THEORY OF TRACKS IN NUCLEAR RESEARCH EMULSIONS

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Summary

The process of the formation of a visible track in a nuclear research emulsion is approximated by a simplified model which reduces it to a one-dimensional problem. Within this basic approximation, there are included many different detailed models, including all models so far proposed in the literature. We then consider the following quantities characteristic of the track: the number of gaps longer than some minimum length \( r \) contained in a cell length \( t \) of the track, \( n_r(t) \); and the sum of the excess (over and above \( r \)) lengths of these \( n_r \) gaps, \( x_r(t) \). We derive complete theoretical results for the mean values and standard deviations of these quantities, in terms of a small number of definite integrals. We then specialize to particular detailed models, and make a quick comparison against experiment. The model of O'Coallaigh fails to agree with experimental data, and so do all straightforward generalizations of that model, such as the model of Happ, Hull, and Morrish. The reason for this failure is traced to the neglect of the "graininess" of the emulsion before development. The model of Herz, which takes the extreme opposite point of view, comes much closer to fitting the data. We also give a reasonable modification of the Herz model. A more detailed comparison against experiment must still be made.

I. INTRODUCTION, DEFINITIONS, AND BASIS APPROXIMATIONS

A photographic emulsion consists of crystals of roughly spherical shape suspended in gelatin. In G5 emulsions, roughly half the volume is taken up by the crystals, the other half is gelatin. A charged particle passing through the emulsion activates some of the crystals through which it travels. When the emulsion is developed, each "activated" crystal grows into a "grain" of considerably larger radius (approximately twice the radius of the undeveloped crystal under usual conditions of development), while the inactive crystals dissolve away. If two activated crystals are close enough together, the final grains can overlap and form a "blob". Since the size of the final grains is close to the limit of optical resolution, it is generally difficult to tell under the microscope whether a blob consists of one, two, or three grains; a single large grain may be mistaken for a two-grain blob, for example. Hence it has become customary to do "grain-counting" by counting blobs (or, equivalently, by counting the gaps between the blobs). It is then necessary to develop a theoretical expression which relates the observed number of gaps per unit length of the track, or some other measurable quantity characteristic of the track, to the fundamental physical characteristics of the emulsion (initial crystal

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size, distribution of initial crystals in space, final grain size) and of the charged particle (probability of activation of a traversed crystal). Furthermore, since the process of track formation is statistical in character, we need to know the standard deviations which must be attached to the measured track characteristics. Indeed, the original impetus for this study arose from the experimental observation that the standard deviation of the number of gaps in a certain length of track $t$ (the "cell length") is less than the standard deviation in a Poisson distribution, and furthermore seems to depend on the character of the track ("thin", "medium", or "thick"). It is the purpose of this paper to provide the necessary theoretical expressions to analyse experimental data.

The main difficulty in such a calculation lies in the spatial distribution of the initial grains. If we could assume that the initial grains traversed by the charged particle all have their centres along the path of the particle, the theory would be greatly simplified. We now present qualitative arguments to show that this is a very good approximation under normal conditions. Figure 1 (a)

![Figure 1](image)

Figure 1 (a) gives a schematic picture of the emulsion as it may actually be, and indicates the path of the charged particle. Let us assume that some but not all of the crystals along this path are activated; the activated ones are indicated by shading. Figure 1 (b) shows this same track after development. Each activated crystal has grown to roughly twice the original size, the other crystals have dissolved. Some of the final grains have combined into blobs, others have remained single.

Figure 2 (a) shows the same track in the "linear approximation", in which we assume that the centre of each activated grain lies directly on the path of the charged particle (since the inactive grains are never seen, no assumption has to be made about their centres). Figures 1 (a) and 2 (a) differ appreciably. However, let us now look at the result after development. Figure 2 (b) shows the same track after development, again in the linear approximation. We see that the growth of the crystals into grains has had the result of decreasing the difference between Figures 1 (b) and 2 (b) compared to the difference between Figures 1 (a) and 2 (a). Indeed, the same grains which had previously merged
into blobs still do so in the linear approximation, except for a few borderline cases, perhaps. The larger the growth of the crystals during the process of development, the less important is the exact location of the centre of each grain. Hence we conclude that the linear approximation is adequate for this problem if there is appreciable growth of the crystals during development. Conversely, if it were possible to find a method of developing the emulsion in such a way that the activated crystals would not grow appreciably, the linear approximation would be of very doubtful validity.

For present methods of development, which result in a grain growth of roughly a factor of two, we feel that the linear approximation is entirely adequate. Indeed, a first approximation to the necessary correction terms can be obtained by observing that the random sideways displacements of the centres of activated crystals from the nearest point along the path of the charged particle are roughly equivalent to random changes in the final size of the grown grains. Since there is a distribution of final grain sizes in any case, the linear approximation is even more accurate than the comparison between Figures 1 and 2 indicates. From now on we shall adopt the linear approximation throughout this paper.

In this approximation, the visible characteristics of the track depend upon two probability distributions only:

1. The distribution-in-final-size of the grains after development. We let $\gamma$ stand for the radius of the developed grain, and we let $G(\gamma)d\gamma$ be the probability that this radius lies between $\gamma$ and $\gamma + d\gamma$.

2. The "survival probability" $Q(t)$; this is the probability that the distance between the centre points of adjacent activated grains exceeds $t$. Alternatively, $Q(t)$ is the probability that the charged particle, after activating a crystal along its path, "survives" for a distance larger than $t$ before activating the next crystal.

We shall solve the statistical problems for two quantities of interest in the experimental study of tracks, for completely arbitrary functions $G(\gamma)$ and $Q(t)$. By specializing to various particular functions $G(\gamma)$ and $Q(t)$, we can then deduce results for particular models of track formation, and compare them with experiment.
Since the distribution in final grain size \( G(\gamma) \) is experimentally a very sharp distribution, it is not necessary to explore all possible distribution functions. We shall give results for a delta function (infinitely sharp) \( G(\gamma) \), and for a particular family of functions \( G(\gamma) \) which is chosen so that (a) the integrals can be evaluated in closed form, (b) there are two adjustable parameters in \( G(\gamma) \) which can be used to fit the mode and the standard deviation of the observed distribution in sizes of single grains (as seen, for example, by studying a very thin track).

The simplest assumption we can make about the survival probability \( Q(t) \) is the one made by O'Ceallaigh (1953) as well as by Happ, Hull, and Morrish (1952). These models differ only in the distribution \( G(\gamma) \), which is assumed to be infinitely sharp by O'Ceallaigh but not by Happ, Hull, and Morrish. The survival probability in both models is the same, and is obtained by assuming that there is a constant probability \( \lambda dt \) per path length \( dt \) for activating the "next" crystal. The resulting survival probability is

\[
Q(t) = e^{-\lambda t}, \quad \text{"model A"}. \quad \text{............. (1.1)}
\]

This model neglects the correlations between positions of crystal centres in the emulsion. It is obviously a good approximation if the emulsion is very dilute, or (equivalently) if the probability of the charged particle activating any crystal it traverses is very low. In either case the final appearance of the track is that of a "thin" track, with wide spaces separating the individual grains and only little blob formation.

The extreme opposite model was chosen by Herz (Herz and Davis 1955), who assumes that the crystals are arrayed along the path of the charged particle in a regular lattice of definite lattice spacing. The charged particle has a probability \( p \) for activating each crystal it traverses. Hence the survival probability \( Q(t) \) is a step function:

\[
Q(t) = \begin{cases} 
1, & 0 < t < \beta, \\
1 - p, & \beta < t < 2\beta, \\
(1 - p)^2, & 2\beta < t < 3\beta,
\end{cases} \quad \text{"model B"}. \quad \text{..... (1.2)}
\]

The true situation is surely somewhere between these two extreme assumptions. If each crystal is a sphere of diameter \( \alpha \), then clearly \( Q(t) = 1 \) for \( 0 < t < \alpha \), in contradiction to model A. On the other hand, the emulsion is essentially amorphous, not crystalline, hence there can not be any long-range order. At distances of several crystal diameters, the "memory" of the location of the initial crystal is lost, and the survival probability becomes asymptotically equal to (1.1), in contradiction to model B. A schematic picture of \( Q(t) \) for models A and B and for the true emulsion is given in Figure 3.

