TWISTED MAGNETIC FIELDS IN CONDUCTING FLUIDS

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Summary

The formation of loops in the lines of force of a twisted magnetic field confined within a cylinder of radius R, first suggested by Alfvén (1950*a*), is discussed by the method of normal modes. The model first becomes unstable with respect to modes which do not lead to the formation of loops. Ignoring this, the condition obtained for loop formation is that the pitch of the twisted field be less than πR.

The velocity of Alfvén waves in this model is also discussed.

I. INTRODUCTION

There are several reasons why the study of twisted magnetic fields should be relevant to solar physics. For example, some prominences show filaments which give the impression of being twisted. Alfvén (1950*a*) drew attention to the formation of loops in a string when it is twisted beyond a certain limit, and pointed out that this will also happen to a magnetic field. While Alfvén was concerned with the origin of the terrestrial and solar magnetic fields, this phenomenon might also be important in prominences and perhaps in the generation of sunspots, although it is not involved in Alfvén's (1950*b*) own theory of sunspots.

Alfvén (1950*a*) discussed the model in which a tube of force of radius R containing a uniform magnetic field is given a uniform twist. By consideration of the magnetic energy he showed that the condition for loop formation is approximately

$$R>(\sqrt{5}-1)p, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where $2\pi p$ is the pitch of the twist. Lundquist (1951) investigated the stability of the general twisted field, which is represented in cylindrical coordinates by the two components H_φ and H_z, these being arbitrary functions of r. He found the field to be unstable if

$$\int H_\varphi^2 r\mathrm{d}r > 2\int \mathrm{H}_z^2 r\mathrm{d}r,$$

which for Alfvén's model becomes

$$R>2p. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

However, this result is obtained by considering the stability with respect to a specific type of displacement which, as Lundquist points out, is not the most critical. In the present paper the stability of Alfvén's model is discussed by the method of normal modes.

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II. The Normal Modes

We first formulate the basic magneto-hydrodynamic equations in the case of a general twisted field

$$\mathbf{H}=(0, H_\varphi(r), H_z(r)), \qquad (3)$$

contained wholly within a cylindrical tube of force of radius R. From a consideration of small oscillations of frequency ω in the conducting medium, we obtain the criterion for stability by putting $\omega=0$. It can be shown that the stability does not depend on the density or compressibility of the conducting fluid, which is thus taken to have uniform mass density μ and to be incompressible. As usual, the electrical conductivity is taken to be infinite. A small material velocity $\mathbf{v}$ then causes a small change $\mathbf{h}$ in the magnetic field $\mathbf{H}$ determined by

$$\frac{\partial \mathbf{h}}{\partial t}=(\mathbf{H}\,.\,\mathrm{grad})\mathbf{v}-(\mathbf{v}\,.\,\mathrm{grad})\mathbf{H}. \qquad (4)$$

The effect of the magnetic force on the motion is represented to the first order by

$$4\pi\mu\frac{\partial \mathbf{v}}{\partial t}=-\mathrm{grad}\ \psi+(\mathbf{H}\,.\,\mathrm{grad})\mathbf{h}+(\mathbf{h}\,.\,\mathrm{grad})\mathbf{H}, \qquad (5)$$

where $\psi/4\pi$ is the variation in " total pressure ", that is, gas pressure plus magnetic pressure ($=H^2/8\pi$).

