HYDROMAGNETIC STABILITY OF A CURRENT LAYER

By R. E. LOUGHHEAD*

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Summary

The hydromagnetic stability of a uniform current flowing along a magnetic field and confined within a pair of parallel planes is discussed by the method of normal modes. The condition for marginal stability is derived and discussed with reference to two special cases.

It is also shown that the velocity of Alfvén waves along a magnetic field in a region bounded by parallel planes is reduced due to the inertia of the surrounding medium.

I. INTRODUCTION

In a recent paper Dungey and Loughhead (1954) have considered the stability of a current of uniform density flowing coaxially along a magnetic field contained within an infinitely long cylinder, the resultant magnetic field having twisted lines of force. The present paper deals with the similar problem of the stability of a uniform current flowing in a perfectly conducting fluid within a region bounded by parallel planes. Using the principle of exchange of stabilities the condition for marginal instability is derived, and then discussed in some detail for two special cases.

The first is that of the stability of a uniform current flowing along a magnetic field, the current and field both vanishing outside a region bounded by a pair of parallel planes. It is concluded that, for any layer of finite thickness, there are modes of disturbance for which the current distribution is unstable.

The second case is an idealization of the "pinch" effect, in which the magnetic field in the fluid is everywhere perpendicular to the current. It is found that the system possesses unstable modes of disturbance when the magnitude of the field inside exceeds that of the field outside the current layer. However, in conformity with the result of Kruskal and Schwarzchild (1954), the current system is shown to be stable so long as the magnitude of the field outside exceeds that inside the layer.

II. EQUATIONS FOR SMALL PERTURBATIONS

Suppose that, referred to Cartesian axes Ox,y,z, a uniform current flows along an imposed magnetic field H_z in the region bounded by the planes $x=\pm a$. Associated with the current there is a transverse component of the magnetic field

$$H_y = A.x, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where A is a constant related to the current density. Outside the current layer the magnetic field is taken to have the uniform transverse components

$$\left.\begin{aligned} H_y = A_+, &\quad \text{for } x > a, \\ A_-, &\quad \text{for } x < -a. \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

* Division of Physics, C.S.I.R.O., University Grounds, Sydney.

For generality the components of the magnetic field in the direction of the current are written

$$\left.\begin{aligned} H_z &= B, && \text{for } -a<x<a, \\ &= B_+, && \text{for } x>a, \\ &= B_-, && \text{for } x<-a, \end{aligned}\right\} \qquad (3)$$

where B, B_+, and B_- are all constants.

The results of small perturbations of the current system are therefore described by the hydromagnetic equations for an ionized fluid in the presence of a steady magnetic field of the form

$$\mathbf{H}=[0,\ H_y(x),\ H_z(x)], \qquad (4)$$

where $H_y(x)$ and $H_z(x)$ are specified functions of x. Since it may be shown that the stability does not depend on either the density or compressibility of the fluid (see Appendix I) it is supposed that the medium is incompressible and has the uniform density μ. The electrical conductivity of the fluid is taken to be infinite.

In Gaussian units the equations governing small perturbations $\mathbf{h}$ and $\mathbf{v}$ in the magnetic field and fluid velocity, respectively, may be written in the forms :

$$\frac{\partial \mathbf{h}}{\partial t}=(\mathbf{H}\,.\,\mathrm{grad})\mathbf{v}-(\mathbf{v}\,.\,\mathrm{grad})\mathbf{H}, \qquad (5)$$

$$4\pi\mu\frac{\partial \mathbf{v}}{\partial t}=-\mathrm{grad}\ \psi+(\mathbf{H}\,.\,\mathrm{grad})\mathbf{h}+(\mathbf{h}\,.\,\mathrm{grad})\mathbf{H}, \qquad (6)$$

$$\mathrm{div}\ \mathbf{v}=0, \qquad (7)$$

where $\psi/4\pi$ denotes the variation in the total pressure (gas pressure plus $\mathbf{H}^2/8\pi$).