It might be thought that the main error in model A consists of the neglect of the size of the initial crystal; hence we have also calculated various quantities measured on tracks with a third model, which we shall call model C:

\[
Q(t) = \begin{cases} 
1, & 0 < t < \alpha, \\
\exp \left[ -\lambda(t - \alpha) \right], & t > \alpha.
\end{cases} \quad \text{"model C"}. \quad \text{..... (1.3)}
\]
We now define the "number of gaps longer than \( r \)", \( n_r(t) \), in a track of length \( t \). We are free to define this quantity in such a way as to make the subsequent mathematics easy, and we shall do so. It should be emphasized that our definition of a "countable gap", while simple enough to learn, is not identical with the usual one. The "last" gap is sometimes included in the count, sometimes excluded. This makes little difference if the cell length \( t \) is long enough, but it is important when comparing the theory against experiments with a short cell length. Since the experiments on the variance in the number of gaps necessarily use rather short cell lengths, this caution is important.

![Diagram showing countable gaps](image)

Our definition of a "countable gap" consists of two parts:

1. The initial point \( t=0 \) of each cell length must be inside a blob.
2. The grain terminating each "countable" gap must have its centre point within the cell length \( t \).
Hence, for example, the situation depicted in Figure 4 (a) corresponds to no gap at all. In general, our convention will result in omitting the "last" gap in thin tracks, while counting it in thick tracks.

The quantity $n_r(t)$ is defined as the number of countable gaps, each one of length equal to or larger than $r$, in a cell length $t$. The quantity $x_r(t)$ is the total length of these $n_r$ gaps, counting for each gap only the excess of its length over $r$.

In Section II we give formal solutions, in terms of definite integrals, for the distribution-in-$n_r$ and the distribution-in-$x_r$. This section is mathematical in character and can be omitted if desired. Sections III and IV contain the results in usable form, as well as a short discussion of the differences between the models and a short comparison with experimental data. An extended experimental check is now under way in this Laboratory, and will be the subject of a future publication.

II. A FORMAL SOLUTION USING THE REGENERATION POINT METHOD

Let $\pi_n(r, t)$ be the probability that the number $n_r(t)$ of countable gaps longer than $r$ in a cell length $t$ is equal to the integer $n$. We now derive an equation for $\pi_n(r, t)$ by using the "regeneration point method" of Bellman and Harris (1948) and Janossy (1950). We concentrate our attention on the position $t'$ of the "next" activated crystal. The "position" of any crystal is that of its centre point, by definition, no matter which point within the crystal was actually activated. The assumption is that during development each activated crystal grows as a whole, so that the precise point of activation within the crystal does not matter.

The probability $q(t')dt'$ that the centre of the "next" activated crystal lies between $t'$ and $t'+dt'$ is simply related to the survival probability $Q(t')$; namely,

$$q(t') = -\frac{dQ}{dt'}.$$  \hspace{1cm} (2.1)

We shall use (2.1) formally even when the process of differentiation is not permissible mathematically (e.g. when $Q(t)$ is the step function of model B). A rigorous mathematical treatment can be given in terms of Stieltje's integrals, but the final result is the same.

Depending upon the position $t'$ of the next activated crystal, and upon the final radii $\gamma_1$ and $\gamma_2$ of the initial and the "next" grains, there are three possibilities to consider:

(1) The grains growing from the initial and the next crystal merge to form a blob, or are so close that the resultant gap is of length smaller than our minimum length $r$. This happens when $t' < \gamma_1 + \gamma_2 + r$. In this case the remaining distance, $t-t'$, must contain $n$ gaps of length greater than $r$, in order to have a contribution to $\pi_n(r, t)$. This contribution is therefore

$$\int_0^\infty d\gamma_1 \int_0^\infty d\gamma_2 \int_0^{\gamma_1 + \gamma_2 + r} dt' q(t')G(\gamma_1)G(\gamma_2)\pi_n(r, t-t').$$
(2) The "next" crystal is far enough away to form a gap of length greater than \( r \), but its centre is still within our cell length, that is, we have \( \gamma_1 + \gamma_2 + r < t' < t \). The remaining distance, \( t - t' \), must now contain \( n - 1 \) gaps of length greater than \( r \), in order to contribute to \( \pi_n(r, t) \). The contribution to \( \pi_n(r, t) \) from this range of values of \( t' \) is

\[
\int_0^\infty d\gamma_1 \int_0^\infty d\gamma_2 \int_{\gamma_1 + \gamma_2 + r}^t dt' q(t') G(\gamma_1) G(\gamma_2) \pi_{n-1}(r, t-t').
\]

(3) The final possibility is that the next activated crystal lies outside the cell length altogether, that is, \( t' < t \). According to our conventions for "countable" gaps, we then say that there is no gap at all in our cell length. Hence possibility (3) makes a contribution to \( 7\pi_n(r, t) \) only.

We define the quantity \( \delta_n \) to equal unity when \( n = 0 \), to equal zero otherwise. Then the contribution of this range of \( t' \) to \( \pi_n(r, t) \) is

\[
\int_t^\infty dt' q(t') \delta_n = Q(t) \delta_n.
\]

We now observe that the possibilities (1), (2), and (3) above correspond to mutually exclusive events, of which exactly one must happen. Hence we can add the probabilities in order to get \( \pi_n(r, t) \), giving the regeneration point equation

\[
\pi_n(r, t) = \int_0^\infty d\gamma_1 \int_0^\infty d\gamma_2 \int_{\gamma_1 + \gamma_2 + r}^t dt' q(t') G(\gamma_1) G(\gamma_2) \pi_{n-1}(r, t-t') + Q(t) \delta_n.
\]

The actual ranges of integration over \( t' \) are restricted somewhat more than indicated in (2.2) by the condition*

\[
\pi_n(r, t) = 0, \quad \text{unless } t > nr.
\]

We now solve (2.2) by means of Laplace transforms. We introduce

\[
p_n(r, s) = \int_0^\infty e^{-st} \pi_n(r, t) dt.
\]

We multiply both sides of (2.2) by \( e^{-st} \) and integrate over \( t \). We then interchange the orders of integration with respect to \( t \) and \( t' \) on the right side. It is useful to introduce the notations:

\[
h(s) = s \int_0^\infty e^{-st} Q(t) dt = 1 - \int_0^\infty e^{-st} q(t) dt,
\]

\[
g(s) = \int_0^\infty d\gamma_1 \int_0^\infty d\gamma_2 \int_{\gamma_1 + \gamma_2 + r}^t dt e^{-st} q(t') G(\gamma_1) G(\gamma_2).
\]

* This condition can be made sharper if the final grain radii \( \gamma \) cannot be smaller than a minimum value \( \gamma_{\text{min}} \), i.e. if \( G(\gamma) = 0 \) for \( \gamma < \gamma_{\text{min}} \). Then the condition on \( \pi_n(r, t) \) is:

\[
t > n(r + 2\gamma_{\text{min}}).
\]
We then get the following Laplace transform equation from (2.2):

\[ p_n(r, s) = (1 - \hat{h} - \hat{g})p_n + np_{n-1} + s^{-1}h(s)\delta_n. \]  

......................... (2.7)

We set \( n = 0 \) to get

\[ p_0(r, s) = \frac{h(s)}{s[h(s) + g(r, s)]}. \]  

......................... (2.8)

We introduce the notation

\[ K(r, s) = \frac{g(r, s)}{h(s) + g(r, s)}, \]  

......................... (2.9)

and obtain the complete formal solution

\[ p_n(r, s) = [K(r, s)]^n p_0(r, s). \]  

......................... (2.10)

The equation (2.10) together with the defining equations (2.5), (2.6), (2.8), and (2.9) constitutes a formal solution to the statistical problem of the number of gaps larger than \( r \). The problem has been reduced to the computation of certain definite integrals, \( h(s) \) and \( g(r, s) \), and the subsequent inversion of the Laplace transforms (2.10). In practice the integrals \( h(s) \) and \( g(r, s) \) present little difficulty, but the inversion of the Laplace transform is usually not possible in closed form in terms of tabulated functions.

