ELECTROMAGNETIC RADIATION FROM ELECTRONS ROTATING IN AN IONIZED MEDIUM UNDER THE ACTION OF A UNIFORM MAGNETIC FIELD

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Summary

A theory is given for the radiation from a fast electron rotating, under the action of an external magnetic field, in an ionized plasma. It is shown that, although the radiation is emitted predominantly in the extraordinary mode, the ordinary mode is also weakly excited, even in the limiting case in which the density of the background plasma is vanishingly small. At the *harmonics* of the gyro frequency of the fast electron the power radiated in the ordinary mode is a few per cent. of that radiated in the extraordinary mode. This ratio is independent of v_0, the velocity of the fast electron, as long as v_0 is sufficiently small compared with c, the velocity of light. However, at the *fundamental* gyro frequency the power radiated in the ordinary mode is lower by a factor $\simeq 10^{-2}(v_0/c)^4$ than that radiated in the extraordinary mode and indeed is significantly smaller than that radiated, in either mode, at the third harmonic.

The gyro theory of the non-thermal radiation from the Sun is discussed in the light of these results and it is argued that this mechanism cannot explain the phenomena associated with the bursts of spectral types II and III. However, it is conceivable that the radiation on spectral type I may be of gyro origin, though even in this case there are serious objections to this explanation.

I. Introduction

The possibility that the non-thermal radiation from the Sun might be due to gyro radiation from fast electrons was suggested at a very early stage in the development of radio astronomy (Kiepenheuer 1946), but in its initial form this hypothesis seemed subject to several very serious objections. The most important of these was that gyrating electrons would radiate only in the so-called extraordinary mode. At the fundamental frequency at which, it was thought, the majority of the radio energy would be generated, it was argued that this mode could not escape from the Sun (Ryle 1950). It has recently been suggested (Kruse, Marshall, and Platt 1956) that this objection is not conclusive, since in fact a gyrating electron will radiate both in the ordinary and in the extraordinary mode except in directions parallel and perpendicular to the magnetic field. However, in estimating the magnitude of the radiation that might be expected in a practical case, Kruse, Marshall, and Platt assumed that an electron gyrating in a plasma would radiate as much energy as in free space, and that all the energy would be in the ordinary mode. These assumptions are not valid, and before one

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can decide whether gyro radiation plays any significant part in the non-thermal solar radio emission one must have a quantitative theory for radiation by electrons in an ionized medium.

In the present paper we consider the idealized case of a single charged particle of arbitrary energy rotating in a circular orbit in a uniform ionized medium of zero temperature and we shall assume that the motions of the ions are too small to be significant. The more general case where the ionized medium is at a finite temperature and where there is a distribution of fast rotating particles with arbitrary axial velocities has yet to be analysed.

II. The Fundamental Equations and Method of Solution

In the small signal theory in which the quadratic terms involving the time-dependent field and space-charge quantities are negligible, the Maxwell-Lorentz equations which determine the generation and propagation of electromagnetic waves in an ionized medium become linear. When the ionized medium is a neutral plasma at zero temperature with infinitely massive ions acted upon by a uniform external magnetic field, then the fundamental equations determining the radiation from a rotating charged particle consist of the vector Maxwell equations

$$\nabla \wedge \mathbf{E} = -\partial \mathbf{B}/\partial t\,; \quad \nabla \wedge \mathbf{H} = \rho_0 \mathbf{v} + \partial \mathbf{D}/\partial t + \mathbf{I}_0, \qquad \ldots\ldots\ldots \quad (1)$$

where

$$\mathbf{B} = \mu_0 \mathbf{H}, \quad \mathbf{D} = \varepsilon_0 \mathbf{E}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

the equation representing the conservation of charge

$$\partial \rho/\partial t + \rho_0 \nabla \mathbf{v} = 0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

and the Lorentz force equation

$$\partial(m_0 \mathbf{v})/\partial t = -e(\mathbf{E} + \mathbf{v} \wedge \mathbf{a} B_0). \qquad \ldots\ldots\ldots\ldots \quad (4)$$

Here

ρ_0, ρ are the average and time-dependent charge densities respectively of the background plasma,

$\mathbf{v}$ is the time-dependent velocity of the background plasma electrons,

$-e$, m_0 are the electronic charge and rest mass respectively,

B_0 is the flux density of the external axial magnetic field,

$\mathbf{a}$ is the unit axial vector (0, 0, 1), and

$\mathbf{I}_0$ is the current density associated with the rotating charged particle.

These equations are subject to the boundary condition that the solution should consist only of outward going waves at distances sufficiently far from the origin.

The solution which we shall give here is based directly upon the analysis given in an earlier paper (Twiss 1952), hereafter referred to as I, in which it was shown that the general solutions of the Maxwell-Lorentz equations in an ionized medium composed of arbitrarily many beams of charged particles and pervaded by a uniform external magnetic field could be represented by an integral expansion over a complete orthogonal set of elementary vectors. These vectors are given by the orthogonal triad **L**, **M**, **N**, where

$$\mathbf{L}=\nabla\psi(x^1, x^2), \quad \mathbf{M}=\nabla\psi(x^1, x^2)\wedge\mathbf{a}, \quad \mathbf{N}=\psi(x^1, x^2)\mathbf{a}, \quad \ldots \quad (5)$$

where the scalar quantity $\psi(x^1, x^2)$ satisfies the two-dimensional wave equation

$$\nabla^2\psi(x^1, x^2)+p^2\psi(x^1, x^2)=0. \quad \ldots\ldots\ldots\ldots \quad (6)$$

Here (x^1, x^2) are the coordinates transverse to the magnetic field and p, the transverse wave number, is a real quantity.

In the case of radiation by a rotating charged particle it is natural to take the origin at the centre of rotation and use circular cylindrical coordinates (r, φ, z) with the plane $z=0$ as the plane of rotation. Then, since the observable quantities will be finite on the axis $r=0$ we may, without loss of generality, take

$$\psi(x^1, x^2)\equiv\psi_{p,m}(r, \varphi)=\mathrm{J}_m(pr)\exp(-im\varphi) \quad \ldots\ldots\ldots \quad (7)$$

as a typical solution of equation (6).

Subject to certain conditions of boundedness and integrability, which will be valid in any physically realizable case, an arbitrary vector field $\mathbf{F}(r, \varphi, z, t)$ can be expanded by

$$\mathbf{F}(r, \varphi, z, t)=\sum_{m=-\infty}^{\infty}\int_0^\infty \mathrm{d}p[F_1(p, m, z, t)\nabla\psi_{p,m} + F_2(p, m, z, t)\nabla\psi_{p,m}\wedge\mathbf{a}+F_3(p, m, z, t)\psi_{p,m}\mathbf{a}], \quad \ldots \quad (8)$$

where $\psi_{p,m}(r, \varphi)$ is given by equation (7). Similarly, an arbitrary scalar field $U(r, \varphi, z, t)$ can be expanded in the form

$$U(r, \varphi, z, t)=\sum_{m=-\infty}^{\infty}\int_0^\infty \mathrm{d}p u_0(p, m, z, t)\psi_{p,m}. \quad \ldots\ldots \quad (9)$$

The quantities $F_l(p, m, z, t)$ $(l=1, 2, 3)$ and $u_0(p, m, z, t)$ can be found explicitly in terms of $\mathbf{F}(r, \varphi, z, t)$ and $U(r, \varphi, z, t)$ by the Fourier-Bessel inversion formulae which state that if

$$G(p)=\int_0^\infty rg(r)\mathrm{J}_m(pr)\mathrm{d}r, \quad \ldots\ldots\ldots\ldots\ldots \quad (10)$$

then

$$g(r)=\int_0^\infty pG(p)\mathrm{J}_m(pr)\mathrm{d}p, \quad \ldots\ldots\ldots\ldots\ldots \quad (11)$$

and vice versa, provided that

$$\int_0^\infty r^{\frac{1}{2}}\,|\,g(r)\,|\,\mathrm{d}r$$

exists in the sense of Lebesgue.