Applying the method of normal modes solutions of these equations are sought in which the variables are proportional to

$$\exp \mathrm{i}(\omega t+my+nz), \qquad (8)$$

where ω, m, and n refer to an arbitrary mode. Inserting (8) in (5), (6), and (7) and defining

$$K=mH_y+nH_z, \qquad (9)$$

the following Cartesian equations are obtained :

$$\mathrm{i}\omega h_x=\mathrm{i}Kv_x, \qquad (10)$$

$$\mathrm{i}\omega h_y=\mathrm{i}Kv_y-\frac{\partial H_y}{\partial x}\,.\,v_x \qquad (11)$$

$$\mathrm{i}\omega h_z=\mathrm{i}Kv_z-\frac{\partial H_z}{\partial x}\,.\,v_x, \qquad (12)$$

$$4\pi\mu\mathrm{i}\omega v_x=-\frac{\partial\psi}{\partial x}+\mathrm{i}Kh_x, \qquad (13)$$

$$4\pi\mu\mathrm{i}\omega v_y=-\mathrm{i}m\psi+\mathrm{i}Kh_y+\frac{\partial H_y}{\partial x}\,.\,h_x, \qquad (14)$$

$$4\pi\mu\mathrm{i}\omega v_z=-\mathrm{i}n\psi+\mathrm{i}Kh_z+\frac{\partial H_z}{\partial x}\,.\,h_x, \qquad (15)$$

$$\frac{\partial v_x}{\partial x}+\mathrm{i}mv_y+\mathrm{i}nv_z=0. \qquad (16)$$

After some reduction one obtains the first order equations

$$(4\pi\mu\omega^2-K^2)\frac{\partial v_x}{\partial x}-\mathrm{i}\omega(m^2+n^2)\psi=0, \qquad (17)$$

$$\omega\frac{\partial\psi}{\partial x}+\mathrm{i}(4\pi\mu\omega^2-K^2)v_x=0, \qquad (18)$$

in the two variables v_x and ψ. Clearly, from equations (**17**) and (**18**), v_x and ψ must be continuous across the boundary planes $x=\pm a$.

By elimination it is found that inside the region $-a<x<a$, where K takes the value

$$K=mAx+nB, \qquad (19)$$

ψ is determined by the equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{2mAK}{4\pi\mu\omega^2-K^2}\cdot\frac{\partial\psi}{\partial x}-(m^2+n^2)\psi=0. \qquad (20)$$

Outside this region, where K assumes the values

$$\left.\begin{aligned} &K_+=mA_++nB_+,\\ \text{or}\quad &K_-=mA_-+nB_-, \end{aligned}\right\} \qquad (21)$$

according as $x>a$ or $x<-a$, ψ is governed by the equation

$$\frac{\partial^2\psi}{\partial x^2}-(m^2+n^2)\psi=0, \qquad (22)$$

Before proceeding to the critical case $\omega=0$, the above analysis may be applied to a discussion of the propagation of Alfvén waves along a uniform magnetic field H_z contained wholly within the region bounded by the planes $x=\pm a$.

III. Propagation of Alfvén Waves in a Region Bounded by Parallel Planes

In this section the discussion is generalized to allow for different densities μ_1 inside, and μ_2 outside, the region bounded by parallel planes. To describe the propagation of Alfvén waves in the region between the planes $x=\pm a$ pervaded by the uniform magnetic field H_z let the current in the z-direction be zero. Thus $A=0$, and equation (**20**) becomes

$$\frac{\partial^2\psi}{\partial x^2}-\varkappa^2\psi=0, \qquad (23)$$

where

$$\varkappa^2=m^2+n^2. \qquad (24)$$

Imposing the condition that ψ remain finite as $x\to\pm\infty$ the solution of (**23**) can be written in the form

$$\left.\begin{aligned} \psi&=C_1e^{\varkappa x}+C_2e^{-\varkappa x}, && -a<x<a,\\ &=C_3e^{-\varkappa x}, && x>a,\\ &=C_4e^{\varkappa x}, && x<-a. \end{aligned}\right\} \qquad (25)$$

where C_1, C_2, C_3, and C_4 are arbitrary constants. With the aid of equation (**18**) the corresponding solution for v_x is found to be given by

$$\left.\begin{aligned} v_x &= \frac{i\omega}{4\pi\mu_1\omega^2 - n^2B^2}\varkappa(C_1 e^{\varkappa x} - C_2 e^{-\varkappa x}), & -a<x<a, \\ &= \frac{-i\omega}{4\pi\mu_2\omega^2}\varkappa C_3 e^{-\varkappa x}, & x>a, \\ &= \frac{i\omega}{4\pi\mu_2\omega^2}\varkappa C_4 e^{\varkappa x}, & x<-a. \end{aligned}\right\} \quad \text{(26)}$$