In practice, experimentalists are not interested in the probability \( \pi_n(r, t) \) of getting exactly \( n \) gaps of length in excess of \( r \). They would like to know the mean number of these gaps, and the standard deviation around this mean. These quantities are related to the moments of the distribution in \( n_r(t) \). We introduce the factorial moments \( F_k(r, t) \) by

\[ F_k(r, t) = \langle n_r(n_r - 1)(n_r - 2) \ldots (n_r - k + 1) \rangle_{\text{Ave}} = \sum_{n=0}^{\infty} n(n-1) \ldots (n-k+1)\pi_n(r, t). \]  

......................... (2.11)

The first few factorial moments are:

\[ F_0(r, t) = 1, \]  

......................... (2.12)

\[ F_1(r, t) = \langle n_r(t) \rangle_{\text{Ave}}, \]  

......................... (2.13)

\[ F_2(r, t) = \langle n_r^2(t) \rangle_{\text{Ave}} - \langle n_r(t) \rangle_{\text{Ave}}^2 \]  

......................... (2.14)

and the standard deviation is given by

\[ \sigma^2 = \bar{n}^2 - (\bar{n})^2 = F_2 - F_1. \]  

......................... (2.15)

The Laplace transforms of the factorial moments are given by

\[ f_k(r, s) = \int_0^\infty e^{-st} F_k(r, t) dt = \sum_{n=0}^{\infty} n(n-1) \ldots (n-k+1)\pi_n(r, s). \]  

......................... (2.16)
and substitution of (2.10) into (2.16) gives

\[
f_k(r, s) = k! p_0(r, s) \frac{[K(r, s)]^k}{[1-K(r, s)]^{k+1}}
\]

\[
= \frac{k!}{s} \left[ \frac{g(r, s)}{h(s)} \right]^k.
\]  \hspace{1cm} \text{(2.17)}

The Laplace transform (2.17) has a particularly simple structure, as a result of certain properties of the functions \(h(s)\) and \(g(r, s)\) which we shall now point out. We see from the definition (2.5) of \(h(s)\) that \(h(s) = 0\) when \(s = 0\); hence (2.17) has a pole of order \(k+1\) at the origin in the \(s\)-plane. Next we observe that \(|h(s)| > 0\) as long as the real part of \(s\) is greater than zero. Also \(h(s^*) = [h(s)]^*\) so that all its zeros must occur in complex conjugate pairs. The function \(g(r, s)\) defined by (2.6) has no poles in the positive half-plane of \(s\), nor at \(s = 0\). \(g\) may have poles along the negative \(s\)-axis, or pairs of complex conjugate poles in the negative half-plane of \(s\). We therefore conclude that \(f_k(r, s)\) has a pole of order \(k+1\) at \(s = 0\), no poles in the positive half-plane of \(s\), and (perhaps) pairs of conjugate complex poles in the negative half-plane of \(s\).

When the Laplace transform (2.17) is inverted by the method of residues, a pole at a point \(s_1\) contributes a term proportional to \(e^{s_1 t}\), unless \(s_1 = 0\), in which case we get a polynomial in \(t\). Since all poles except the one at \(s = 0\) lie in the negative half-plane of \(s\), their contributions are damped exponentials in \(t\); hence if \(t\) is large enough, these contributions can all be neglected.

We therefore restrict ourselves to the contribution of the pole at \(s = 0\), and obtain

\[
F_k(r, t) = \frac{d^k}{ds^k} \left[ \frac{sg(r, s)}{h(s)} \right]_{s=0}^k
\]

\[
= \frac{k!}{s} \left[ \frac{g(r, s)}{h(s)} \right]^k, \hspace{1cm} \text{(2.18)}
\]

Thus the problem of finding the moments of the distribution-in-\(n_r(t)\) has been reduced to the evaluation of the definite integrals (2.5) and (2.6) and simple differentiations. We emphasize that this has been achieved without any major loss of generality: the distribution \(G(\gamma)\) of final grain radii and the survival probability \(Q(t)\) are still completely arbitrary and at our disposal, subject only to the (not very stringent) condition that the integrals (2.5) and (2.6) can be done in closed form.

We now turn our attention to the distribution in the total gap length \(x_r(t)\). \(x_r(t)\) is defined as the sum of the "excess" lengths of all countable gaps longer than \(r\). We emphasize that gaps shorter than \(r\) make no contribution to \(x_r(t)\), while gaps longer than \(r\) (provided they are "countable" by our conventions) contribute not their whole length, but merely the excess of their length over \(r\). We now let \(\pi(r, x, t)dx\) be the probability that the total gap length \(x_r(t)\) lies in the range \(x, x+dx\). Since there exists a non-zero probability of having no gaps at all of length greater than \(r\), \(\pi(r, x, t)\) contains a singular contribution equal to \(\pi_0(r, t)\delta(x)\), where \(\pi_0(r, t)\) is the probability of finding no gaps at all (longer
than \( r \) in the cell length \( t \), and \( \delta(x) \) is Dirac's delta function. We shall show the existence of this contribution in our solution later on.

By arguments entirely analogous to the ones used in deriving equation (2.2), we find the following regeneration point equation for \( \pi(r, x, t) \):

\[
\pi(r, x, t) = \int_0^\infty d\gamma_1 \int_0^\infty d\gamma_2 \int_0^{\gamma_1 + \gamma_2 + r} dt' q(t') G(\gamma_1) G(\gamma_2) \pi(r, x, t - t')
\]

\[+ \int_0^\infty d\gamma_1 \int_0^\infty d\gamma_2 \int_0^{\gamma_1 + \gamma_2 + r} dt' q(t') G(\gamma_1) G(\gamma_2) \pi(r, x - t' + \gamma_1 + \gamma_2 + r, t - t')
\]

\[+ Q(t) \delta(x). \] ........................................ (2.19)

We solve this equation by Laplace transforms; this time we need a double Laplace transform, on the variable \( x \) as well as on the variable \( t \). We define

\[
\varphi(r, u, s) = \int_0^\infty dx \int_0^\infty dt \exp(-ux-st) \pi(r, x, t). \] ........ (2.20)

We multiply both sides of (2.19) by \( \exp(-ux-st) \) and integrate over \( x \) and \( t \). In addition to the functions \( h(s) \) and \( g(r, s) \) encountered previously, we need to define an additional function

\[
v(r, u, s) = \int_0^\infty d\gamma_1 \int_0^\infty d\gamma_2 \int_0^{\gamma_1 + \gamma_2 + r} dt' q(t') G(\gamma_1) G(\gamma_2)
\]

\[\times \exp[(\gamma_1 + \gamma_2 + r)_u - (u + s)t]. \] ................. (2.21)

We observe that

\[
v(r, 0, s) = g(r, s), \] .............................. (2.22)

and

\[
\lim_{u \to \infty} v(r, u, s) = 0. \] .......................... (2.23)

The Laplace transform image of equation (2.19) then is

\[
\varphi(r, u, s) = [1 - h(s) - g(r, s)] \varphi(r, u, s) + v(r, u, s) \varphi(r, u, s) + s^{-1}h(s),
\]

which gives the formal solution

\[
\varphi(r, u, s) = \frac{h(s)}{s[h(s) + g(r, s) - v(r, u, s)]}, \] .......................... (2.24)

In the general case, the function \( v(r, u, s) \) may be difficult to find in closed form, and the subsequent inversion of the double Laplace transform (2.24) can be very difficult indeed. Hence (2.24) as it stands is not directly a practical solution. However, we shall show that information of practical value can be obtained by using (2.24).

First of all, we perform two checks on (2.24):

(1) When \( u = 0 \), \( \varphi(r, 0, s) \) must be the Laplace transform of the probability of getting any gap length \( x \) whatever, by comparison with (2.20); substitution of (2.22) into (2.24) gives \( \varphi(r, 0, s) = s^{-1} \), which is indeed the Laplace transform of unity.
(2) \( \varphi(r, u, s) \) must contain the Laplace transform of the singular contribution 
\( \pi_0(r, t) \delta(x) \), that is, the quantity \( p_0(r, s) \), equation (2.8). We can 
isolate this contribution by going to the limit of large positive \( u \), since 
all non-singular functions of \( x \) have Laplace transforms which vanish 
for large positive \( u \). Substitution of (2.23) into (2.24) shows that 
\[
\lim_{u \to \infty} \varphi(r, u, s) = p_0(r, s),
\]
as it should be.

Next, we observe that we are primarily interested in the moments of the 
distribution in \( x_r(t) \). Multiplication of \( \pi(r, x, t) \) by \( x^k \) and integration over \( x \) 
has a simple Laplace transform image: \( k \)-fold differentiation with respect to 
the transform variable \( u \), multiplication by \(-1\)^k, and setting \( u=0 \). We 
therefore define the quantity \( \varphi_k(r, s) \) by 
\[
\varphi_k(r, s) = (-1)^k \left[ \frac{\partial^k}{\partial u^k} \varphi(r, u, s) \right]_{u=0} \quad \ldots \quad (2.25)
\]
Then \( \varphi_1(r, s) \) is the Laplace transform of \( x_r(t) \), \( \varphi_2(r, s) \) is the Laplace transform 
of \( x_r^2(t) \), and so on. Since higher moments give rather complicated expressions, 
we shall restrict ourselves to the first two moments, which are sufficient to give 
the mean value and the standard deviation.