The method of solution given in I is to express the vector quantities **E, H, v,** and $\mathbf{I}_0$ which appear in equations (**1**)-(**4**) by the vector expansions

$$E(r, \varphi, z, t)=\sum_{m=-\infty}^{\infty}\int_0^{\infty} \mathrm{d}p[E_1(p, m, z, t)\nabla\psi_{p,m}+E_2(p, m, z, t)\nabla\psi_{p,m}\wedge \mathbf{a} +E_3(p, m, z, t)\psi_{p,m}\mathbf{a}], \quad \text{(12a)}$$

$$H(r, \varphi, z, t)=\sum_{m=-\infty}^{\infty}\int_0^{\infty} \mathrm{d}p[H_1(p, m, z, t)\nabla\psi_{p,m}+H_2(p, m, z, t)\nabla\psi_{p,m}\wedge \mathbf{a} +H_3(p, m, z, t)\psi_{p,m}\mathbf{a}], \quad \text{(12b)}$$

$$v(r, \varphi, z, t)=\sum_{m=-\infty}^{\infty}\int_0^{\infty} \mathrm{d}p[v_1(p, m, z, t)\nabla\psi_{p,m}+v_2(p, m, z, t)\nabla\psi_{p,m}\wedge \mathbf{a} +v_3(p, m, z, t)\psi_{p,m}\mathbf{a}], \quad \text{(12c)}$$

$$I_0(r, \varphi, z, t)=\sum_{m=-\infty}^{\infty}\int_0^{\infty} \mathrm{d}p[I_1(p, m, z, t)\nabla\psi_{p,m}+I_2(p, m, z, t)\nabla\psi_{p,m}\wedge \mathbf{a} +I_3(p, m, z, t)\psi_{p,m}\mathbf{a}], \quad \text{(12d)}$$

and the scalar quantity $\rho(r, \varphi, z, t)$ by the scalar expansion

$$\rho(r, \varphi, z, t)=\sum_{m=-\infty}^{\infty}\int_0^{\infty} \mathrm{d}p\sigma(p, m, z, t)\psi_{p,m}. \quad \text{(13)}$$

Now because of equation (**6**) it was shown in I that all the vector-yielding operations in equations (**1**)-(**4**) leave the general form of a vector

$$\mathbf{V}=a(z, t)\mathbf{L}+b(z, t)\mathbf{M}+c(z, t)\mathbf{N}$$

invariant, while all the scalar-yielding operations in equations (**1**)-(**4**) when applied to a vector **V** reduce to a form linearly dependent upon $\psi_{p,m}$. Hence all the vector equations of equations (**1**)-(**4**) can be expressed in the general form

$$\sum_{m=-\infty}^{\infty}\int_0^{\infty} \mathrm{d}p[\alpha(p, m, z, t)\mathbf{L}+\beta(p, m, z, t)\mathbf{M}+\gamma(p, m, z, t)\mathbf{N}]=0,$$

while all the scalar equations are of the form

$$\sum_{m=-\infty}^{\infty}\int_0^{\infty} \mathrm{d}p\,\omega(p, m, z, t)\psi_{p,m}=0.$$

If these equations are to be satisfied for all z, t we must have

$$\alpha(p, m, z, t)=\beta(p, m, z, t)=\gamma(p, m, z, t)=\omega(p, m, z, t)=0, \quad \text{(14)}$$

and these latter equations can be solved for the components $E_l(p, m, z, t)$ ($l=1, 2, 3$) of the electric field in terms of the components of the "driving function" $I_0(r, \varphi, z, t)$ and of the initial conditions along the lines given in I.

III. The Vector Expansion for the Driving Function

In terms of the Dirac δ-function we may write the "driving function" $I_0(r, \varphi, z, t)$ in the form

$$I_0(r, \varphi, z, t)=-\sum_{n=0}^{\infty} e_s\delta(z)\delta(r-r_0)\delta(t-nt_0-t_0\varphi/2\pi)\mathbf{b}, \quad \text{(15)}$$

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where

t_0 is the rotation period defined by

$$2\pi/t_0 = |e_s B_0/m_s| = |v_0|/r_0, \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{16})$$

v_0 is the velocity of the particle, with charge $-e_s$, in its orbit of radius r_0,
b is the unit vector $(0, 1, 0)$, and
m_s is the relativistic transverse mass of the rotating particle.

The summation in equation (**15**) arises because φ is confined to the range $0 \leqslant \varphi < 2\pi$ by the need to keep all functions of position single valued; we have assumed that the fast particle is " started up " at time $t=0$ so that at a later stage we can use the Laplace transform analysis.

In a practical case e_s, m_s will have the values appropriate to the electron, but we shall not make this restriction to begin with.

As a first step in the solution outlined in the previous section, we must find explicit expressions for $I_l(p, m, z, t)$ $(l=1, 2, 3)$ in equation (**12**).

From equation (**15**), we have immediately

$$I_3(p, m, z, t) = 0.$$

To find I_1, we operate on both sides of equation (**12**) with the scalar operator

$$(\nabla - \mathbf{a}\partial/\partial z),$$

and get

$$(\nabla - \mathbf{a}\partial/\partial z) \,.\, \mathbf{I}_0(r, \varphi, z, t) = \sum_{m=-\infty}^{\infty} \int_0^\infty \mathrm{d}p I_1(p, m, z, t) \nabla^2 \psi_{p,m} \,.$$

Now from equations (**6**), (**10**), (**11**), and the Fourier series inversion formulae, we have immediately

$$\begin{aligned} I_1(p, m, z, t) &= \frac{-1}{2\pi p} \int_0^{2\pi} \mathrm{d}\varphi \int_0^\infty \mathrm{d}r\, r \left[\frac{1}{r} \frac{\partial}{\partial \varphi} \left\{ -\sum_{n=0}^{\infty} e_s \delta(z) \delta(r - r_0) \right.\right. \\ &\qquad \left.\left. \times \delta\left(t - n t_0 - \frac{t_0 \varphi}{2\pi} \right) \right\} \mathrm{J}_m(pr) \exp(im\varphi) \right] \\ &= -\frac{e_s \delta(z) \mathrm{J}_m(p r_0)}{2\pi p} \cdot \frac{2\pi i m}{t_0} \exp\left(\frac{2\pi i m t}{t_0} \right), \quad t \geqslant 0. \quad \ldots\ldots \quad (\mathbf{17}) \end{aligned}$$

To find $I_2(p, m, z, t)$ we take the curl of both sides of equation (**12d**) and then take the scalar product with **a** to get

$$\mathbf{a} \,.\, [\nabla \wedge \mathbf{I}_0(r, \varphi, z, t)] = -\sum_{m=-\infty}^{\infty} \int_0^\infty \mathrm{d}p I_2(p, m, z, t) \nabla^2 \psi_{p,m},$$

whence

$$\begin{aligned} I_2(p, m, z, t) &= \frac{1}{2\pi p} \int_0^{2\pi} \mathrm{d}\varphi \int_0^\infty \mathrm{d}r\, r \left[\frac{1}{r} \frac{\partial}{\partial r} \left\{ -r \sum_{n=0}^{\infty} e_s \delta(z) \delta(r - r_0) \right.\right. \\ &\qquad \left.\left. \times \delta\left(t - n t_0 - \frac{t_0 \varphi}{2\pi} \right) \right\} \mathrm{J}_m(pr) \exp(im\varphi) \right] \\ &= -\frac{e_s \delta(z) p r_0 \mathrm{J}'_m(p r_0)}{2\pi p} \cdot \frac{2\pi}{t_0} \exp\left(\frac{2\pi i m t}{t_0} \right), \quad t \geqslant 0. \quad \ldots \quad (\mathbf{18}) \end{aligned}$$

