligned}
&\int_{\Omega} \begin{pmatrix} (\nabla \times \vec{\mathrm{w}}) \cdot \boldsymbol{\mu}^{-1} \cdot \left(\nabla \times \vec{E}^{\,s}\right) + i\omega\vec{\mathrm{w}} \cdot \boldsymbol{\sigma} \cdot \vec{E}^{\,s} \\ (\nabla \times \vec{\mathrm{w}}) \cdot \boldsymbol{\sigma}^{-1} \cdot \left(\nabla \times \vec{H}^{\,s}\right) + i\omega\vec{\mathrm{w}} \cdot \boldsymbol{\mu} \cdot \vec{H}^{\,s} \end{pmatrix} d\Omega \\
&= c_{\Gamma} \int_{\Omega} \begin{pmatrix} \vec{\mathrm{w}} \cdot \left(\hat{\mathrm{n}} \times \vec{H}_0^{\,s}\right) \\ \vec{\mathrm{w}} \cdot \left(\vec{E}_0^{\,s} \times \hat{\mathrm{n}}\right) \end{pmatrix} d\Gamma + \int_{\Omega} \begin{pmatrix} \vec{\mathrm{w}} \cdot \vec{s}^{\,E} \\ \vec{\mathrm{w}} \cdot \vec{s}^{\,H} \end{pmatrix} d\Omega
\end{aligned} \tag{2}$$

The model domain Ω, including topography and subsurface interfaces, e.g. Fig. 1, is discretized with hexahedral elements, and extension zones, Fig. 2. Within each hexahedral element, the unknown field $\vec{F}^{(e)}$ can be obtained from a vector basis function eq. (3), where ME is the number of edges in the element. The vector basis function, eq. (4), is defined in terms of a nodal basis function of global coordinates $N_{\alpha}^{(e)}(\vec{r})$, and a normalized edge vector $\vec{w}_{\alpha}$. This edge vector is obtained from the covariant projections $\vec{\xi}$ in local coordinates (Crowley, et al., 1988). In order to obtain edge vectors that can be shared with adjacent elements, the contribution of nodes i and j that defines the edge must be taken into account. Using the numerical field eq. (3) in eq. (2) for $\vec{\mathrm{w}}$ and $\vec{E}^{\,s}$ or $\vec{H}^{\,s}$, then the integrals can be numerically solved by Gauss Quadrature. This numerical application yields to a sparse system of equations, which can be stored as a non-zero entries array. In this algorithm, MUMPS package is used to solve the system of equations.

$$\vec{F}^{(e)}(\vec{r}) = \sum_{\alpha=1}^{ME} F_{\alpha}^{(e)} \vec{V}_{\alpha}^{(e)}(\vec{r}) \tag{3}$$

$$\begin{aligned}
&\vec{V}_{\alpha}^{(e)}(\vec{r}) = N_{\alpha}^{(e)}(\vec{r})\vec{w}_{\alpha} \\
&N_{\alpha}^{(e)}(\vec{r}) = \sum_{l=0}^{2}\sum_{m=0}^{2}\sum_{n=0}^{2} c_{lmn}\, x^{l} y^{m} z^{n} \\
&\vec{w}_{\alpha} = \frac{\vec{\xi}_i + \vec{\xi}_j}{\left|\vec{\xi}_i + \vec{\xi}_j\right|}; \quad \vec{\xi}_i = \frac{\partial \vec{r}}{\partial \xi}
\end{aligned} \tag{4}$$

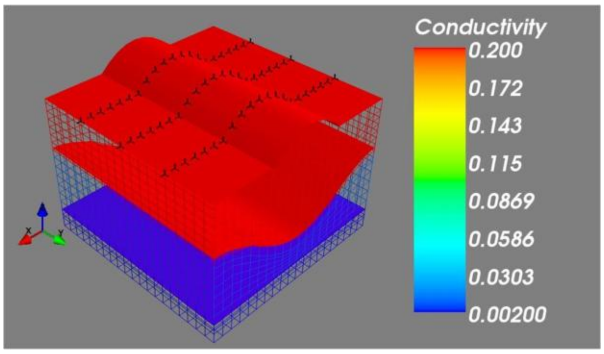

Figure 1 Inner model example with topography and subsurface interfaces.