The requirement that v_x and ψ be continuous across the planes $x=a$ and $x=-a$ imposes the following relations on the four constants C_1, C_2, C_3, and C_4:

$$\left.\begin{aligned} C_1 e^{\varkappa a} + C_2 e^{-\varkappa a} - C_3 e^{-\varkappa a} &= 0, \\ C_1 e^{\varkappa a} - C_2 e^{-\varkappa a} + C_3 e^{-\varkappa a} Q &= 0, \\ C_1 e^{-\varkappa a} + C_2 e^{\varkappa a} - C_4 e^{-\varkappa a} &= 0, \\ C_1 e^{-\varkappa a} - C_2 e^{\varkappa a} - C_4 e^{-\varkappa a} Q &= 0, \end{aligned}\right\} \quad \text{(27)}$$

where Q has the value

$$Q = \frac{\mu_1}{\mu_2}\left\{1 - \frac{V_a^2}{V^2}\right\}, \quad \text{(28)}$$

and

$$V_a = \frac{B}{\sqrt{(4\pi\mu_1)}} \quad \text{(29)}$$

is Alfvén's velocity, V being the velocity of propagation of waves in the z-direction. From (**27**), the condition for the existence of non-trivial solutions may be expressed in the form

$$\begin{vmatrix} e^{\varkappa a} & e^{-\varkappa a} & -e^{-\varkappa a} & 0 \\ e^{\varkappa a} & -e^{-\varkappa a} & e^{-\varkappa a} Q & 0 \\ e^{-\varkappa a} & e^{\varkappa a} & 0 & -e^{-\varkappa a} \\ e^{-\varkappa a} & -e^{\varkappa a} & 0 & -e^{-\varkappa a} Q \end{vmatrix} = 0. \quad \text{(30)}$$

This may be rewritten

$$Q^2(e^{2\varkappa a} - e^{-2\varkappa a}) + 2Q(e^{2\varkappa a} + e^{-2\varkappa a}) + (e^{2\varkappa a} - e^{-2\varkappa a}) = 0.$$

Corresponding to the two roots Q of this equation we obtain the expressions

$$V = \pm\left(1 + \frac{\mu_2}{\mu_1}\tanh \varkappa a\right)^{-\frac{1}{2}} . V_a, \quad \text{(31)}$$

and

$$V = \pm\left(1 + \frac{\mu_2}{\mu_1}\coth \varkappa a\right)^{-\frac{1}{2}} . V_a. \quad \text{(32)}$$

These relations express the reduction of the wave velocity due to the inertia of the surrounding medium. Thus equation (**31**) shows that the magnitude of V decreases steadily from the value V_a when $\varkappa a=0$ to the value $(1+\mu_2/\mu_1)^{-\frac{1}{2}}V_a$ as $\varkappa a \to \infty$, while equation (**32**) implies that the magnitude of V increases steadily from the value 0 when $\varkappa a=0$ to the limiting value $(1+\mu_2/\mu_1)^{-\frac{1}{2}}. V_a$ as $\varkappa a \to \infty$. This behaviour is illustrated in Figure 1 for the special case $\mu_1=\mu_2$.

The above discussion shows that, when the effect of boundary conditions is taken into account, the velocity of propagation of hydromagnetic disturbances is not the simple Alfvén velocity V_a. The actual velocity of any mode of disturbance is reduced, due to the inertia of the surrounding medium, by a factor which, for magnetic regions of large extent (a large), approximates to $(1+\mu_2/\mu_1)^{-\frac{1}{2}}$.