When $\omega\neq 0$ equations (4) and (5) may be solved with the aid of the relation

$$\mathrm{div}\ \mathbf{v}=0. \qquad (6)$$

By seeking solutions proportional to

$$\exp\ \mathrm{i}(\omega t+kz+m\varphi), \qquad (7)$$

and writing

$$K=kH_z+\frac{m}{r}H_\varphi, \qquad (8)$$

we obtain from (4), (5), and (6) the following equations :

$$\mathrm{i}\omega h_r=\mathrm{i}Kv_r, \qquad (9)$$

$$\mathrm{i}\omega h_\varphi=\mathrm{i}Kv_\varphi-v_r\frac{\partial H_\varphi}{\partial r}+v_r\frac{H_\varphi}{r}, \qquad (10)$$

$$\mathrm{i}\omega h_z=\mathrm{i}Kv_z-v_r\frac{\partial H_z}{\partial r}, \qquad (11)$$

$$4\pi\mu\mathrm{i}\omega v_r=-\frac{\partial\psi}{\partial r}+\mathrm{i}Kh_r-2\frac{H_\varphi}{r}h_\varphi, \qquad (12)$$

$$4\pi\mu\mathrm{i}\omega v_\varphi=-\mathrm{i}m\psi/r+\mathrm{i}Kh_\varphi+\left(\frac{\partial H_\varphi}{\partial r}+\frac{H_\varphi}{r}\right)h_r, \qquad (13)$$

$$4\pi\mu\mathrm{i}\omega v_z=-\mathrm{i}k\psi+\mathrm{i}Kh_z+\frac{\partial H_z}{\partial r}h_r, \qquad (14)$$

$$\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\mathrm{i}mv_\varphi/r+\mathrm{i}kv_z=0. \qquad (15)$$

After some elimination, we then find

$$(4\pi\mu\omega^2-K^2)\left(\frac{\partial v_r}{\partial r}+\frac{v_r}{r}\right)+\frac{2mKH_\varphi}{r^2}v_r=\mathrm{i}\omega\left(k^2+\frac{m^2}{r^2}\right)\psi, \quad \ldots\ldots \quad (16)$$

$$\omega\frac{\partial\psi}{\partial r}+\mathrm{i}(4\pi\mu\omega^2-K^2)v_r+2\frac{H_\varphi}{r}\left\{\mathrm{i}v_r\frac{\partial H_\varphi}{\partial r}-\frac{\mathrm{i}H_\varphi v_r}{r}\left(\frac{4\pi\mu\omega^2+K^2}{4\pi\mu\omega^2-K^2}\right)-\frac{\omega Km\psi}{(4\pi\mu\omega^2-K^2)r}\right\}=0. \quad \ldots\ldots \quad (17)$$

From these equations the necessary boundary conditions for the components v_r and ψ at $r=R$ readily follow ; for it is evident from the form of equation (**16**), and is in any case physically obvious, that v_r must be continuous across the boundary $r=R$. Also, the discontinuous change $\delta\psi$ in ψ across the boundary is related to the value of v_r at $r=R$ through the term involving $\partial H_\varphi/\partial r$ in (**17**), which yields

$$v_r=\mathrm{i}\omega\,.\,\frac{R}{H_\varphi{}^2}\delta\psi. \quad \ldots\ldots \quad (18)$$

In Alfvén's model, where H_z is constant and H_φ takes the form

$$H_\varphi=Ar, \quad \ldots\ldots \quad (19)$$

appropriate to a uniformly twisted field,

$$K=kH_z+mA$$

is a constant for $r<R$ and vanishes for $r>R$. Eliminating v_r between equations (**16**) and (**17**) we obtain the modified Bessel equation

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\,\frac{\partial\psi}{\partial r}-\left(k'^2+\frac{m^2}{r^2}\right)\psi=0, \quad \ldots\ldots \quad (20)$$

where

$$k'^2=k^2\left\{1-\frac{4K^2A^2}{(4\pi\mu\omega^2-K^2)^2}\right\}. \quad \ldots\ldots \quad (21)$$

Then, since ψ remains finite at the origin, the variation of the total pressure inside the tube of force is given by

$$\psi=\alpha I_m(k'r)e^{\mathrm{i}(\omega t+kz+m\varphi)}, \quad \ldots\ldots \quad (22)$$

where $I_m(k'r)$ is the modified Bessel function (Jeffreys 1950) of the first kind of order m and α is an arbitrary constant.