In differentiating (2.24) with respect to \( u \), we notice that the only quantity 
depending on \( u \) is \( v(r, u, s) \). It helps to introduce a notation for the derivatives 
of \( v \). We define 
\[
g_k(r, s) = (-1)^k \left[ \frac{\partial^k v(r, u, s)}{\partial u^k} \right]_{u=0} = 
\int_0^\infty d\gamma_1 \int_0^\infty d\gamma_2 \int_0^\infty \int_0^\infty dt \, (t-\gamma_1-\gamma_2-r)^k q(t) G(\gamma_1) G(\gamma_2) e^{-st}. \quad \ldots \quad (2.26)
\]
Clearly we have 
\[
g_0(r, s) = g(r, s). \quad \ldots \quad (2.27)
\]
We then find from (2.24) and (2.25), using (2.22), 
\[
\varphi_1(r, s) = \frac{g_1(r, s)}{s h(s)}, \quad \ldots \quad (2.28)
\]
and 
\[
\varphi_2(r, s) = \frac{g_2(r, s)}{s h(s)} + \frac{2}{s} \left[ \frac{g_1(r, s)}{h(s)} \right]^2. \quad \ldots \quad (2.29)
\]
We now observe that these Laplace transforms again have poles at \( s=0 \) 
and no poles in the positive half-plane of \( s \). We can again restrict ourselves to 
the contribution from the pole at \( s=0 \). This contribution is always a polynomial 
in the cell length \( t \). We introduce some more notation, to simplify writing, 
\[
u_{kk'}(r) = \frac{d^k}{ds^k} \left[ \frac{g_k(r, s)}{h(s)} \right]_{s=0} \quad \ldots \quad (2.30)
\]
We then get the following explicit expressions:
\[
\overline{x_r(t)} = w_{01} t + w_{11}, \quad \ldots \quad (2.31)
\]
\[
\overline{x_r^2(t)} = (w_{01})^2 t^2 + (4 w_{01} w_{11} + w_{02}) t + [2(w_{11})^2 + 2 w_{01} w_{21} + w_{12}]. \quad \ldots \quad (2.32)
\]
A quantity of more interest than the mean square of \( x(t) \) is the variance, which is given by

\[
\overline{\delta^2(t)} - \overline{(x(t))^2} = (2w_0w_{11} + w_{12})t + [(w_{11})^2 + 2w_0w_{21} + w_{12}]. \tag{2.33}
\]

We now continue the reduction of these expressions to practical form. We have already neglected the contributions from poles other than the dominant pole at \( s=0 \). We now make an additional approximation: since the cell length \( t \) is in general much larger than the average final grain radius \( \overline{\gamma} \), the constant terms in (2.31) and (2.33), as well as in corresponding expressions for the mean value and variance of the number of gaps, are small corrections to the terms proportional to \( t \). It turns out that the terms proportional to \( t \) involve fewer differentiations, and are therefore easier to evaluate practically, than the constant terms. On the other hand, the differentiations necessary to find the coefficient of \( t \) in the variance already suffice to determine also the constant term in the mean value. Hence we shall make the following approximation: we give the mean values exactly (i.e. neglecting only the contributions of other poles, which we shall show to be an excellent approximation), but we shall give only the coefficient of \( t \) (i.e. we shall neglect the constant term) for the variance. Thus the expressions for the variance will be somewhat less accurate than for the mean values; this is sufficient for comparison with experiment, since the variance is much harder to determine accurately anyway. In the limit of very large cell length \( t \) the expression for the variance is accurate; in practice, the fractional error is of order \((2\overline{\gamma} + r)/t\), where \( \overline{\gamma} \) is the mean radius of developed grains, \( r \) is the minimum gap length chosen for counting, and \( t \) is the cell size. Thus, in G5 emulsions, we have \( \overline{\gamma} = 0.3 \) \( \mu \) and the error is less than 2 per cent. if \( r=0 \) and the cell length \( t \) exceeds 30 \( \mu \) (as it always does). An error of 2 per cent. in the variance is generally of no significance.

Making these approximations, it turns out that all the information of practical value can be reduced to the evaluation of the following seven definite integrals:

\[
Q_0 = \int_0^\infty Q(t)dt, \tag{2.34}
\]

\[
Q_1 = \int_0^\infty tQ(t)dt, \tag{2.35}
\]

\[
g_{00} = \int_0^\infty d\gamma_1 \int_0^\infty d\gamma_2 \int_{\gamma_1 + \gamma_1 + r}^\infty dt \quad q(t)G(\gamma_1)G(\gamma_2), \tag{2.36}
\]

\[
g_{01} = \int_0^\infty d\gamma_1 \int_0^\infty d\gamma_2 \int_{\gamma_1 + \gamma_1 + r}^\infty dt \quad tq(t)G(\gamma_1)G(\gamma_2), \tag{2.37}
\]

\[
g_{10} = \int_0^\infty d\gamma_1 \int_0^\infty d\gamma_2 \int_{\gamma_1 + \gamma_1 + r}^\infty dt \quad (t-\gamma_1 - \gamma_2 - r)q(t)G(\gamma_1)G(\gamma_2), \tag{2.38}
\]

\[
g_{11} = \int_0^\infty d\gamma_1 \int_0^\infty d\gamma_2 \int_{\gamma_1 + \gamma_1 + r}^\infty dt \quad t(t-\gamma_1 - \gamma_2 - r)q(t)G(\gamma_1)G(\gamma_2), \tag{2.39}
\]

\[
g_{20} = \int_0^\infty d\gamma_1 \int_0^\infty d\gamma_2 \int_{\gamma_1 + \gamma_1 + r}^\infty dt \quad (t-\gamma_1 - \gamma_2 - r)^2q(t)G(\gamma_1)G(\gamma_2). \tag{2.40}
\]
In practice these integrals are evaluated most easily by Laplace transform methods; details are given in Appendix I. In terms of these integrals, the final results for the mean values are:

\[
\bar{n}(t) = \frac{g_{10}}{Q_0} + \frac{g_{20} - g_{10}Q_0}{(Q_0)^2}, \quad \ldots \quad (2.41)
\]

\[
\bar{x}(t) = \frac{g_{10}}{Q_0} + \frac{g_{10}Q_1 - g_{11}Q_0}{(Q_0)^2} \quad \ldots \quad (2.42)
\]

We shall describe the fluctuations around these mean values by the fluctuation parameters \( \mu_n \) and \( \mu_x \) which are defined to be the ratios of the standard deviations to the square roots of the mean values; for a Poisson distribution \( \mu_n = \mu_x = 1 \). We then get, restricting ourselves to the coefficients of \( t^1 \):

\[
(\mu_n)^2 = \frac{(\bar{n}_r^2) - (\bar{n}_r)^2}{\bar{n}_r} = 1 + 2\frac{g_{20}Q_1 - g_{10}Q_0}{(Q_0)^2} + \text{order } \left( \frac{2\gamma + r}{t} \right), \quad \ldots \quad (2.43)
\]

\[
(\mu_x)^2 = \frac{(\bar{x}_r^2) - (\bar{x}_r)^2}{\bar{x}_r} = \left( \frac{g_{10} + 2g_{10}Q_1 - g_{11}Q_0}{(Q_0)^2} \right) \left( \frac{1}{t} + \text{order } \left( \frac{2\gamma + r}{t} \right) \right). \quad \ldots \quad (2.44)
\]

The main error made in these expressions, indeed the only error in (2.41) and (2.42), is the neglect of the contribution of the other poles in the complex \( s \)-plane. This error becomes small (as we shall show in Section III) if there are many developed grains in the cell length \( t \), on the average, so that \( t \) contains many "mean free paths" for grain activation. If we set \( r = 0 \), the quantity \( \bar{x}(t) = \bar{x}_0(t) \) is the expectation value of the total length of the gaps; hence \( t - \bar{x}_0(t) \) is the expectation value of the total length occupied by the blobs. If this length is much larger than a mean grain diameter \( 2\gamma \), there are many developed grains (on the average) within the cell length \( t \). Hence our criterion of validity reads

\[
t - \bar{x}_0(t) \gg 2\gamma, \quad \ldots \quad (2.45)
\]

or, to the extent that we can neglect the constant term in (2.42),

\[
t \gg \frac{2\gamma Q_0}{Q_0 - g_{10}}, \quad \ldots \quad (2.46)
\]

where \( g_{10} \) is evaluated for \( r = 0 \).