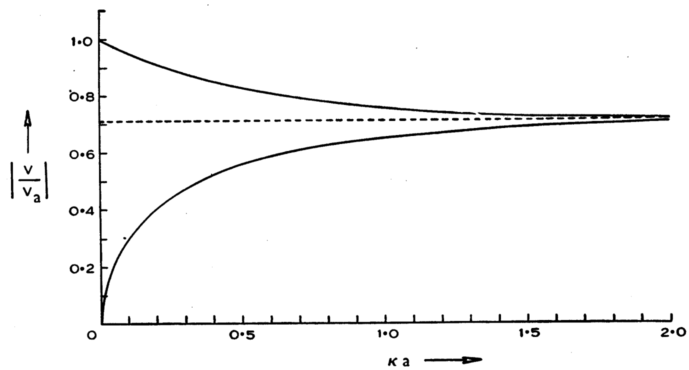

Fig. 1.—Variation of wave velocity with a for the case $\mu_1=\mu_2$.

IV. The Critical Case $\omega=0$

The condition for instability in any conservative system is the existence of a solution for which the period is infinite. To determine whether any such solutions exist for different physical conditions of the system let $\omega=0$ in the equations for the normal modes. In this case, however, equation (**5**) requires $\mathbf{v}=0$, so that equation (**6**) must now be supplemented by the relation

$$\operatorname{div} \mathbf{h}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{33})$$

to obtain a determinate set of equations. One thus obtains the equations

$$\frac{\partial\psi}{\partial x}=\mathrm{i}Kh_x, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{34})$$

$$\mathrm{i}m\psi=\mathrm{i}Kh_y+\frac{\partial H_y}{\partial x}\cdot h_x, \quad \ldots\ldots\ldots\ldots \quad (\mathbf{35})$$

$$\mathrm{i}n\psi=\mathrm{i}Kh_z+\frac{\partial H_z}{\partial x}\cdot h_x, \quad \ldots\ldots\ldots\ldots \quad (\mathbf{36})$$

$$\frac{\partial h_x}{\partial x}+\mathrm{i}mh_y+\mathrm{i}nh_z=0. \quad \ldots\ldots\ldots\ldots \quad (\mathbf{37})$$

After some reduction these yield the first order equations

$$K\frac{\partial h_x}{\partial x}+\mathrm{i}(m^2+n^2)\psi=\frac{\partial K}{\partial x}\cdot h_x, \quad \ldots\ldots\ldots\ldots \quad (\mathbf{38})$$

$$\frac{\partial\psi}{\partial x}=\mathrm{i}Kh_x, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{39})$$

in the two variables h_x and ψ. Equations (**38**) and (**39**) imply respectively the continuity of h_x/K and ψ across the boundary planes $x=\pm a$.

Inside the current layer where K takes the value (**19**), ψ is found from (**38**) and (**39**) to be governed by the equation

$$\frac{\partial^2\psi}{\partial x^2}-\frac{2mA}{K}\cdot\frac{\partial\psi}{\partial x}-(m^2+n^2)\psi=0. \qquad \textbf{(40)}$$

On changing the independent variable to K and writing

$$\varphi^2=(m^2+n^2)/m^2A^2, \qquad \textbf{(41)}$$

this equation becomes

$$\frac{\partial^2\psi}{\partial K^2}-\frac{2}{K}\cdot\frac{\partial\psi}{\partial K}-\varphi^2\psi=0, \qquad \textbf{(42)}$$

whose general solution is

$$\psi=C_1(\varphi K+1)\mathrm{e}^{-\varphi K}+C_2(\varphi K-1)\mathrm{e}^{\varphi K}, \qquad \textbf{(43)}$$

C_1 and C_2 being arbitrary constants. By defining

$$\varkappa=\sqrt{m^2+n^2}, \qquad \textbf{(44)}$$

and

$$\gamma=\varkappa\cdot\frac{nB}{mA}, \qquad \textbf{(45)}$$

the solution may be expressed in the form

$$\psi=C_1(\varkappa x+\gamma+1)\mathrm{e}^{-(\varkappa x+\gamma)}+C_2(\varkappa x+\gamma-1)\mathrm{e}^{\varkappa x+\gamma}. \qquad \textbf{(46)}$$