Outside the tube of force, $k'=k$, and then the solution of (**20**), vanishing at infinity, is

$$\psi=\beta K_m(kr)e^{\mathrm{i}(\omega t+kz+m\varphi)}, \quad \ldots\ldots \quad (23)$$

where $K_m(kr)$ is the modified Bessel function of the second kind of order m and β is an arbitrary constant. Generalizing to allow for different densities μ_1 for

$r<R$ and μ_2 for $r>R$ and substituting from (22) and (23) we find that the boundary requirements at $r=R$ are satisfied if

$$\frac{(4\pi\mu_1\omega^2-K^2)\dfrac{x}{I_m(x)}\dfrac{\mathrm{d}I_m(x)}{\mathrm{d}x}-2mKA}{(4\pi\mu_1\omega^2-K^2)^2-4K^2A^2}=\frac{y\cdot\dfrac{\mathrm{d}K_m(y)}{\mathrm{d}y}}{4\pi\mu_2\omega^2K_m(y)-A^2y\dfrac{\mathrm{d}K_m(y)}{\mathrm{d}y}}, \quad \ldots\ldots\ldots\ldots (24)$$

where $x=k'R$ and $y=kR$.

Before proceeding to the case $\omega=0$ it is of interest to apply the above general formulae to the propagation of Alfvén waves along a uniform untwisted tube of force ($A=0$).

III. Alfvén Waves in a Uniform Untwisted Tube of Force

In this case the velocity of torsional waves in an infinite conducting fluid of density μ_1 follows from the fundamental equations (9)–(15) by putting

$$v_r=0. \quad \ldots\ldots\ldots\ldots (25)$$

Then $\psi=0$ and $4\pi\mu_1\omega^2-K^2=0$. Hence for the velocity of torsional waves we obtain

$$V=\pm V_a,$$

where V_a is Alfvén's velocity

$$V_a=\frac{H_z}{\sqrt{4\pi\mu_1}}. \quad \ldots\ldots\ldots\ldots (26)$$

If, however, we drop the condition (25) and seek instead solutions which satisfy the general boundary condition (24) we find that the solution for $r<R$ is now

$$\psi=\alpha I_m(kr)\mathrm{e}^{\mathrm{i}(\omega t+kz+m\varphi)}.$$

Substitution in (24) yields

$$V_m=\pm V_a\left\{1-\frac{\mu_2}{\mu_1}\cdot\frac{I_m^{-1}(y)\dfrac{\mathrm{d}I_m(y)}{\mathrm{d}y}}{K_m^{-1}(y)\dfrac{\mathrm{d}K_m(y)}{\mathrm{d}y}}\right\}^{-\frac{1}{2}}, \quad \ldots\ldots (27)$$

where $y=kR$ and V_m is the velocity of waves parallel to the z-axis for any given mode m. Equation (27) shows the reduction of wave velocity due to the inertia of the surrounding fluid. Simple consideration of the expansions of the Bessel functions for large and small y shows that

$$V_m\to\pm\frac{1}{\sqrt{2}}V_a$$

as kR becomes very large or very small. This conclusion is true for all m except $m=0$, which is exceptional in that $V_m\to\pm V_a$ for small kR. The variation of wave velocity with kR is illustrated in Figure 1 for the case $m=1$, with $\mu_1=\mu_2$.

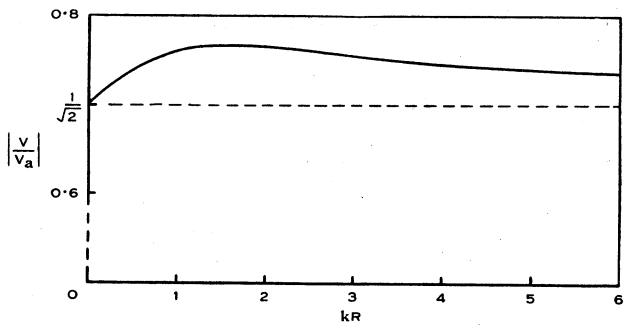

Fig. 1

IV. The Critical Case $\omega=0$

When $\omega=0$, equation (4) requires $\mathbf{v}=0$, so that the fundamental equation (5) must now be supplemented by the relation