III. RESULTS FOR SPECIFIC MODELS

(a) The Model of O'Ceallaigh

This model is characterized by the survival probability (1.1) together with the assumption of a unique value \( \gamma \) of the radius of the developed grains, that is, the distribution function \( G(\gamma) \) is a delta function. We shall also use this model to test the validity of the approximations made in Section II, in particular the sufficiency of condition (2.46) on the minimum cell length required. This can
be done because the Laplace transforms can be inverted exactly. The exact results are

\[ n_r(t) = 0, \quad \text{for } t < 2\gamma + r, \]
\[ = \lambda (t - 2\gamma - r) \exp \left[ -\lambda (2\gamma + r) \right], \quad \text{for } t > 2\gamma + r, \]

(3.1)

\[ \bar{x}_r(t) = 0, \quad \text{for } t < 2\gamma + r, \]
\[ = (t - 2\gamma - r - \lambda^{-1}) \exp \left( \left[ -\lambda (2\gamma + r) \right] + \lambda^{-1} \exp (-\lambda t) \right), \quad \text{for } t > 2\gamma + r, \]

(3.2)

whereas the results obtained by the use of (2.41) and (2.42) are

\[ \bar{n}_r(t) = \lambda (t - 2\gamma - r) \exp \left[ -\lambda (2\gamma + r) \right], \quad \ldots \ldots \quad (3.1a) \]
\[ \bar{x}_r(t) = (t - 2\gamma - r - \lambda^{-1}) \exp \left[ -\lambda (2\gamma + r) \right]. \quad \ldots \ldots \quad (3.2a) \]

The difference between (3.1) and (3.1a) occurs only for \( t < 2\gamma + r \); actually there is no real difference, we have simply not been careful in the general treatment about specifying the smallest value of \( t \) for which the pole at \( s = 0 \) makes a contribution. As we see from (3.1) this is not a significant restriction for any reasonable cell length \( t \). The difference between (3.2) and (3.2a), for \( t > 2\gamma + r \), is due to the contribution of a pole at \( s = -1 \). Condition (2.46) (or (2.45), from which it is derived) becomes

\[ t > \frac{2\gamma}{1 - e^{-2\gamma\lambda}}. \quad \ldots \ldots \quad (3.3) \]

We now assert that the term \( \lambda^{-1}e^{-\lambda t} \) in (3.2) is indeed negligible whenever (3.3) is satisfied. To see this, we take the extreme cases of a very thin and a very thick track. A thin track has \( 2\gamma\lambda < 1 \), hence (3.3) becomes \( t > \lambda^{-1} \), \( \exp (-\lambda t) < 1 \). A very thick track has \( 2\gamma \lambda > 1 \), hence (3.3) is equivalent to \( t > 2\gamma \), and thus \( e^{-\lambda t} < e^{-2\gamma \lambda} < 1 \), which establishes what we want. We also observe that condition (2.45) becomes trivial for thick tracks, and is always satisfied in practice. Even for thin tracks, (2.45) merely asserts that the cell length \( t \) must contain many developed grains, and hence many gaps. A cell length containing as few as 5 gaps on the average is sufficient to make the \( \lambda^{-1}e^{-\lambda t} \) term in (3.2) less than 1 per cent. of the main term. Since no experimentalist does track statistics with such short cell lengths, condition (2.45) in no sense a practical restriction.

The fluctuation parameters in the O'Ceallaigh model are given by

\[ \mu_n^2 = -2\lambda (2\gamma + r) \exp \left[ -\lambda (2\gamma + r) \right] + \text{order } [(2\gamma + r)/t], \quad \ldots \ldots \quad (3.4) \]
\[ \mu_r^2 = (2\gamma)\{1 - [1 + \lambda (2\gamma + r)] \exp \left[ -\lambda (2\gamma + r) \right] + \text{order } [(2\gamma + r)/t]\}. \quad \ldots \ldots \quad (3.5) \]

For the purpose of testing the predictions of this model against experiment, the following relation (which follows from (3.1a) and (3.2a)) is very useful:

\[ x_r(t)/n_r(t) = \lambda^{-1}[1 - \text{order } (1/t)]. \quad \ldots \ldots \quad (3.6) \]

For all practical cell lengths \( \lambda t > 1 \), so that (3.6) can be used to compute \( \lambda \) directly. Since all the subsequent formulas depend on the value of \( \lambda \), this is a convenient procedure.
The ratio on the left side of (3.6) is approximately equal to the mean length of individual gaps.* In the O'Ceallaigh model, the parameter \( \lambda \) can assume any value; according to (3.1) and (3.2), very thick tracks are described by very large values of \( \lambda \). We see from (3.6) that the mean gap width approaches zero for very thick tracks, on the O'Ceallaigh model. Indeed, we shall see that this is true for all "continuum models", that is, all models which ignore the "graininess" of the emulsion before development.

(b) The Model of Happ, Hull, and Morrish

This model has the same survival probability, (1.1), as the O'Ceallaigh model, but introduces a distribution \( G(\gamma) \) of final grain radii \( \gamma \). Rather than adopting the particular distribution function \( G(\gamma) \) used by Happ, Hull, and Morrish, we shall work with a family of functions \( G(\gamma) \) for which the integrals can be done in closed form. Since the experimentally observed distribution in \( \gamma \) is very sharp, the detailed shape of \( G(\gamma) \) is of no great importance. We merely must allow ourselves two parameters in the functional form of \( G(\gamma) \) which can be used to fit the mean value \( \bar{\gamma} \) and the spread in the distribution-in-\( \gamma \). We shall adopt the functional form

\[
G(\gamma) = (k!)^{-1}(k/\gamma_0)^{k+1} \gamma^k \exp\left(-k\gamma/\gamma_0\right). \tag{3.7}
\]

For sharp distributions, that is, high values of \( k \), \( \gamma_0 \) is close to the mean value \( \bar{\gamma} \). More precisely,

\[
\gamma_0 = \bar{\gamma} - \frac{\gamma^2}{\bar{\gamma} - \gamma_0}, \quad k = \frac{\gamma_0}{\gamma - \gamma_0}. \tag{3.8}
\]

For typical experimental distributions of radii of single grains (not lengths of blobs which may contain several grains) \( k \) is in the neighbourhood of 10.

The quantities of experimental interest are then given by:

\[
n_r(t) = \frac{\lambda e^{-\lambda r}}{[1 + (\lambda \gamma_0/k)]^{2k+2}} \left[t - r - 2\gamma_0 \left(\frac{k+1}{k+\lambda \gamma_0}\right)\right], \tag{3.9}
\]

\[
av_r(t) = \frac{e^{-\lambda r}}{[1 + (\lambda \gamma_0/k)]^{2k+2}} \left[t - r - 2\gamma_0 \left(\frac{k+1}{k+\lambda \gamma_0}\right) \lambda^{-1}\right], \tag{3.10}
\]

\[
\mu_2 = 1 - \frac{2\lambda e^{-\lambda r}}{[1 + (\lambda \gamma_0/k)]^{2k+2}} \left[r + 2\gamma_0 \left(\frac{k+1}{k+\lambda \gamma_0}\right) + \text{order} \left((2\gamma_0 + r)/t, \right)\right], \tag{3.11}
\]

\[
\mu_2 = (2/\lambda) \left[1 - \frac{e^{-\lambda r}}{[1 + (\lambda \gamma_0/k)]^{2k+2}} \left[1 + \lambda r + 2\lambda \gamma_0 \left(\frac{k+1}{k+\lambda \gamma_0}\right) + \text{order} \left(2\gamma_0 + r\right)/t\right]\right]. \tag{3.12}
\]

It can be seen by inspection that these expressions reduce to the ones of the O'Ceallaigh model in the limit \( k \to \infty \). Furthermore, equation (3.6) is still valid and can again be used for a quick determination of the parameter \( \lambda \).

* Strictly speaking, the mean length of individual gaps is the average value of \( (x/n) \), not the ratio of the averages \( x/n \). However, if the fluctuations are small the difference is insignificant. This is always so for long enough cell lengths \( t \).
These expressions differ from the ones obtained by Happ, Hull, and Morrish because we concentrate our attention on different track parameters. In particular, the total excess gap length \( x_e(t) \) was not considered by them (Happ, Hull, and Morrish 1952). We shall show in Section IV of this paper that it is impossible to fit both the total gap length and the mean number of gaps with values of \( \gamma_0 \) and \( k \) in agreement with experimental measurements on the emulsion. Hence the model of Happ, Hull, and Morrish will turn out to be inadequate. This fact could not have been seen from a study of the fluctuations in \( n(t) \); the fluctuation parameter \( \mu_n \) agrees with experiment.

(c) A Modified O’Ceallaigh Model

As a first step towards taking into account the “graininess” of the emulsion before development, we can use the survival probability (1.3) rather than (1.1). The modified survival probability takes cognizance of the fact that each crystal has a non-zero radius, and hence two crystals can never be closer together than some minimum distance \( \alpha \); \( \alpha \) corresponds to the diameter (not the radius) of the crystals before development. Since this model will also fail to fit the data, we shall give only some of the results, namely the coefficients of \( t \) in \( n_r(t) \) and \( x_r(t) \); these are:

\[
n_r(t) = \frac{\lambda t}{1 + \alpha} \exp[\lambda(\alpha - 2\gamma - r)] + \text{constant}, \quad \ldots \ldots \ (3.13)
\]

\[
x_r(t) = \frac{t}{1 + \alpha} \exp[\lambda(\alpha - 2\gamma - r)] + \text{constant}. \quad \ldots \ldots \ (3.14)
\]

The constant terms in each case are of relative order of magnitude \( (2\gamma + r)/t \). We observe that equation (3.6) is still valid.