Outside the current layer, where K takes the values defined by (**21**), ψ is governed by the equation

$$\frac{\partial^2\psi}{\partial x^2}-\varkappa^2\psi=0, \qquad \textbf{(47)}$$

which has the solutions

$$\left.\begin{aligned}\psi&=C_3\mathrm{e}^{-\varkappa x}, & x&>a,\\ &=C_4\mathrm{e}^{\varkappa x}, & x&<-a,\end{aligned}\right\} \qquad \textbf{(48)}$$

finite as $x\to\pm\infty$ respectively. C_3 and C_4 are arbitrary constants.

With the aid of (**39**) the corresponding solutions for h_x are found to be

$$\left.\begin{aligned}h_x&=\frac{\mathrm{i}\varkappa}{K}(\varkappa x+\gamma)\{C_1\mathrm{e}^{-(\varkappa x+\gamma)}-C_2\mathrm{e}^{\varkappa x+\gamma}\}, & -a<x&<a,\\ &=\frac{\mathrm{i}\varkappa}{K_+}C_3\mathrm{e}^{-\varkappa x}, & x&>a,\\ &=-\frac{\mathrm{i}\varkappa}{K_-}C_4\mathrm{e}^{\varkappa x}, & x&<-a,\end{aligned}\right\} \qquad \textbf{(49)}$$

The requirement that h_x/K and ψ be continuous across the planes $x=\pm a$ imposes four relations on the constants $C_1, \ldots, C_4$. and it may be shown that the condition for the existence of non-trivial solutions is satisfied for all finite values of $\varkappa a$ such that

$$\mathrm{e}^{4\varkappa a}=\frac{\{(\varkappa a+\gamma+1)-(\varkappa a+\gamma)\alpha^2\}\{(\varkappa a-\gamma+1)+(\varkappa a-\gamma)\beta^2\}}{\{(\varkappa a+\gamma-1)+(\varkappa a+\gamma)\alpha^2\}\{(\varkappa a-\gamma-1)-(\varkappa a-\gamma)\beta^2\}} \qquad \textbf{(50)}$$

where

$$\left.\begin{aligned}\alpha&=\frac{mA_{+}+nB_{+}}{mAa+nB},\\ \beta&=\frac{mA_{-}+nB_{-}}{-mAa+nB}.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\quad(51)$$

The condition (50) determines the stability of the current layer in the general case where the magnetic field in the fluid has the form specified by (19) and (21). The effect of the magnetic field on the stability enters into the condition (50) through the parameters α, β, and γ. In the following sections the implications of (50) are illustrated by reference to two special cases.

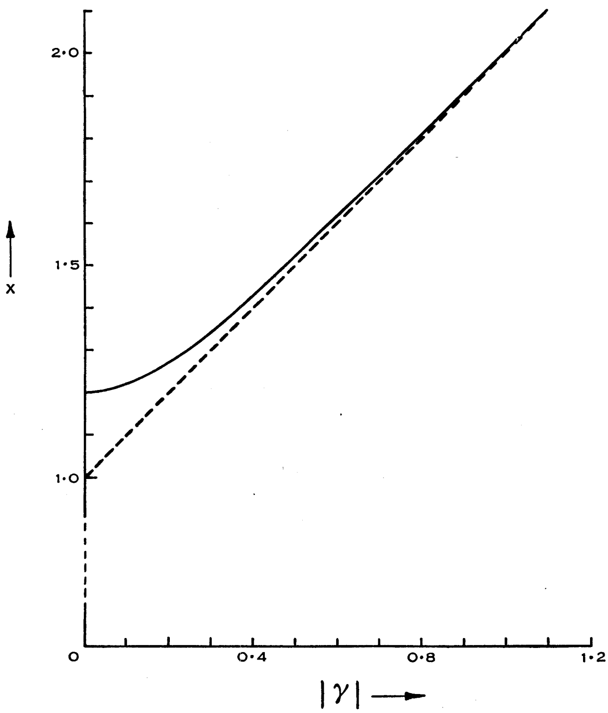

Fig. 2.—Variation of real positive root of equation (52) with $|\gamma|$.