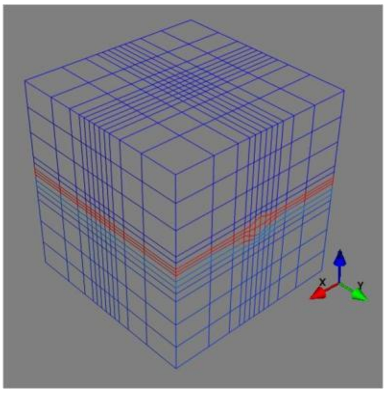

Figure 2 Domain discretization with extension zones.

Three different order of basis function has been developed. These are, Linear (12 Edges, Fig. 3a), Quadratic (24 Edges, Fig. 3b), and Cubic (48 Edges, Fig. 3c) vector basis functions. The accuracy, time and computational load required for these basis function cases will be studied.

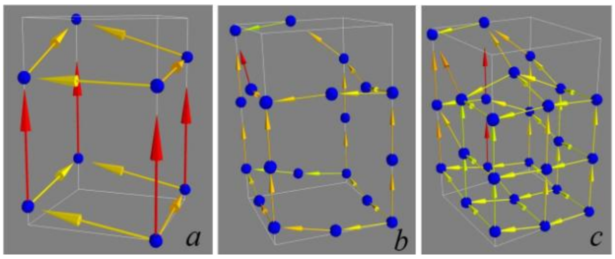

Figure 3 a) Linear element defined with 8 nodes and 12 edges. b) Quadratic element defined with 20 nodes and 24 edges. c) Cubic element defined with 26 nodes and 48 edges.

BOUNDARY CONDITIONS

For MT modelling, the most common boundary condition is Dirichlet boundary. This condition considers that boundaries lie far away from the inhomogeneities, so the fields vanish at the boundary. This is applied by assuming a value of zero for the tangential fields (e.g. $\hat{n} \times \vec{H}_0^{\,s} = 0$) on the boundaries of the model domain (i.e. at the boundary of the extension zones). In this case we don't need to consider the boundary integration in eq. (2), so $c_{\Gamma} = 0$. The problem with this boundary condition is that sometimes is not computationally possible to extend the domain until the condition is achieved. This brings some field reflection from the boundaries back to the inhomogeneities, making the solution inaccurate.

To deal with this problem, Berenger (1994) proposed an absorbing boundary zone where the fields decay with distance until it vanish at the boundary of the domain. This Perfect Matched Layer (PML) absorbs the waves that strike it, without reflecting it backwards. Moreover, Fang and Wu (1996) offered a PML scheme that absorbs also evanescent waves; this scheme is called the Generalized Perfect Matched Layer. MT waves are of evanescent nature, so the GPML is used in this methodology. This application is done by introducing a complex coordinate stretching factor $h^{(x_i)}$ on the governing equations, eq. (5), (Zhou and Greenhalgh, 2011).

$$\begin{aligned} &\partial_{\tilde{x}_j} = \left(1 / h^{(x_j)} \right) \partial_{x_j}, \quad h^{(x_j)} = 1 - ib\left(x_j\right) \\ &b(x_j) = c_0 \left(\left| \vec{r}(x_j) - \vec{r}_a \right| / \left| \vec{r}_b - \vec{r}_a \right| \right)^n \end{aligned} \tag{5}$$

In eq. (5), c_0 and n are constant and are to be selected in terms of the thickness $\left|\vec{r}_b - \vec{r}_a\right|$, where $\vec{r}_a$ and $\vec{r}_b$ are the ending points of the PML. By applying the coordinate stretching into the governing equations, we obtain a new formulation in term of the complex coordinate derivatives, eq. (6).

$$\begin{aligned} &\tilde{\nabla} \times \left(\mathbf{v}^{-1} \cdot \tilde{\nabla} \times \vec{F} \right) + i\omega \boldsymbol{\kappa} \cdot \vec{F} = \vec{s} \\ &\tilde{\nabla} = \left(\partial_{\tilde{x}_1}, \partial_{\tilde{x}_2}, \partial_{\tilde{x}_3} \right) \end{aligned} \tag{6}$$

OUTCOMES

A multi-order VFEM for hexahedral elements is being developed. The main outcome of this project will be a 3D MT forward modelling technique in frequency domain. This technique can be applied for isotropic and anisotropic media, with complex topography and subsurface interfaces. This method will be validated with an analysis of convergence of the resulting model with analytical data. The analytical data for this validation will be the solution of a homogeneous earth and a layered earth. With this analysis, sampling densities, and air layer height that yields to a faster and accurate model will be obtained. In addition a comparison of different edge vector orders will be carried out, and a comparison of different boundary conditions. This method will be used to understand the topography effect and anisotropy effect on the solution of MT modelling.

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