IV. The Solution in Terms of the Driving Function

If we substitute from equations (**12**), (**13**) in equations (**1**)-(**4**), then from equations of the form of equation (**14**) it can be shown that the Maxwell equations reduce to

$$\left.\begin{array}{ll} -\mu_0\partial H_1/\partial t=\partial E_2/\partial z, & \partial H_2/\partial z=\rho_0 v_1+\varepsilon_0\partial E_1/\partial t+I_1, \\ -\mu_0\partial H_2/\partial t=E_3-\partial E_1/\partial z, & -\partial H_1/\partial z+H_3=\rho_0 v_2+\varepsilon_0\partial E_2/\partial t+I_2, \\ -\mu_0\partial H_3/\partial t=p^2 E_2, & p^2 H_2=\rho_0 v_3+\varepsilon_0\partial E_3/\partial t. \end{array}\right\} \quad (19)$$

From the charge conservation equation we get

$$\partial\sigma/\partial t=-\rho_0(-p^2 v_1+\partial v_3/\partial z), \quad (20)$$

and from the Lorentz force equations

$$\left.\begin{array}{l} \partial(m_0 v_1)/\partial t=-eE_1+m_0\omega_H v_2, \\ \partial(m_0 v_2)/\partial t=-eE_2-m_0\omega_H v_1, \\ \partial(m_0 v_3)/\partial t=-eE_3, \end{array}\right\} \quad (21)$$

where

$$\omega_H=eB_0/m_0 \quad (22)$$

is the gyro angular frequency of the background plasma electrons.

Except for the forcing terms I_1, I_2 in equation (**19**) this set of equations is a special case of those derived in I, in which the solution was obtained in terms of arbitrary initial conditions by means of successive Laplace transforms firstly with respect to the time coordinate t and secondly with respect to the axial coordinate z. To apply this procedure to the present case of an unbounded medium we introduce a cut over the plane $z=0$ and consider the solutions in the regions $z>0$, $z<0$ separately.

The double Laplace transform $F^{z,t}(k, \omega)$ of a quantity $F(z, t)$ can be defined in the region $z>0$, $t>0$ by

$$F^{z,t}(k, \omega)=\int_0^\infty \mathrm{d}z \exp(-\mathrm{i}kz)\int_0^\infty \mathrm{d}tF(z, t)\exp(-\mathrm{i}\omega t), \quad (23)$$

where

$$F(z, t)=\frac{1}{(2\pi)^2}\int_{-\infty-\mathrm{i}\gamma_1}^{\infty-\mathrm{i}\gamma_1} \mathrm{d}k \exp(\mathrm{i}kz)\int_{-\infty-\mathrm{i}\gamma_2}^{\infty-\mathrm{i}\gamma_2} \mathrm{d}\omega F^{z,t}(k, \omega)\exp(\mathrm{i}\omega t), \quad (24)$$

and where γ_1, γ_2 are real positive numbers such that all the poles and singularities of $F^{z,t}(k, \omega)$ and $F^t(k, \omega)$ lie above the lines

$$\mathrm{Im}\,(k)+\gamma_1=0, \qquad \mathrm{Im}\,(\omega)+\gamma_2=0$$

in the complex k- and ω-planes respectively.

In the region $z<0$, $t>0$, $F^{z,t}(k,\omega)$ is defined by a similar equation in which all the singularities of $F^{z,t}(k,\omega)$ lie *below* the line

$$\mathrm{Im}\,(k)+\gamma_1'=0$$

in the complex k-plane.

Now that the plane $z=0$ has been excluded by a cut, the quantities I_1, I_2 in equation (**19**), may be taken as zero in the regions $z>0$ and $z<0$ and we can use the solution, derived in I, for $E_l^{z,t}(p,m,k,\omega)$ ($l=1,2,3$) in terms of the initial conditions. In the present case the disturbance at $t=0$ may be taken as identically zero in the regions $z>0$ and $z<0$ and under these conditions we have immediately from equation (**3.52**) of I

$$E_l^{z,t}(k,\omega)=\sum_{r=1}^{3} C_r^*(p,m,k,\omega)\mathbf{A}_{lr}(p,m,k,\omega)/\det\mathbf{A}. \quad .. \quad (25)$$

Here $\mathbf{A}_{lr}(p,m,k,\omega)$ is the minor of $a_{lr}(p,m,k,\omega)$ in the matrix $\mathbf{A}$ defined by equation (**3.47**) in I which, in the present case, may be written

$$\mathbf{A}=\begin{bmatrix} k^2-\dfrac{\omega^2}{c^2}+\dfrac{\omega^2\omega_0^2}{c^2(\omega^2-\omega_H^2)}, & \dfrac{-i\omega\,\omega_H\,\omega_0^2}{c^2(\omega^2-\omega_H^2)}, & ik \\ \dfrac{i\omega\,\omega_H\,\omega_0^2}{c^2(\omega^2-\omega_H^2)}, & p^2+k^2-\dfrac{\omega^2}{c^2}+\dfrac{\omega^2\omega_0^2}{c^2(\omega^2-\omega_H^2)}, & 0 \\ ikp^2, & 0, & \dfrac{\omega^2-\omega_0^2}{c^2}-p^2 \end{bmatrix}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (26)$$

where

$$\omega_0=(-e\rho_0/\varepsilon_0 m_0)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (27)$$

is the angular plasma frequency associated with the background electrons.

The column matrix $C^*(p,m,k,\omega)$ defined by equation (**3.49**) of I reduces in the present case to the simple form given, for $z>0$, by

$$\left.\begin{aligned} C_1^*(p,m,k,\omega)&=-ikE_1^t(+0)-i\omega\mu_0H_2^t(+0),\\ C_2^*(p,m,k,\omega)&=-ikE_2^t(+0)+i\omega\mu_0H_1^t(+0),\\ C_3^*(p,m,k,\omega)&=p^2E_1^t(+0).\end{aligned}\right\} \quad \ldots. \quad (28)$$

It therefore depends only upon the transverse components of the electric and magnetic fields at $z=+0$.

The expression for $C^*(p,m,k,\omega)$ in the region $z<0$ is of exactly similar form and involves the initial conditions $E_1^t(-0)$, $E_2^t(-0)$, $H_1^t(-0)$, and $H_2^t(-0)$.