(d) A Modification of the Model of Happ, Hull, and Morrish

This model has the same survival probability (1.3), as the preceding model, but assumes a distribution in final grain radius \( G(\gamma) \). It turns out to be convenient to use the following modification of the distribution (3.7):

\[
G(\gamma) = 0, \quad \text{for} \quad \gamma < \frac{1}{2} \alpha, \quad \ldots \ldots \ (3.15)
\]

\[
=k^{-1}(k/\gamma_0)^{k+1}(\gamma - \frac{1}{2} \alpha)^k \exp \left(-k(\gamma - \frac{1}{2} \alpha)/\gamma_0\right), \quad \text{for} \quad \gamma > \frac{1}{2} \alpha. \quad \ldots \ldots \ (3.15)
\]

Physically this means that we assume that grains never shrink during the process of development. Since the mean grain radius after development, \( \gamma \), exceeds the initial crystal radius, \( \frac{1}{2} \alpha \), by a considerable margin, and since the distribution function \( G(\gamma) \) is quite sharp in any case, the few grains which perhaps shrink during development have no appreciable influence on the final results. The mathematical evaluation of the various integrals is simplified greatly by (3.15). The relationship of the parameters \( \gamma_0 \) and \( k \) to the measured distribution of sizes of single grains is

\[
\gamma_0 = \gamma - \frac{1}{2} \alpha, \quad k = \frac{\gamma_0}{\gamma - \frac{1}{2} \alpha - \gamma_0}, \quad \ldots \ldots \ (3.16)
\]
For the same reasons as before, we give only the coefficients of $t$ in the mean values $n_r(t)$ and $x_r(t)$; these are:

\[ n_r(t) = \frac{\lambda t e^{-\lambda r}}{(1 + \lambda z)[1 + (\lambda \gamma_0/k)]^{2k+2} + \text{constant}}, \quad \ldots \quad (3.17) \]

\[ x_r(t) = \frac{t e^{-\lambda r}}{(1 + \lambda z)[1 + (\lambda \gamma_0/k)]^{2k+2} + \text{constant}}. \quad \ldots \quad (3.18) \]

Again the relative contribution of the constant terms is of order $(2\gamma_0 + r)/t$; again equation (3.6) can be used to find $\lambda$. Although (3.17) and (3.18) look very similar to (3.9) and (3.10), it must be remembered that the parameters $\gamma_0$ and $k$ determined from (3.16) are quite different from $\gamma_0$ and $k$ determined from (3.8), for the same emulsion.

(e) The Model of Herz

This model goes to the opposite extreme in treating the "graininess" of the emulsion. It is characterized by the survival probability (1.2) together with the assumption of a unique final grain radius $\gamma$.* We introduce the integer $R$ as follows:

\[ R = \text{largest integer less than } (2\gamma + r)/\beta. \quad \ldots \quad (3.19) \]

We then get:

\[ n_r(t) = p(1-p)^R(t/\beta - R), \quad \ldots \quad (3.20) \]

\[ x_r(t) = p(1-p)^R[\bar{w}(t/\beta - R) - \beta(1-p)/p^2], \quad \ldots \quad (3.21) \]

where the "mean gap width" $\bar{w}$ is defined by

\[ \bar{w} = \beta R - (2\gamma + r) + (\beta/p). \quad \ldots \quad (3.22) \]

The fluctuations in this model are given by

\[ \mu_n^2 = 1 - (2R + 1)p(1-p)^R + \text{order } (2\gamma + r)/t, \quad \ldots \quad (3.23) \]

\[ \mu_x^2 = \frac{w_0^2}{w} \left[ 1 - pqR(1 + 2R) \right] + 2\beta(w_0^2/\bar{w})qR^2 + [1 + 2R - 4Rq - (q^2/p)] \]

\[ + \left( \frac{\bar{w}^2}{w} \right)(q/p)[1 + 2(q/p) - qR^2 - (3 + 2R + 2q/p)] \], \quad \ldots \quad (3.24) \]

where we have used the abbreviations

\[ q = 1 - p, \quad \ldots \quad (3.25) \]

and

\[ w_0 = \lim_{p \to 1} \bar{w} = (R + 1)\beta - (2\gamma + r). \quad \ldots \quad (3.26) \]

In the model of Herz the mean gap width $\bar{w}$ does not approach zero as the activation probability $p$ comes close to unity; rather it approaches the length

* The quantity " $\gamma$ " of Herz and Davis (1955) equals our $2\gamma/\beta$. For the special case $r=0$, that is, all gaps counted, our $R$ equals the $\Gamma$ of Herz.
of the "zero order gap", \( w_0 \), expression (3.26). This is a characteristic difference between the Herz model and all models discussed so far. The difference can be traced directly to the assumption of Herz that the undeveloped crystals are spaced in a uniform lattice, that is, to the special form (1.2) of the survival probability \( Q(t) \). This means that the grains either merge into a blob, or else have gap spacings equal to

\[
w = w_0 + m\beta, \quad m = \text{integral}. \quad (3.27)
\]

As the track becomes more dense, the gaps move farther and farther apart, and a larger proportion of them become "zero order" gaps, that is, gaps with width \( w = w_0, m = 0 \) in (3.27). In the limit of a very dense track, all gaps are zero order gaps.

In the continuum models discussed so far, the mean gap width is given by equation (3.6), and thus approaches zero for very dense tracks (\( \lambda \) approaching infinity). The difference between the continuum models and the Herz model is unimportant for thin tracks, but becomes increasingly more important as the track gets heavier.

(f) A Modification of the Herz Model

This modification consists in retaining the survival probability (1.2) of the Herz model, but allowing the final grain size to vary according to a statistical distribution function \( G(\gamma) \). The function \( G(\gamma) \) is chosen to be (3.15), with \( z \) replaced by \( \beta \), again because the integrals can be done in closed form, and the precise form of \( G(\gamma) \) is unimportant in any case. The integrations with this model are not altogether trivial, and are discussed in some detail in Appendix I. Since the results are rather complicated, we shall restrict ourselves here to giving the coefficients of \( t \) in the mean values for \( r = 0 \), that is, in \( \bar{n}_0 \) and \( \bar{x}_0 \). If the model checks sufficiently well against experiment (which has not been determined as yet) the fluctuations can also be computed by the methods given in Appendix I. The mean values, omitting constant terms, are

\[
\bar{n}_0(t) = p \left[ 1 - p \sum_{m=0}^{2k+1} \frac{(k\beta/\gamma_0)^m}{m!} T_m(y) \right] t + \text{constant}, \quad (3.28)
\]

where \( k \) and \( \gamma_0 \) are the parameters in \( G(\gamma) \), equation (3.15), \( \beta \) is the lattice constant in the Herz model, \( T_m(y) \) is a function defined in Appendix I, and \( y \) is an abbreviation for

\[
y = (1 - p) \exp \left( -k\beta/\gamma_0 \right). \quad (3.29)
\]

The mean total gap length, counting all gaps, is

\[
\bar{d}_0(t) = \left[ 1 - p - \frac{2(k+1)\gamma_0 p}{k\beta} + \frac{\gamma_0 p^2}{k\beta} \sum_{m=0}^{2k+1} (2k+2-m) \right] t + \text{constant}. \quad (3.30)
\]

Numerical evaluation of these expressions and comparison against experimental data will form the subject of a later publication.
IV. COMPARISON WITH EXPERIMENT

The purpose of this paper has been to derive and state the theoretical formulas necessary to analyse experimental data. A detailed analysis of data on tracks in nuclear research emulsions will be presented in a later publication. However, it is perhaps of some interest to point out one qualitative result of this analysis at this stage. This is the inadequacy of the continuum models of track formation, that is, of all models which replace the survival probability by a completely smoothed-out function (usually an exponential).

In order to obtain a quick test of the various models, we choose one rather dense track in a G5 emulsion.* For this track the experimental values of the mean number of gaps and the total gap length are \( r = 0, i.e. \) all gaps are counted

\[
\begin{align*}
\bar{n} &= 0.383t \quad \text{(in microns),} \\
\bar{x} &= 0.177t.
\end{align*}
\]

In the same emulsion, the mean radius of developed single grains was \( 0.325 \mu \).