$$\operatorname{div} \mathbf{h}=0, \qquad (28)$$

which, when $\omega\neq0$, is a consequence of (4). We thus obtain the equations

$$\frac{\partial\psi}{\partial r}=\mathrm{i}Kh_r-\frac{2H_\varphi}{r}\,.\,h_\varphi,$$

$$\frac{\mathrm{i}m\psi}{r}=\mathrm{i}Kh_\varphi+\left(\frac{\partial H_\varphi}{\partial r}+\frac{H_\varphi}{r}\right)h_r,$$

$$\mathrm{i}k\psi=\mathrm{i}Kh_z+\frac{\partial H_z}{\partial r}\,.\,h_r,$$

$$\frac{\partial h_r}{\partial r}+\frac{h_r}{r}+\mathrm{i}mh_\varphi/r+\mathrm{i}kh_z=0,$$

which, if $H_\varphi=Ar$, reduce to the forms

$$K\left(\frac{\partial h_r}{\partial r}+\frac{h_r}{r}\right)-\frac{2mA}{r}h_r+\mathrm{i}\left(k^2+\frac{m^2}{r^2}\right)\psi=\frac{\partial K}{\partial r}h_r \qquad (29)$$

and

$$K\frac{\partial\psi}{\partial r}+\frac{2mA}{r}\psi-\mathrm{i}(K^2-4A^2)h_r=-\mathrm{i}r\frac{\partial A^2}{\partial r}h_r. \qquad (30)$$

With Alfvén's model, the right-hand sides of (29) and (30) vanish for $r<R$, and we can obtain by elimination Bessel's equation giving the solution

$$\psi=\alpha I_m(k'r)e^{\mathrm{i}(\omega t+kz+m\varphi)}, \qquad (31)$$

where

$$k'^2=k^2\left(1-\frac{4A^2}{K^2}\right). \qquad (32)$$

The discontinuous change in ψ at $r=R$ can be deduced from the derivatives on the right-hand sides of (29) and (30) by noting that, for $r>R$, $K=A=0$

and hence $\psi=0$. Equation (**29**) implies the continuity of h_r/K at $r=R$, and then equation (**30**) shows that

$$\psi=-\mathrm{i}\,.\,R\,.\,A^2\,.\,\frac{h_r}{K}. \qquad \qquad (33)$$

This is the boundary condition at $r=R$ and, with the aid of (**30**), may be written in the form

$$\left\{\frac{r}{\psi}\cdot\frac{\partial\psi}{\partial r}\right\}_{r=R}=4-\left(\frac{K}{A}\right)^2-\frac{2mA}{K}, \qquad \qquad (34)$$

provided that $K^2\neq 4A^2$.

The latter condition may be further restricted; for if $K^2>4A^2$, k' is real and then

$$\frac{r}{I_m(k'r)}\cdot\frac{\mathrm{d}I_m(k'r)}{\mathrm{d}r}\geqslant|\,m\,|. \qquad \qquad (35)$$

However, with $K^2>4A^2$, the right-hand side of (**34**) is less than $|\,m\,|$ and hence there are no solutions with $K^2>4A^2$.

The possibility of a solution with $K^2=4A^2$ requires separate investigation. From (**30**) it follows that mA/K must be negative and ψ proportional to $r^{|m|}$. Then equation (**29**) yields

$$2Kh_r=-\mathrm{i}\psi\left\{\frac{|\,m\,|}{r}+\frac{k^2r}{|\,m\,|+1}\right\},$$

which requires

$$\mathrm{i}Kh_r/\psi>0$$

at $r=R$ and thus contradicts the boundary condition (**34**). Hence no such solution exists.