The transverse fields in the region $z>0$ and $z<0$ are connected by the continuity conditions which, for the electric field components, give

$$\left.\begin{aligned} E_1^t(+0)&=E_1^t(-0),\\ E_2^t(+0)&=E_2^t(-0).\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (29)$$

The continuity conditions on the tangential magnetic fields are complicated by the fact that the driving function consists of a current sheet in the plane $z=0$. However, by applying Stokes's theorem

$$\oint \mathbf{H} \cdot \mathrm{d}\mathbf{s} = \int \mathbf{J} \cdot \mathrm{d}\mathbf{S}.$$

it can be shown that

$$\left.\begin{aligned} H_1^t(+0)-H_1^t(-0) &= +\int_{-0}^{+0} \mathrm{d}z I_2^t(p, m, z, \omega) \\ &= \frac{-e_s}{2\pi p \mathrm{i}} \cdot \frac{2\pi/t_0}{\omega-2\pi m/t_0} p r_0 \mathrm{J}_m'(p r_0), \\ H_2^t(+0)-H_2^t(-0) &= -\int_{-0}^{+0} \mathrm{d}z I_1^t(p, m, z, \omega) \\ &= \frac{+e_s}{2\pi p} \cdot \frac{2\pi m/t_0}{\omega-2\pi m/t_0} \mathrm{J}_m(p r_0). \end{aligned}\right\} \quad \ldots\ldots (30)$$

Two further relations between $E_1^t(+0)$, $E_2^t(+0)$, $H_1^t(+0)$, and $H_2^t(+0)$ are provided by the boundary conditions which require that the solution in the region $z>0$ should consist only of outward-going waves. Two similar relations will exist between the transverse field components at $z=-0$ but in the present case these need not be set up explicitly since, from symmetry considerations, we have at once

$$\left.\begin{aligned} H_1^t(+0) = -H_1^t(-0) &= -\tfrac{1}{2} \cdot \frac{e_s}{2\pi p \mathrm{i}} \cdot \frac{2\pi/t_0}{\omega-2\pi m/t_0} p r_0 \mathrm{J}_m'(p r_0), \\ H_2^t(+0) = -H_2^t(-0) &= +\tfrac{1}{2} \cdot \frac{e_s}{2\pi p} \cdot \frac{2\pi m/t_0}{\omega-2\pi m/t_0} \mathrm{J}_m(p r_0). \end{aligned}\right\} \quad \ldots (31)$$

The components at $z=-0$ can therefore be eliminated by means of equations (**29**) and (**31**).

(*a*) *The Inversion from the Complex* k-*Plane onto the Real* z-*Axis*

In the complex k-plane the singularities of $E_l^{z,t}(p, m, k, \omega)$ consist of simple poles at the zero of det $\mathbf{A}$. In the present case det $\mathbf{A}$ may be expanded in the biquadratic form given by

det $\mathbf{A}\equiv$

$$k^4\frac{\omega^2-\omega_0^2}{c^2}+k^2\left[\frac{\omega^2-\omega_0^2}{c^2}\left\{p^2-\frac{2\omega^2}{c^2}\left(1-\frac{\omega_0^2}{\omega^2-\omega_H^2}\right)\right\}+\frac{p^2\omega^2}{c^2}\left(1-\frac{\omega_0^2}{\omega^2-\omega_H^2}\right)\right]$$

$$-\frac{\omega^2}{c^2}\left(\frac{\omega^2-\omega_0^2}{c^2}-p^2\right)\left\{p^2\left(1-\frac{\omega_0^2}{\omega^2-\omega_H^2}\right)-\frac{\omega^2}{c^2}\left(1-\frac{\omega_0^2}{\omega^2-\omega_H^2}\right)^2+\frac{\omega_H^2}{c^2}\cdot\frac{\omega_0^4}{(\omega^2-\omega_H^2)^2}\right\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (32)$$

which may be written in the equivalent form

$$\det \mathbf{A} \equiv \{(\omega^2-\omega_0^2)/c^2\}[(k-k_1)(k-k_2)(k-k_3)(k-k_4)]. \quad \ldots . \quad (33)$$

The inversion from the complex k-plane onto the real z-axis expresses $E_l^t(p, m, z, \omega)$ as the sum of the residues of

$$\mathrm{i}E_l^{z,t}(p, m, k, \omega) \exp \mathrm{i}kz$$

at the poles given by the zeros of det $\mathbf{A}$. Hence $E_l^t(p, m, z, \omega)$ may be represented as the sum of four partial waves of the form

$$\sum_{r=-2}^{2} G_l(k_r) \exp (\mathrm{i}k_r z).$$

Two of these partial waves will be reverse waves excited only by reflection and their propagation constants will be designated by $+k_1$, $+k_2$, where for the reasons given in I, the sign of $k_1(\omega)$, $k_2(\omega)$ must be chosen so that

$$\lim_{\omega \to \infty} \frac{k_1(\omega)}{\omega} = \lim_{\omega \to \infty} \frac{k_2(\omega)}{\omega} = +\frac{1}{c},$$

when ω tends to infinity along a line in the complex ω-plane which lies below all the singularities of $k_1(\omega)$, $k_2(\omega)$.

To meet the boundary conditions at infinity the amplitudes $G_l(k_1)$, $G_l(k_2)$ of these reverse partial waves must be identically zero. From equation (**25**) this implies that

$$\sum_{r=1}^{3} C_r^*(p, m, k_s, \omega) \mathbf{A}_{lr}(p, m, k_s, \omega) = 0 \quad \ldots\ldots\ldots \quad (34)$$

for any† choice of l.

The amplitudes of the forward partial waves with propagation constants $-k_1$, $-k_2$ are proportional to

$$\sum_{r=1}^{3} C_r^*(p, m, -k_s, \omega) \mathbf{A}_{lr}(p, m, -k_s, \omega), \quad s=1, 2.$$

By inspection of equation (**26**) we see that $\mathbf{A}_{l1}$, $\mathbf{A}_{l2}$ are even functions of k and $\mathbf{A}_{l3}$ is an odd function of k for $l=1, 2$ and vice versa for $l=3$. Hence, from equations (**28**) and (**34**), we find that the amplitudes of the forward waves are proportional to

$$-2\mathrm{i}\omega\mu_0[H_2^t(+0)\mathbf{A}_{l1}(p, m, -k_s, \omega) - H_1^t(+0)\mathbf{A}_{l2}(p, m, -k_s, \omega)], \quad s=1, 2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (35)$$

† It is a well-known consequence of the theory of determinants that the three linear equations obtained by taking the three possible choices of l are linearly dependent when det $\mathbf{A}(p,m,k_s,\omega)=0$ so that this condition gives only two independent equations for $s=1, 2$.