We first compare against the continuum models \((a), (b), (c), \) and \((d)\) of Section III). In all these models the parameter \( \lambda \) is given by equation (3.6), so that

\[
\lambda = 2.16_6 \mu^{-1}. \quad \text{.................. (4.2)}
\]

We now insert this value of \( \lambda \) and the grain size \( \gamma = 0.325 \mu \) into formula (3.1a) to get, ignoring the constant term,

\[
\bar{n} = 0.530t, \quad \text{O'Ceallaigh model.} \quad \text{........... (4.3)}
\]

Since this is quite different from (4.1), we turn to the model of Happ, Hull, and Morrish. For this emulsion, reasonable values of \( \gamma_0 \) and \( k \) are: \( \gamma_0 = 0.300 \mu, \) \( k = 10. \) Substitution in (3.9) gives, again ignoring the constant term,

\[
\bar{n} = 0.543t, \quad \text{model of Happ, Hull, and Morrish} \quad \text{.... (4.4)}
\]

Hence the statistics of final grain sizes has made little difference (as is to be expected for \( k = 10 \)) and in so far as it had an effect, the effect went in the wrong direction. We therefore turn to the modifications which take into account the initial crystal sizes to a first approximation (model C of Section I). In this emulsion, a reasonable value of \( \beta \) is \( 0.290 \mu. \) Substitution in (3.13) gives

\[
\bar{n} = 0.616t, \quad \text{modified O'Ceallaigh model.} \quad \text{........... (4.5)}
\]

Thus the modified O'Ceallaigh model is even worse than the original one. Finally, we put in the statistics for the final grain radii, this time with \( \gamma_0 = 0.125 \mu \) and \( k = 2.8. \) The result is the worst yet:

\[
\bar{n} = 0.652t, \quad \text{modified model of Happ, Hull, and Morrish.} \quad \text{.... (4.6)}
\]

* The data for this track were supplied to the author by Dr. A. J. Herz.
While we have picked only one track for this comparison, it should be noted that the results are typical for all thick tracks in nuclear research emulsions. Of course, the theory could be brought in agreement with experiment by choosing a different value of $\gamma$. To see what $\gamma$ would have to be, we take the most favourable case (i.e. the one which came nearest to agreement before), that of the O'Ceallaigh model. We use $\lambda$ as in (4.2) and determine $\gamma$ from (4.1) and (3.1a). The result is

$$\gamma = 0.40 \mu. \quad \text{(4.7)}$$

The experimental distribution of radii of single grains depends on the resolving power of the microscope, the adjustment of the microscope, the state of the emulsion, and the particular observer. For any combination of optical conditions and observer, however, the average grain diameter is a clearly defined quantity, and two adjacent grains will be counted as a single blob if their boundaries, as seen by that particular observer under these particular conditions, overlap. Hence, if the blob counting and gap measuring is carried out under the same conditions and by the same observer as the measurement of developed grain sizes, then the theory should apply, and the optical conditions and peculiarities of the observer need not be considered.

The same observer who measured the track listed in (4.1) also measured, in the same plate and under the same optical conditions, the distribution of blob sizes in a thin track, where most blobs correspond to single grains. The distribution of blob sizes in this thin track has a clearly defined peak corresponding to single-grain blobs. The most probable grain diameter is $0.65 \mu$, and 43 per cent. of the grains are contained in the peak between $0.62$ and $0.68 \mu$. A grain diameter of $0.80 \mu$ is definitely in the tail of the distribution; the distribution function $G(\gamma)$ has dropped to about $1/7$th of its peak value by the time grain diameters $2\gamma = 0.80 \mu$ are reached.

Hence we conclude that the models which ignore the approximate lattice structure of the spatial distribution of undeveloped crystals in the emulsion are definitely in disagreement with experimental data. For the same track already discussed, Herz had no trouble in fitting the data to his model. Furthermore, Herz was able to fit several tracks in the same emulsion using identical values of lattice spacing $\beta$ and final grain radius $\gamma$, varying only the activation probability $p$.

A more detailed comparison against experimental data will form the subject of a later publication.

V. ACKNOWLEDGMENT

The author would like to thank Dr. A. J. Herz for many valuable discussions.

VI. REFERENCES


Some Methods for Evaluating the Integrals

The integrals (2.34)-(2.40) must be evaluated, preferably in closed form, before the theory makes useful statements. In general there is no difficulty with (2.34) and (2.35), which are just the first two moments of the survival probability $Q(t)$. We shall therefore restrict ourselves to a discussion of the integrals $g_{ij}$, (2.36)-(2.40).

It turns out to be very convenient to introduce the Laplace transform of the survival probability:

$$P(s) = \int_0^\infty e^{-st}Q(t)dt,$$  \hspace{1cm} (A1)

as well as its derivative

$$P'(s) = \int_0^\infty e^{-st}Q(t)dt.$$  \hspace{1cm} (A2)

Clearly the following relations hold

$$P(0) = Q_0, \quad P'(0) = -Q_1.$$  \hspace{1cm} (A3)

Consider now the integral $g_{00}$, expression (2.36). We observe that the integration over $t$ simply yields the function $Q(\gamma_1 + \gamma_2 + r)$, for which we write

$$Q(\gamma_1 + \gamma_2 + r) = (2\pi i)^{-1} \int_C ds \exp[s(\gamma_1 + \gamma_2 + r)]P(s).$$  \hspace{1cm} (A4)

The contour in (A4) is the standard Bromwich contour, to the right of all poles of $P(s)$. We then interchange orders of integration over $\gamma_1$, $\gamma_2$, and $s$. We introduce the Laplace transform of the distribution-in-final-grain-radius $G(\gamma)$:

$$H(s) = \int_0^\infty e^{-s\gamma}G(\gamma)d\gamma,$$  \hspace{1cm} (A5)

in order to get

$$g_{00}(r) = (2\pi i)^{-1} \int_C ds e^{sr}H^2(-s)P(s),$$  \hspace{1cm} (A6)

where the contour $C$ must be restricted in such a way that $H(-s)$ represents a convergent integral rather than the analytic continuation of a convergent integral. That is, the contour $C$ must lie to the left of all poles of $H(-s)$ as well as to the right of all poles of $P(s)$. These conditions may turn out to be contradictory, in which case this method of evaluating the integrals fails; but so far it has been quite easy to satisfy both conditions simultaneously.

Next consider $g_{01}$, equation (2.37). We introduce for the moment the notation

$$T = \gamma_1 + \gamma_2 + r,$$  \hspace{1cm} (A7)

and we ask for the Laplace transform of the function of $T$ obtained by performing the integration over $t$, that is,

$$U(s) = \int_0^\infty dT \int_T^\infty dte^{-st}q(t).$$  \hspace{1cm} (A8)
We interchange orders of integration to get

\[ U(s) = s^{-1} \int_0^\infty dt \left( 1 - e^{-st} \right) q(t), \]

and integrate by parts to obtain

\[ U(s) = s^{-1} \left[ Q_0 - P(s) - sP'(s) \right]. \quad \text{(A9)} \]

From here on the steps are just as before, leading to the result

\[ g_{01}(r) = (2\pi)^{-1} \int_C ds \, e^{\sigma H^2(-s)} \frac{Q_0 - P(s) - sP'(s)}{s}. \quad \text{(A10)} \]

Contrary to the first impression, the integrand of (A10) is regular at \( s=0 \); this can be seen by using (A3), or more directly by observing that (A8) is by definition a convergent integral when \( s=0 \).

The same approach can be used on the other three integrals \( g_{ij} \) and yields:

\[ g_{10}(r) = (2\pi)^{-1} \int_C ds \, e^{\sigma H^2(-s)} \frac{P(s) + sP'(s) - Q_0 + 2sQ_1}{s^2}. \quad \text{(A11)} \]

\[ g_{11}(r) = (2\pi)^{-1} \int_C ds \, e^{\sigma H^2(-s)} \frac{2[P(s) - Q_0 + sQ_1]}{s^2}. \quad \text{(A12)} \]

\[ g_{20}(r) = (2\pi)^{-1} \int_C ds \, e^{\sigma H^2(-s)} \frac{2P(s) - Q_0}{s^2}. \quad \text{(A13)} \]

In all cases the integrand is regular at \( s=0 \). In most cases the contour can be deformed towards the left, so that the relevant poles are the poles of \( P(s) \) (which are of course at the same points as the poles of \( P'(s) \)). In the continuum models, with survival probabilities (1.1) or (1.3), \( P(s) \) is very simple and has only one pole, at \( s = -\lambda \).

The rest of this appendix is devoted to the evaluation of the integrals in the model of Herz and in the generalization of that model discussed in Section III.