V. Zero Magnetic Field Outside the Current

If the magnetic field vanishes everywhere outside the region bounded by the planes $x=\pm a$, the parameters α and β in (50) are both zero and the stability condition assumes the limiting form

$$e^{4\varkappa a}=\frac{(\varkappa a+1)^2-\gamma^2}{(\varkappa a-1)^2-\gamma^2}.\quad\ldots\ldots\ldots\ldots\ldots\quad(52)$$

It may be shown that this equation has just one positive real root $\varkappa a=X$, say, corresponding to each value of $|\gamma|$. The variation of X with $|\gamma|$ is sketched in Figure 2. It is clear that X always exceeds $1+|\gamma|$, but converges rapidly to this value with increasing $|\gamma|$.

To an excellent approximation, therefore, the solution of (**52**) may be written in the form

$$\varkappa a = 1 + |\gamma|. \qquad \text{(53)}$$

Thus the relation (**52**) requires

$$a = \frac{1}{\sqrt{m^2+n^2}} + \left|\frac{nB}{mA}\right| \qquad \text{(54)}$$

for instability. Since m and n may have any values between $-\infty$ and $+\infty$, there will clearly be a range of values of m and n for which any layer of finite thickness is unstable, and hence such a layer can have no permanent existence.

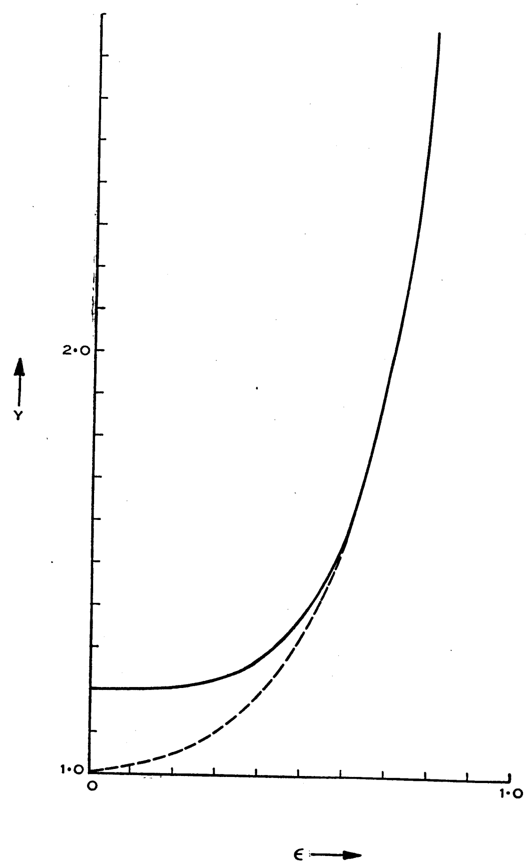

Fig. 3.—Variation of real positive root of equation (**57**) with ε.

VI. Zero Magnetic Field in Direction of Current

In a recent paper Kruskal and Schwarzchild (1954) have examined the stability of a uniform current flowing through a region bounded by parallel planes, the magnetic field being taken to have no component in the direction of the current. On the assumption that the transverse field is greater outside the region than within, these authors conclude that the system is neutral to perturbations in which the two defining surfaces are distorted into sinusoidal wave forms.

In this section the stability condition (50) is applied to the case where the transverse field outside the layer bears an arbitrary ratio to that inside. Suppose that the field inside is

$$H_y=\frac{\eta}{a}\cdot x, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (55)$$

and the field outside

$$H_y=\pm\varepsilon\eta, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (56)$$

where η is a constant and ε may take any positive value. In effect ε measures the ratio of the magnitude of the transverse field outside the region to that at the boundary of the current layer, and it is to be noted that the discussion is equally applicable when H_y has the same or opposite sign when $x>a$ and $x<-a$.