and using equations (**25**), (**26**), (**28**), and (**31**) we get

$$\left.\begin{aligned}
E_1^t(p,m,z,\omega) &= -\sum_{s=1}^{2}\frac{(\omega^2-\omega_0^2)/c^2-p^2}{(\omega^2-\omega_0^2)/c^2}\cdot\frac{\omega\mu_0 e_s pm}{t_0 p^2(\omega-2\pi m/t_0)}\\
&\times\left[\left\{\left(p^2+k_s^2-\frac{\omega^2}{c^2}+\frac{\omega^2\omega_0^2}{c^2(\omega^2-\omega_H^2)}\right)\mathrm{J}_m(pr_0)\right.\right.\\
&\left.\left.-\frac{\omega\,\omega_H\,\omega_0^2}{c^2(\omega^2-\omega_H^2)}\cdot\frac{pr_0}{m}\mathrm{J}_m'(pr_0)\right\}\frac{(-)^{s-1}\exp(-\mathrm{i}k_s z)}{2k_s(k_1^2-k_2^2)}\right],\\
E_2^t(p,m,z,\omega) &= \sum_{s=1}^{2}\frac{(\omega^2-\omega_0^2)/c^2-p^2}{(\omega^2-\omega_0^2)/c^2}\cdot\frac{\mathrm{i}\omega\mu_0 e_s pm}{t_0 p^2(\omega-2\pi m/t_0)}\\
&\times\left[\left\{\left(\frac{p^2k_s^2}{(\omega^2-\omega_0^2)/c^2-p^2}+k_s^2-\frac{\omega^2}{c^2}+\frac{\omega^2\omega_0^2}{c^2(\omega^2-\omega_H^2)}\right)\frac{pr_0}{m}\mathrm{J}_m'(pr_0)\right.\right.\\
&\left.\left.-\frac{\omega\,\omega_H\,\omega_0^2}{c^2(\omega^2-\omega_H^2)}\mathrm{J}_m(pr_0)\right\}\frac{(-)^{s-1}\exp(-\mathrm{i}k_s z)}{2k_s(k_1^2-k_2^2)}\right],\\
E_3^t(p,m,z,\omega) &= -\sum_{s=1}^{2}\frac{p^2}{(\omega^2-\omega_0^2)/c^2}\cdot\frac{\mathrm{i}\omega\mu_0 e_s pm}{t_0 p^2(\omega-2\pi m/t_0)}\\
&\times\left[\left\{\left(p^2+k_s^2-\frac{\omega^2}{c^2}+\frac{\omega^2\omega_0^2}{c^2(\omega^2-\omega_H^2)}\right)\mathrm{J}_m(pr_0)\right.\right.\\
&\left.\left.-\frac{\omega\,\omega_H\,\omega_0^2}{c^2(\omega^2-\omega_H^2)}\cdot\frac{pr_0}{m}\mathrm{J}_m'(pr_0)\right\}\frac{(-)^{s-1}\exp(-\mathrm{i}k_s z)}{2k_s(k_1^2-k_2^2)}\right].
\end{aligned}\right\}\qquad\ldots\ldots\ldots\ldots\ (36)$$

In the complex ω-plane the electric field components $E_l^t(p,m,z,\omega)$ $(l=1,2,3)$ all have a simple pole at

$$\omega=2\pi m/t_0\equiv\omega_m,\qquad\ldots\ldots\ldots\ldots\ (37)$$

that is, at the mth harmonic of the gyro frequency of the fast charged particle. The residue of this pole represents the steady state solution in which we are interested* in this paper. Thus, ignoring the decaying transient terms, we have

$$E_l(p,m,z,t)=\mathrm{i}(\omega-2\pi m/t_0)E_l^t(p,m,z,\omega)\exp(2\pi\mathrm{i}m/t_0).\quad\ldots\ (38)$$

V. The Asymptotic Solution at Large Distances from the Source

We shall consider the solution at very large distances from the source in a direction making an angle χ with the z-axis at the point

$$r=R\sin\chi,\quad z=R\cos\chi,\quad R\to\infty.$$

* It might appear from equation (**36**) that $E_l^t(p,m,z,\omega)$ would have poles at $\omega=\omega_0$ and $\omega=\omega_H$, that is, at the natural resonances associated with the plasma and gyro frequencies of the medium. However, a more detailed discussion involving the behaviour of $k_1(\omega)$, $k_2(\omega)$ at these frequencies shows that this is not the case. This latter result is to be expected on physical grounds since, if it did not hold, it would imply that the medium would oscillate with finite amplitude at its natural resonances whatever the nature of the initial stimulus. The transient terms, arising from the integrals around the cuts in the complex ω-plane that need to be inserted to eliminate the branch points of $E_l^t(p,m,z,\omega)$ decay with time and do not contribute to the steady state solution.

Since the solution has been derived for the case $z>0$ we have

$$0\leqslant\chi\leqslant\tfrac{1}{2}\pi,\quad 0\leqslant\tan\chi\leqslant\infty.$$

Now as $r\to\infty$

$$\begin{aligned}\psi_{p,m}(r,\varphi)&=\mathrm{J}_m(pr)\exp(-\mathrm{i}m\varphi)\\&\sim\{1/\sqrt{(\tfrac{1}{2}\pi pr)}\}\cos(pr-\tfrac{1}{2}m\pi-\tfrac{1}{4}\pi)\exp(-\mathrm{i}m\varphi),\end{aligned}\quad\ldots\ (\mathbf{39a})$$

as long as $p>0$, while

$$\left.\begin{aligned}\nabla\psi_{p,m}(r,\varphi)&\sim(-\mathrm{i}p\psi,0,0),\\\nabla\psi_{p,m}(r,\varphi)\wedge\mathbf{a}&\sim(0,\mathrm{i}p\psi,0),\end{aligned}\right\}\quad\ldots\ldots\ldots\ (\mathbf{39b})$$

Since $(1/r)\partial\psi/\partial\varphi\sim r^{-3/2}$ and may be neglected.

Accordingly, the lth component of the electric field associated with the sth mode is of the general form

$$\begin{aligned}&E_{ls}(R\sin\chi,m,R\cos\chi,t)\\&\sim\int_0^\infty\frac{\mathrm{d}pW_{ls}(p)}{\sqrt{(\tfrac{1}{2}\pi pR\sin\chi)}}\cos(pR\sin\chi-\tfrac{1}{2}m\pi-\tfrac{1}{4}\pi)\\&\quad\times\exp[-\mathrm{i}R\cos\chi\,k_s(p)],\end{aligned}\quad\ldots\ldots\ldots\ (\mathbf{40})$$

where s may take the value 1, 2 corresponding to the ordinary and extraordinary mode respectively, and where

$$(W_{1s}(p),W_{2s}(p),W_{3s}(p))\equiv(-\mathrm{i}pE_{1s},\mathrm{i}pE_{2s},E_{3s}).\quad\ldots\ (\mathbf{41})$$

For very large values of R, the only effective contribution to the integral in equation (**40**) comes from values of p near the point of " stationary phase " at which

$$p=p_0\ \text{and}\ \tan\chi=\pm\partial k_s/\partial p_0.\quad\ldots\ldots\ (\mathbf{42})$$

That is, the only Fourier-Bessel components of the driving function which contribute to the power radiated in directions close to χ are those with radial wave numbers near p_0 defined by equation (**42**).

Since $0<\chi<\tfrac{1}{2}\pi$, equation (**42**) can be satisfied for only one choice of the ambiguous sign and in the present case where $-k_s(p)$ is the propagation constant of an outward-going wave, we have

$$\begin{aligned}&E_{ls}(R\sin\chi,m,R\cos\chi,t)\sim\frac{\exp[\mathrm{i}\theta(p_0)]W_{ls}(p_0)}{\sqrt{(2\pi p_0R\sin\chi)}}\\&\times\int_0^\infty\mathrm{d}p\exp\left[-\mathrm{i}\left\{\frac{\partial^2k_s}{\partial p_0^2}\cdot\frac{R\cos\chi}{2}(p-p_0)^2+0(p-p_0)^3\right\}\right],\end{aligned}\quad\ldots\ (\mathbf{43})$$

where

$$\theta(p_0)=-p_0R\sin\chi-k_s(p_0)R\cos\chi+(m+\tfrac{1}{2})\pi.$$

The absolute value of the integral in equation (**43**) does not depend upon the sign of $\partial^2k_s/\partial p_0^2$, and since

$$\int_{-\infty}^\infty\frac{\exp(-\mathrm{i}u)}{2u^{\frac{1}{2}}}\mathrm{d}u=(\sqrt{\pi})\exp(-\tfrac{1}{4}\mathrm{i}\pi)=-\mathrm{i}\int_{-\infty}^\infty\frac{\exp(\mathrm{i}u)}{2u^{\frac{1}{2}}}\mathrm{d}u,\quad\ldots\ (\mathbf{44})$$

we have from equations (**12**), (**39**), and (**43**)