We introduce the notation

\[ q = 1 - p, \quad \text{(A14)} \]

where \( p \) is the probability of activating a traversed crystal. The Laplace transform of the survival probability \( Q(t) \), expression (1.2), is

\[ P(s) = \sum_{k=0}^{\infty} \frac{q^k}{k!} \int_0^{(k+1)\beta} e^{-st} \, dt \]

\[ = s^{-1} \left[ 1 - e^{-s\beta} \right] \sum_{k=0}^{\infty} [qe^{-s\beta}]^k. \quad \text{(A15)} \]
We can sum the infinite series to get the form
\[ P(s) = \frac{1 - e^{-s\beta}}{s[1 - qe^{-s\beta}]} \] ........................... (A16)

However, we shall find that (A15) is often a more convenient form to work with. The poles of \( P(s) \) are located at
\[ s_n = -\beta^{-1} \ln \left( \frac{1}{q} + n(2\pi i) \right), \quad n = \text{integer}. \] ........................... (A17)

Since \( q \) is by definition less than unity, it suffices to take our contour \( C \) to the right of the origin.

To illustrate the method, let us first evaluate \( g_{00}(r) \) for the Herz model. In this model \( G(\gamma) \) is a delta function, so that
\[ H(s) = e^{-s\gamma_0}. \] ........................... (A18)

Substitution of (A15) and (A18) into (A6) yields
\[ g_{00}(r) = (2\pi i)^{-1} \int_C \frac{ds}{s} \exp \left[ \frac{1 - \exp \left( -s\beta \right)}{s} \sum_{k=0}^{\infty} q^k \exp \left( -ks\beta \right) \right] \]
\[ = (2\pi i)^{-1} \int_C ds \, s^{-1} \exp \left[ s(r + 2\gamma_0) \right] \left\{ 1 - p \sum_{k=1}^{\infty} q^{k-1} \exp \left( -ks\beta \right) \right\}. \] ........................... (A19)

We now interchange the order of integration and summation; this is possible since the infinite series converges absolutely and uniformly along the entire contour \( C \). The contributions of the various terms \( k \) depend on the magnitude of \( k \). First consider small values of \( k \), for which the coefficient in the exponential \( \exp \left[ s(r + 2\gamma_0 - k\beta) \right] \) is positive. We then close the contour to the left and obtain the contribution from the pole at \( s = 0 \) only. This contribution is \( -p(q)^{k-1} \). But when \( k \) gets large enough that the coefficient in the exponential is negative, we close the contour to the right, and get zero. The critical value of \( k \) is precisely the integer \( R \) defined by (3.19). Hence we get
\[ g_{00}(r) = 1 - p \sum_{k=1}^{R} q^{k-1} = 1 - p \frac{1 - q^R}{1 - q} = 1 - (1 - q^R) = q^R. \] ........................... (A20)

Of course, this integral could have been evaluated much more directly from first principles, but this derivation was intended to show an application of the method of this appendix to a particularly simple case.

We now turn to the modified model of Herz, where the integrals are somewhat less trivial. The distribution-in-final-grain-radius \( G(\gamma) \) is given by (3.15), and its Laplace transform is
\[ H(s) = \frac{\exp \left( -\frac{1}{2}s\beta \right)}{[1 + (\gamma_0 s/k)]^{k+1}}. \] ........................... (A21)

For large values of \( k \) this reduces to
\[ \lim_{k \to \infty} H(s) = \exp \left[ -s(\gamma_0 + \frac{1}{2}\beta) \right]. \] ........................... (A22)
This is of course the same as (A18) if we remember the altered definition of the parameter $\gamma_0$.

We evaluate $g_{00}(r)$ explicitly for the special case $r=0$; it should be noted that it would be very difficult to get a closed result without the use of these Laplace transform methods. Substitution of (A15) and (A21) into (A6) gives

$$
g_{00}(r) = (2\pi i)^{-1} \int_C ds \frac{1 - \exp(-s\beta)}{s[1-(\gamma_0/k)]^{2k+2}} \sum_{n=0}^{\infty} q^n \exp(-ns\beta)$$

$$= (2\pi i)^{-1} \int_C ds \frac{\exp(s\sigma)}{s[1-(\gamma_0/k)]^{2k+2}} \{ \exp(s\beta) - p \sum_{n=0}^{\infty} q^n \exp(-ns\beta) \}. \tag{A23}$$

In order to get formulas of reasonable length, we restrict ourselves now to the special case of the limit $r \to 0$. It should be emphasized, however, that non-zero values of $r$ can be handled by the same methods; the results are merely a little more involved. If we assume that $r$ is an infinitesimal positive number, we see that we must close the contour to the left for the $e^{s\beta}$ term and for the term $n=0$ in (A23), to the right for all terms $n>1$. The contour $C$ runs to the left of the origin but to the right of the pole

$$s = \sigma \equiv k/\gamma_0. \tag{A24}$$

The contribution of the pole at $s=0$ is simple; let us now consider a typical term $n>1$ and its contribution, which arises from the pole $s=\sigma$. This contribution is the negative of the residue at $s=\sigma$, that is, setting $r=0$,

$$+pq^n(-\sigma)^{2k+2} \frac{1}{(2k+1)!} \left\{ \frac{d^{2k+1}}{ds^{2k+1}}[s^{-1} e^{-ns\beta}] \right\}_{s=\sigma}$$

$$= -p \sum_{m=0}^{2k+1} \frac{(n\beta\sigma)^m}{m!} [q e^{-\beta\sigma}]^n.$$  

We therefore get

$$g_{00}(0) = q - p \sum_{n=1}^{\infty} \sum_{m=0}^{2k+1} \frac{(n\beta\sigma)^m}{m!} [q e^{-\beta\sigma}]^n. \tag{A25}$$

At this stage we interchange the order of the two summations, and evaluate the infinite sum over $n$ explicitly. We define

$$y = q \exp(-\beta\sigma) = (1-p) \exp(-k\beta/\gamma_0), \tag{A26}$$

and

$$T_m(y) = \sum_{n=0}^{\infty} n^m y^n = (yd/dy)^m \left( \frac{1}{1-y} \right), \tag{A27}$$

in order to get

$$g_{00}(0) = q - p \sum_{m=0}^{2k+1} \frac{(\beta\sigma)^m}{m!} [T_m(y) - \delta_{m,0}],$$

where $\delta_{m,0}$ is the Kronecker delta; the final result is

$$g_{00}(0) = 1 - p \sum_{m=0}^{2k+1} \frac{(k\beta/\gamma_0)^m}{m!} T_m(y). \tag{A28}$$
This result is not overly simple, but it would of course be unreasonable to expect a simple answer, since we must get a continuous and differentiable function of \( \beta/\gamma_0 \) which in the limit \( k \to \infty \) approaches the discontinuous function (A20).

It may be worth while to say a few words about the functions \( T_m(y) \) defined by (A27). Their calculation can be simplified by expressing the operators \((yd/dy)^m\) as linear combinations of the operators

\[
D_r = y^r(d/dy)^r. \quad \text{(A29)}
\]

We write

\[
(yd/dy)^m = \sum_{r=1}^{m} a_m D_r, \quad m \neq 0, \quad \text{(A30)}
\]

and obtain the recursion relations

\[
a_{m+1, r} = a_{m, r-1} + a_{m, r}, \quad r \neq 1, \quad r \neq m+1,
a_{m+1, 1} = a_{m+1, m+1} = 1. \quad \text{(A31)}
\]

The first few coefficients \( a_{mr} \) are given in Table 1. We now introduce (A30) into (A27) to get

\[
T_m(y) = (1-y)^{-1} \sum_{r=1}^{m} r! a_{mr} \left( \frac{y}{1-y} \right)^r, \quad m \neq 0,
\]

\[
T_0(y) = (1-y)^{-1}. \quad \text{(A32)}
\]

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
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<tbody>
<tr>
<td>VALUES OF THE COEFFICIENTS ( a_{mr} ) IN EQUATION (A32)</td>
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</table>

<table>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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</table>

Expression (A32) together with Table 1 gives a practical way of calculating \( T_m(y) \). It should be noted that all the terms in the sum over \( r \) are positive, so that there is no danger of near cancellation of two large numbers of opposite sign. There is, however, appreciable cancellation of this type between the \( "1" \) in (A28) and the sum over \( m \), so that each \( T_m(y) \) must be calculated to considerably higher accuracy than the accuracy of the final result.

An entirely similar derivation gives

\[
g_{10} = \beta q/p - (2k+2)\gamma_0/k + (p\gamma_0/k)^{2k+1} \sum_{m=0}^{2k+1} (2k+2-m) \left( \frac{k\beta/\gamma_0}{m!} \right)^m T_m(y). \quad \text{(A33)}
\]

These results are sufficient to determine the coefficients of \( t \) in the mean number of gaps and the mean total gap length, quoted in Section III.