When $K^2<4A^2$ it is convenient to define

$$\varkappa^2=k^2\left(\frac{4A^2}{K^2}-1\right), \qquad \qquad (36)$$

and then the solution for $r<R$ takes the more familiar form

$$\psi=\alpha J_m(\varkappa r)\mathrm{e}^{\mathrm{i}(\omega t+kz+m\varphi)}, \qquad \qquad (37)$$

which is now an oscillatory function of r. Since $\varkappa$ can be made large by making K small solutions of the form (**37**) must exist. It is sufficiently general to consider only positive values of k, H_z, and A provided that both positive and negative values of m are allowed. If

$$p=H_z/A, \qquad \qquad (38)$$

then $2\pi p$ is the pitch of the uniformly twisted field. With this notation the inequality $K^2<4A^2$ may be written

$$(kp+m)^2<4,$$

and this eliminates the modes with $m\geqslant 2$.

V. The Conditions of Stability

The critical values of A for stability are obtained from the solutions with $\omega=0$.

Let

$$F_m(x)=\frac{x}{J_m(x)}\cdot\frac{dJ_m(x)}{dx}, \qquad (39)$$

and

$$g_m(kp)=4-(kp+m)^2-\frac{2m}{kp+m}, \qquad (40)$$

then the boundary condition (34) is expressed by the relation

$$F_m(\varkappa R)=g_m(kp). \qquad (41)$$

The following properties of the function $F_m(x)$ are subsequently required

$$\left.\begin{array}{ll} F_m(x)=|m|, & \text{when } x=0, \\ \dfrac{dF_m(x)}{dx}\leqslant 0, & \text{when } x\geqslant 0, \\ \text{and} \quad F_m(x) \text{ has poles at the zeros of } J_m(x). & \end{array}\right\} \qquad (42)$$

For given values of m, k, and p equation (41) determines a set of values of R, the smallest of which will be called R_0.

To determine the ranges of values of A for which imaginary eigenvalues of ω can occur we refer back to equation (20), which has solutions of the form

$$\psi=\alpha J_m(\varkappa' r)e^{i(\omega t+kz+m\varphi)}, \qquad (43)$$

where

$$\varkappa'^2=k^2\left\{\frac{4K^2A^2}{(4\pi\mu\omega^2-K^2)^2}-1\right\}. \qquad (44)$$

When $K^2>4A^2$ a small imaginary value of ω will still leave $\varkappa'$ real and the value of $F_m(\varkappa' R)$ given by (24) will vary only slightly. An eigenvalue of ω is therefore determined by $\varkappa' R\approx\varkappa R_0$ and since, when ω is imaginary, $|\varkappa'|<|\varkappa|$, the condition for instability is expressed by the inequality

$$R>R_0. \qquad (45)$$

It is evident that instability will occur for $m<0$ when $|kp+m|$ is small enough since then $\varkappa$ is large and R_0 small. This type of instability is not relevant to loop formation. If $kp+m=0$, the resultant motion has the same symmetry as the twisted field. Only the mode $m=-1$ can be observed with a string; the string as a whole takes a helical form. It will be seen that this occurs before loop formation, so that a complete study would require an investigation of the new equilibrium configuration of the field, but this seems to be rather intractable. It may also be noted that this type of instability will not always occur for more general models; it is facilitated when the pitch of the twist is independent of r.

The only modes for which the axial line of force moves are those with $m=\pm 1$; this is easily seen by substituting the solution (43) into equation (17), which then shows that $v_r=0$ at $r=0$ unless $m=\pm 1$. The other modes are therefore irrelevant to a discussion of loop formation. If the field is unstable with respect to $m=-1$ only, the tube of force as a whole takes up a helical configuration, but does not form loops. The type of motion required in forming loops occurs only when the field is simultaneously unstable with respect to both the modes $m=\pm 1$. This is therefore the best criterion for loop formation obtainable from the present restricted analysis.