$$\left.\begin{aligned} \mathbf{E}_s(R,\chi,m,t) &\sim \frac{\exp\left[\mathrm{i}\{\theta(p_0)+\frac{1}{2}\pi+\frac{1}{4}\pi(\partial^2k_s/\partial p_0^2)/(|\partial^2k_s/\partial p_0^2|)\}\right]}{R\sqrt{\{p_0\sin\chi\cos\chi\,|\partial^2k_s/\partial p_0^2|\}}} \\ &\quad\times\{-\mathrm{i}p_0E_{1s},\, p_0E_{2s},\, E_{3s}\}, \\ \mathbf{H}_s(R,\chi,m,t) &\sim \frac{\exp\left[\mathrm{i}\{\theta(p_0)+\frac{1}{2}\pi+\frac{1}{4}\pi(\partial^2k_s/\partial p_0^2)/(|\partial^2k_s/\partial p_0^2|)\}\right]}{R\sqrt{\{p_0\sin\chi\cos\chi\,|\partial^2k_s/\partial p_0^2|\}}} \\ &\quad\times\{-\mathrm{i}p_0H_{1s},\, \mathrm{i}p_0H_{2s},\, H_{3s}\}, \end{aligned}\right\} \quad \ldots (\mathbf{45})$$

where

$$\partial k_s/\partial p_0 = -\tan\chi, \quad \ldots\ldots\ldots\ldots (\mathbf{46})$$

provided that*

$$\partial^2 k_s/\partial p_0^2 \neq 0.$$

VI. The Radiation Flux in a Given Direction

From equation (**45**), it follows that the mean energy flux associated with the sth mode at a great distance from the source is given by

$$\operatorname{Re}\tfrac{1}{2}(\mathbf{E}\wedge\mathbf{H}^*)_s = \frac{1}{p_0R^2\sin 2\chi\,|\partial^2k_s/\partial p_0^2|}$$
$$\times\operatorname{Re}\{\mathrm{i}p_0(E_{2s}H_{3s}^*+E_{3s}H_{2s}^*),\ \mathrm{i}p_0(E_{1s}H_{3s}^*+E_{3s}H_{1s}^*),\ -p_0^2(E_{1s}H_{2s}^*-E_{2s}H_{1s}^*)\}. \quad \ldots\ldots (\mathbf{47})$$

From the vector Maxwell equation

$$-\mu_0\partial\mathbf{H}/\partial t = \nabla\wedge\mathbf{E},$$

and from equations (**36**), (**37**), (**38**), and (**47**) we get

$$P_s = \operatorname{Re}\tfrac{1}{2}(\mathbf{E}\wedge\mathbf{H}^*)_s = \frac{1}{\omega_m\mu_0p_0R^2\sin 2\chi\,|\partial^2k_s/\partial p_0^2|}$$
$$\times\left\{p_0\left(p_0^2E_{2s}E_{2s}^*+\frac{\omega_m^2-\omega_0^2}{c^2p_0^2}E_{3s}E_{3s}^*\right),\ 0,\ k_sp_0^2E_{2s}E_{2s}^*+\frac{\omega_m^2-\omega_0^2}{c^2p_0^2}\left(\frac{\omega_m^2-\omega_0^2}{c^2}-p_0^2\right)\frac{E_{3s}E_{3s}^*}{k_s}\right\} \quad \ldots\ldots (\mathbf{48})$$

on using the relation

$$\{(\omega_m^2-\omega_0^2)/c^2-p_0^2\}E_{3s}(p,m,z,t)-\mathrm{i}k_sp_0^2E_{1s}(p,m,z,t)=0, \quad \ldots (\mathbf{49})$$

which follows from equation (**36**).

The power flux $P_{\chi s}\mathrm{d}\chi$ out of an infinite sphere in angle $\mathrm{d}\chi$ is given by

$$P_{\chi s}\mathrm{d}\chi = 2\pi R^2\sin\chi\,\mathrm{d}\chi(P_{rs}\sin\chi+P_{zs}\cos\chi), \quad \ldots\ldots (\mathbf{50})$$

* Equation (**45**) is not valid when $\partial^2k_s/\partial p_0^2=0$ but it can be shown that the zeros of $\partial^2k_s/\partial p_0^2$ form a set of measure zero on the χ-axis, so that they do not contribute to the total radiated power. It remains true, however, that the amplitude of the electromagnetic field and therefore of the radiated power is likely to have a maximum near the zeros of $\partial^2k_s/\partial p_0^2$.

so that

$$P_{\chi_s}\mathrm{d}\chi = \frac{\pi \mathrm{d}\chi}{\omega_m \mu_0 p_0 \mid \partial^2 k_s/\partial p_0^2 \mid} \times \left\{ p_0^2 E_{2s} E_{2s}^* (p_0 \tan \chi + k_s) + \frac{\omega_m^2 - \omega_0^2}{c^2 p_0^2} E_{3s} E_{3s}^* \left(p_0 \tan \chi + \frac{\omega_m^2 - \omega_0^2 - c^2 p_0^2}{c^2 k_s} \right) \right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (51)$$

The total power radiated in the forward direction in both modes is given by

$$P = \sum_{s=1}^{2} \int_0^{\frac{1}{2}\pi} P_{\chi_s} \mathrm{d}\chi. \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (52)$$

The integral involved in equation (52) is extremely complicated in the general case and one must rely on numerical integration to find a quantitative expression for the radiated power. However, it is possible to draw some useful conclusions about the relative power radiated in different harmonics and different modes from the general nature of the solution. As a preliminary to this discussion, it will be necessary to review briefly some of the properties of the propagation constants $-k_1$, $-k_2$ of the ordinary and extraordinary modes respectively with the aid of the familiar magneto-ionic theory.*

(a) *The Propagation Constants*

It follows from relation (32) that the propagation constants for a given value of p_0^2 are given by

$$k^2 = \{b \pm \sqrt{(b^2 - 4ad)}\}/2a, \quad \ldots\ldots\ldots\ldots\ldots \ (53)$$

where

$$\left.\begin{aligned} a &= (\omega_m^2 - \omega_0^2)/c^2, \\ b &= \frac{\omega_m^2 - \omega_0^2}{c^2}\left[\frac{2\omega_m^2}{c^2}\left(1 - \frac{\omega_0^2}{\omega_m^2 - \omega_H^2}\right) - p_0^2\right] - p_0^2 \frac{\omega_m^2}{c^2}\left(1 - \frac{\omega_0^2}{\omega_m^2 - \omega_H^2}\right), \\ d &= \frac{\omega_m^2}{c^2}\left(\frac{\omega_m^2 - \omega_0^2}{c^2} - p_0^2\right)\left[\frac{\omega_m^2}{c^2}\left(1 - \frac{\omega_0^2}{\omega_m^2 - \omega_H^2}\right)^2 - \frac{\omega_H^2}{c^2} \cdot \frac{\omega_0^4}{\omega_m^2 - \omega_H^2} \right. \\ &\qquad \left. - p_0^2\left(1 - \frac{\omega_0^2}{\omega_m^2 - \omega_H^2}\right)\right]. \end{aligned}\right\} \quad \ldots \ (54)$$

One cannot give a general rule for deciding which choice of the ambiguous sign in equation (53) corresponds to the ordinary and which to the extraordinary wave since this depends upon the relative magnitudes of ω_m^2, ω_0^2, and ω_H^2 and each case must be considered separately. In what follows we shall assume that

$$\omega_0^2 < \omega_1^2 < \omega_H^2 < \omega_2^2 - \omega_0^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (55)$$

* The point of view developed in the present paper is rather different from that normally adopted in magneto-ionic theory but the general survey given by Mitra (1947) contains all the results we need below.