For this case the stability condition (50) assumes the form

$$e^{4\varkappa a}=\frac{\{(1+\varepsilon^2)\varkappa a+1\}\{(1-\varepsilon^2)\varkappa a+1\}}{\{(1+\varepsilon^2)\varkappa a-1\}\{(1-\varepsilon^2)\varkappa a-1\}}. \quad \ldots\ldots\ldots\ldots \quad (57)$$

The nature of the possible roots of equation (57) depends on the particular value given to the parameter ε. It may be shown that (57) has no real positive root for any $\varepsilon>1$. A separate examination shows that for the value $\varepsilon=1$ there is also no real positive root. Hence the current layer is stable for all $\varepsilon\geqslant 1$. This agrees with the result obtained by Kruskal and Schwarzchild.

However, when ε lies in the range $0\leqslant\varepsilon<1$, it may be shown that equation (57) has just one positive real root Y, say, for each value of ε. The variation of Y with ε is sketched in Figure 3. It is clear that Y always exceeds $1/(1-\varepsilon^2)$, but converges to this value as $\varepsilon\rightarrow 1$. Hence to a good approximation, the solution of (57), may be written as

$$\varkappa a=\frac{1}{1-\varepsilon^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (58)$$

where $0\leqslant\varepsilon<1$. Equation (58) implies that, for any layer of finite thickness there are modes of disturbance for which the current distribution is unstable. Hence the state of the system passes from one of stability to one of instability when ε falls below unity.

VII. Conclusions

The general condition for the stability of a plane current layer has been derived by the method of normal modes and its implications have been discussed with reference to two special cases. In the first case the magnetic field has been taken to vanish everywhere outside the current layer, and it has been found that, for any layer of finite thickness, there are modes of disturbance for which the current distribution is unstable. The second case is an idealization of the well-known pinch effect, in which the current is self-constricted by a transverse magnetic field. It has been shown that the state of the system changes from one of stability to one of instability when the magnitude of the field outside becomes less than that at the boundary of the current layer. This conclusion is in conformity with results obtained recently by Kruskal and Schwarzchild (1954).

The velocity of Alfvén waves along a magnetic field in a region bounded by parallel planes has been found to be reduced due to the inertia of the surrounding medium. This reduction is by a factor which approximates to $(1+\mu_2/\mu_1)^{-\frac{1}{2}}$, where μ_1 is the mass density of the fluid inside the layer and μ_2 the density outside (cf. Fig. 1).

VIII. Acknowledgment

The author wishes to thank Dr. R. G. Giovanelli for his interest in this work.

IX. References

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Appendix I

In Section II it was pointed out that the stability of the current layer does not depend on the compressibility or density of the conducting fluid. An equivalent result has been used in an earlier paper by Dr. J. W. Dungey and the author (1954), and it may be of some interest to give the proof in the case of the plane current layer.

For a compressible conducting fluid of variable density the equation describing small perturbations may be stated in the forms

$$\frac{\partial \mathbf{h}}{\partial t}=\text{curl }(\mathbf{v}\times\mathbf{H}), \quad \ldots\ldots \quad (\mathbf{59})$$

$$4\pi\mu\frac{\partial \mathbf{v}}{\partial t}=-\text{grad }\psi+(\mathbf{H}\,.\,\text{grad})\mathbf{h}+(\mathbf{h}\,.\,\text{grad})\mathbf{H}, \quad \ldots\ldots \quad (\mathbf{60})$$

$$\frac{\partial \rho}{\partial t}=-\text{div }(\mu\mathbf{v}), \quad \ldots\ldots \quad (\mathbf{61})$$

where ρ is the variation in the mass density μ, which is now a function of x. Applying the method of normal modes, solutions are sought in which the variables $\mathbf{h}$, $\mathbf{v}$, ψ, and ρ are taken proportional to $\exp i(\omega t+my+nz)$. The conditions for instability are then obtained by putting $\omega=0$; but in this case equations (**59**) and (**61**) require $\mathbf{v}=0$, and so equation (**60**) must be supplemented by the divergence relation div $\mathbf{h}=0$ to obtain a determinate solution.

Hence the equations determining the onset of instability in a compressible fluid of variable density are identical with those obtained previously for the case of an incompressible fluid of uniform density. The conditions for instability are thus unaltered when compressibility and non-uniform density are taken into account.