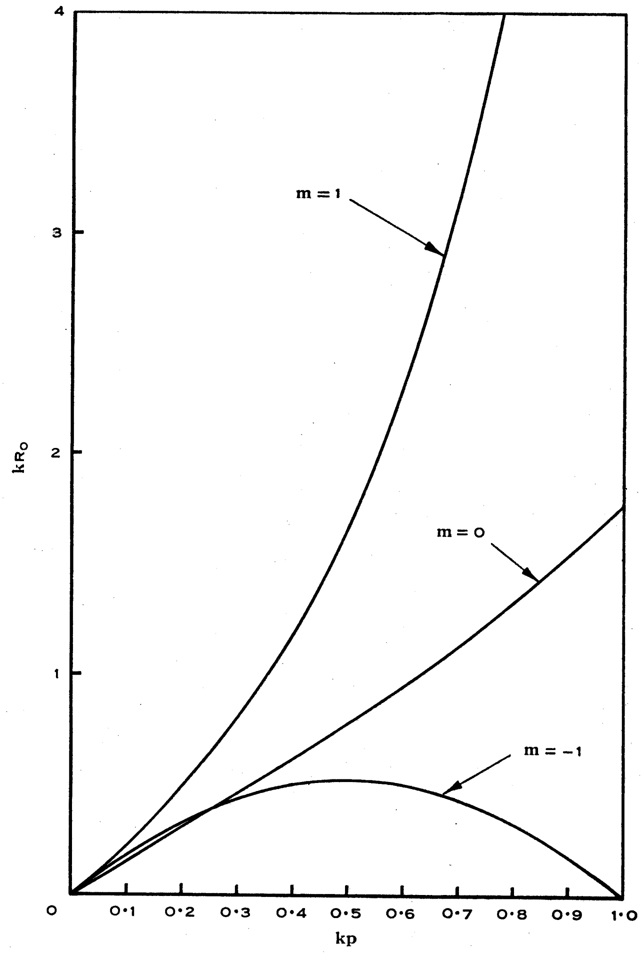

Fig. 2

The values of kR_0 obtained from (**41**) are plotted against kp in Figure 2 for $m=\pm 1$ and 0. It is clear from this diagram that, for any given kp, the corresponding value of kR_0 when $m=1$ exceeds the value when $m=-1$. Hence it follows from (**45**) that when the field is unstable with respect to $m=1$ it is always unstable with respect to $m=-1$, so that the condition for loop formation can be taken as the condition for instability with respect to $m=1$. For small values of kp this condition is

$$R>2p, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (46)$$

which follows from the relation $R_0 \approx 2p$ implied by equation (**41**) when $m = \pm 1$ and kp is small. The criterion (**46**) is the same as that obtained by Lundquist (1951) (cf. equation (**2**)), and this is not altogether surprising, since the system is unstable with respect to many modes and Lundquist's disturbance may be a combination of these.

The graph also shows that loop formation cannot occur with $kp > 1$; this simply means that no more loops can be formed than there are twists. If there is more physical limitation on k, such that k must be an integral multiple of $2\pi/L$ say, the critical value of p for loop formation will be decreased; if, however, $\pi R \ll L$, the decrease will be only slight.

VI. Conclusions

We may summarize the above results as follows:

In the normal modes the disturbance varies as $\exp i(\omega t + kz + m\varphi)$.

The velocity of Alfvén waves in a uniform untwisted tube of force is reduced by the motion in the surrounding material by a factor which approximates to $(1 + \mu_2/\mu_1)^{-\frac{1}{2}}$ for both long and short wavelengths (see Fig. 1).

A twisted tube of pitch $2\pi p$ is unstable with respect to a given mode when $| kp + m |$ is small enough; it is always stable when $| kp + m | > 2$. The formation of loops requires simultaneous instability with respect to two modes with $m = +1$ and -1 and with the same value of k. This requires $p < \frac{1}{2}R$.

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