This case is one of the more interesting from the point of view of solar radiophysics since, when it is valid, the ordinary mode can escape from the Sun at the gyro frequency of the rotating fast electron as well as at all the harmonics, while the extraordinary mode can escape at all the harmonics though not, of course, at the fundamental.* It may be noted that the inequality $\omega_1^2<\omega_H^2$ is a direct consequence of the assumption that the radiating charged particle is an electron rotating under the action of the axial magnetic field in a frame of reference at rest with respect to the background plasma since, in this case,

$$\omega_1^2=(eB_0/m)^2=(eB_0/m_0)^2(1-v_0^2/c^2)=\omega_H^2(1-v_0^2/c^2)<\omega_H^2,$$

where v_0 is the velocity of the fast particle.

We know from magneto ionic theory that the infinity of the propagation constant at $\omega=\omega_H$ is associated with the extraordinary mode. Now, as ω_m tends to ω_H from below it may be seen from equations (**53**), (**54**) that k will tend to infinity when the ambiguous sign is chosen positive. The reverse is true when ω_m tends to ω_H from above, the change of sign arising because the parameter b, defined in equation (**54**), changes from $+\infty$ to $-\infty$ as ω_m changes from $\omega_H-\varepsilon$ to $\omega_H+\varepsilon$. Accordingly, for radiation at the fundamental of the gyro frequency of the fast electron, for which $\omega_m=\omega_1<\omega_H$, the positive choice of sign in equation (**53**) corresponds to the extraordinary mode and vice versa for the higher harmonics of this gyro frequency for which, from the inequality (**55**), $\omega_m>\omega_H$.

It can be shown that k_1^2, k_2^2 both decrease monotonically as p increases from zero to the values at which k_1^2 or k_2^2 become zero, which, it may further be shown, coincide with the two zeros of the parameter d defined in equation (**54**). At these zeros of k_s, $|\partial k_s/\partial p|=\infty$ so that, as one would expect, they correspond to propagation in a direction perpendicular to the axial magnetic field. As p is increased beyond these critical values, k_s^2 becomes negative, so that the associated wave becomes evanescent and a radiation field is produced for only a finite range of values of p which, it can be shown, is defined for the ordinary mode by

$$0\leqslant p^2\leqslant(\omega_m^2-\omega_0^2)/c^2=p_1^2, \qquad \ldots\ldots\ (\mathbf{56})$$

and for the extraordinary mode by

$$0\leqslant p^2\leqslant\frac{\omega_m^2}{c^2}\left[\frac{(\omega_m^2-\omega_H^2-2\omega_0^2)\omega_m^2+\omega_0^4}{\omega_m^2(\omega_m^2-\omega_H^2-\omega_0^2)}\right]=p_2^2. \qquad \ldots\ldots\ (\mathbf{57})$$

(*b*) *Comparison between Radiation in the Ordinary and in the Extraordinary Mode*

One important difference between the radiation emitted in the two possible modes is that the ordinary mode is relatively weakly excited for small values of p, that is, in directions making a small angle with the axial magnetic field. To see this we note that

$$pr_0\mathrm{J}_m'(pr_0)=m\mathrm{J}_m(pr_0)-pr_0\mathrm{J}_{m+1}(pr_0),$$

* The theory in this paper has been developed assuming a uniform medium but will be valid for the solar corona as long as the macroscopic properties of the medium vary sufficiently slowly with distance.

hence, from equation (**36**), the amplitudes of the components of the electric field are proportional to

$$\left[k^2-\frac{\omega_m^2}{c^2}+\frac{\omega_m\omega_0^2}{c^2(\omega_m+\omega_H)}\right]\mathrm{J}_m(pr_0)+\text{terms of order } p^2\mathrm{J}_m(pr_0).$$

Neglecting terms of the order of p^2, it is known that for the ordinary wave

$$k_1^2-\frac{\omega_m^2}{c^2}+\frac{\omega_m\omega_0^2}{c^2(\omega_m+\omega_H)}=0,$$

so that the energy carried by this mode is proportional to $p^4\mathrm{J}_m^2(pr_0)$ for sufficiently small values of p. For the extraordinary mode on the other hand

$$k_2^2-\frac{\omega_m^2}{c^2}+\frac{\omega_m\omega_0^2}{c^2(\omega_m-\omega_H)}=0,$$

so that the energy carried by this mode is proportional to $\mathrm{J}_m^2(pr_0)$ for small values of p.

It can also be shown that the ordinary mode is not excited in a direction perpendicular to the magnetic field. This arises from the fact that the electric vector of the ordinary mode lies parallel to the axial magnetic field in this case and cannot, therefore, be excited by a charged particle of which the velocity and acceleration vectors are both perpendicular to the axial magnetic field.

The fact that the ordinary mode can be excited appreciably over only a limited range of values of the direction angle is one of the reasons for the associated radiation being much less than that in the extraordinary mode; a second and more obvious reason is that the polarization of the extraordinary mode rotates in the same sense as does the radiating electron. This latter fact probably explains why the possibility of the ordinary mode also being generated has been ignored for so long. Furthermore, at the fundamental gyro frequency of the fast electron the relative contribution of the ordinary mode is reduced by an additional effect which arises because the propagation constant of the extraordinary wave near the gyro frequency is very much greater than the propagation constant for the ordinary mode. Thus for the former we have

$$k_2^2\approx\frac{c^2}{v_0^2}\cdot\frac{\omega_0^2}{\omega_H^2-\omega_0^2}\left(2\frac{\omega_H^2-\omega_0^2}{c^2}-p_0^2\right), \quad \ldots\ldots\ldots\ldots \quad (58)$$

for $\omega_m^2=\omega_H^2(1-v_0^2/c^2)$ and for values of p_0^2 which obey the inequality (**56**), at least as long as

$$v_0^2/c^2\ll\omega_0^2/(\omega_H^2-\omega_0^2).$$

The propagation constant for the ordinary mode on the other hand is nearer to the free space value, being given by

$$k_1^2\simeq\frac{\{(2\omega_H^2-\omega_0^2)/c^2-p_0^2\}\{(\omega_H^2-\omega_0^2)/(c^2-p_0^2\}}{2(\omega_H^2-\omega_0^2)/c^2-p_0^2}. \quad \ldots\ldots \quad (59)$$

The characteristic admittance of the extraordinary wave is therefore much greater than that of the ordinary wave with which it is effectively in parallel from the point of view of the source current generator, so that almost all the available power will be generated in the extraordinary mode.

To make this argument more concrete, we note from the relations (**58**), (**59**) that the average value of p_0^2 is of the order of magnitude of ω_H^2/c^2, for radiation at the fundamental gyro frequency of the rotating electron, so that the parameter pr_0 which occurs as the argument of the Bessel function in equation (**36**) is of the order of magnitude of v_0/c. As long as $v_0/c \ll 1$, we have

$$\mathrm{J}_1(pr_0) \simeq pr_0\mathrm{J}_1'(pr_0) \simeq \tfrac{1}{2}v_0/c, \qquad (60)$$

while

$$\mathrm{J}_1(pr_0) - pr_0\mathrm{J}_1'(pr_0) = pr_0\mathrm{J}_2(pr_0) \simeq \tfrac{1}{8}(v_0/c)^3. \qquad (61)$$

Now for the large values of k_2 we see from equation (**51**) that the power radiated in the χ-direction in the extraordinary mode is of the order of magnitude of

$$(\omega_H^2/c^2)E_{22}E_{22}^*k_2 + E_{32}E_{32}^*\omega_H/c.$$

From equation (**36**), (**58**), (**60**), and (**61**), we see that for the extraordinary mode the term $E_{22}E_{22}^*k_2\omega_H^2/c^2$ is of the order of magnitude of v_0^3/c^3, while the term $E_{32}E_{32}^*\omega_H/c$ is of the order of magnitude of v_0^2/c^2. Hence, just as in the free space case, the total power radiated in the extraordinary mode at the first harmonic of the gyro frequency of the fast electron is proportional to v_0^2/c^2. For the ordinary mode, on the other hand, in which k_1 is also of the order of magnitude of ω_H/c, we see from the same equations that E_{21}, E_{31} are both of the order of magnitude of v_0^3/c^3. The total power radiated in the ordinary mode at the first harmonic of the gyro frequency of the fast electron is therefore of the order of magnitude of v_0^6/c^6.

If we now consider the mth harmonic of the gyro frequency, however, at which p and also k_1, k_2 are all of the order of magnitude of $m\omega_H/c$, it follows, from equation (**36**) and the fact that

$$\mathrm{J}_m(pr_0) \simeq (m^m/m!2^m)(v_0/c)^m \quad \text{for } mv_0/c \ll 1,$$

that the power radiated in either mode is proportional, for small enough values of v_0/c, to $(v_0/c)^{2m}$. In particular, it follows that the power radiated in the ordinary mode at the second harmonic of the gyro frequency of the fast electron is larger by a factor $(c/v_0)^2$ than that radiated at the fundamental frequency in the same mode. Finally, it may be noted that the power radiated in the ordinary mode at the fundamental varies in the same way with v_0/c as the power radiated in this extraordinary mode at the third harmonic, but because the latter mode should be generated more efficiently one would expect that it would carry appreciably more power. This is confirmed by the numerical results given below.

(c) *Numerical Results*

The calculations shown in Table 1 below were carried out for the special case

$$\beta^2 = v_0^2/c^2 = 0\cdot 1, \qquad (62)$$

corresponding to a kinetic energy of the rotating fast electron of approximately 27·7 keV. In order to bring out the effect of the background plasma electrons we have compared the powers radiated in different modes and at different harmonics in the two special cases, (*a*) $\omega_0^2/\omega_H^2=0\cdot 8$, (*b*) $\omega_0^2/\omega_H^2=0$. To show how the radiated power varies with the velocity of the fast electron we have expressed the total power radiated at the *m*th harmonic as a numerical quantity times β^{2m}. The results are exact only for the special case of equation **(62)**, but they will be accurate to better than 10 per cent. for values of β^2 in the range

$$0<\beta^2<0\cdot 15.$$

As might be expected, the effect of the plasma electrons is most serious at the fundamental frequency. In every case, however, it may be seen that the power radiated is lower than in the free space case.

TABLE 1

COMPARISON BETWEEN POWERS RADIATED IN DIFFERENT MODES AND AT DIFFERENT HARMONICS FOR TWO VALUES OF THE DENSITY OF THE BACKGROUND PLASMA ELECTRONS

Harmonic	Total Power* Radiated in Forward Direction	Percentage of Power Radiated in Forward Direction in Ordinary Mode	Total Power* Radiated in Forward Direction	Percentage of Power Radiated in Forward Direction in Ordinary Mode
1	$0\cdot 0755\beta^2(1-\beta^2)C$	$\beta^4\times 1\cdot 53\%$	$0\cdot 322\beta^2(1-\beta^2)C$	$\beta^4\times 1\cdot 63\%$
2	$0\cdot 59\beta^4(1-\beta^2)C$	$2\cdot 1\%$	$0\cdot 729\beta^4(1-\beta^2)C$	$2\cdot 5\%$
3	$1\cdot 18\beta^6(1-\beta^2)C$	$6\cdot 5\%$	$1\cdot 68\beta^6(1-\beta^2)C$	$4\cdot 4\%$

* The expressions for the power are in watts when $C=2\cdot 48\times 10^{-22}H^2$ and H is in gauss.

Besides changing the total radiated power, the plasma electrons also modify appreciably the polar diagram, especially at the fundamental frequency. This effect is illustrated in Figures 1 (*a*) and 1 (*b*), where we have plotted the polar diagrams for the radiation at the first three harmonics in the ordinary mode for the case (*a*) $\omega_0^2/\omega_H^2=0\cdot 8$, (*b*) $\omega_0^2/\omega_H^2=0$. In Figures 2 (*a*) and 2 (*b*), we have plotted polar diagrams for the radiation, in the extraordinary mode, into the second and third harmonics : the polar diagram for the first harmonic in the ionized plasma has not been plotted since at this frequency the radiation cannot escape from the Sun.

VII. DISCUSSION AND CONCLUSIONS

The main result of this paper is to confirm that a fast electron rotating with velocity $v_0 \ll c$ in an ionized plasma permeated by a uniform magnetic field will radiate energy both in the ordinary and in the extraordinary modes, but predominantly in the latter. Thus the power radiated at the fundamental gyro frequency in the ordinary mode is proportional to $(v_0/c)^6$ in contrast with the extraordinary mode in which the power radiated is proportional to $(v_0/c)^2$; indeed the first harmonic power in the ordinary mode is appreciably less than that radiated in the third harmonic—at least in the numerical cases considered in this paper. At the mth harmonic, $m>1$, the power is still radiated predominantly in the extraordinary mode, though for both modes it is proportional to $(v_0/c)^{2m}$ as long as $(v_0/c)^2$ is sufficiently small. These conclusions are valid

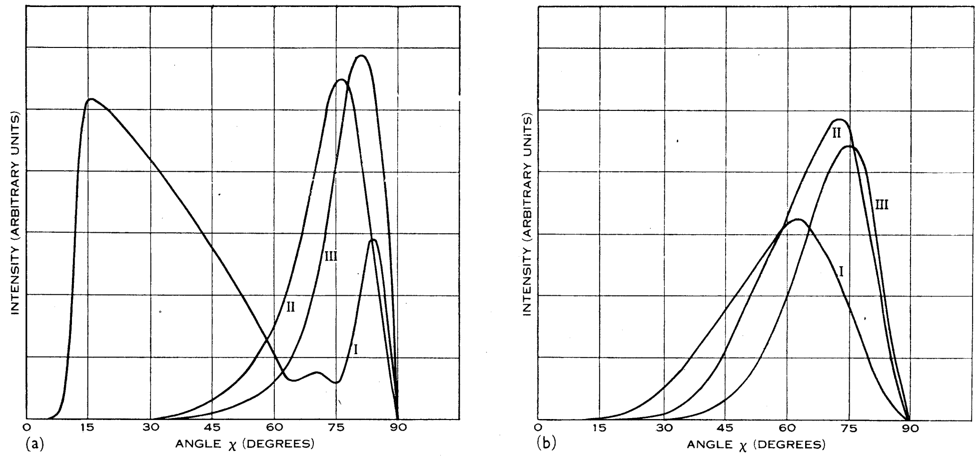

Fig. 1.—Emission polar diagrams for the ordinary mode. (*a*) With $\omega_0^2/\omega_H^2=0{\cdot}8$; (*b*) for free space, $\omega_0^2/\omega_H^2=0$. The emission per unit solid angle is shown as a function of χ, the inclination to the magnetic field. The intensity scale is arbitrary and is different for each curve. Curve I fundamental; curve II, second harmonic; curve III, third harmonic.

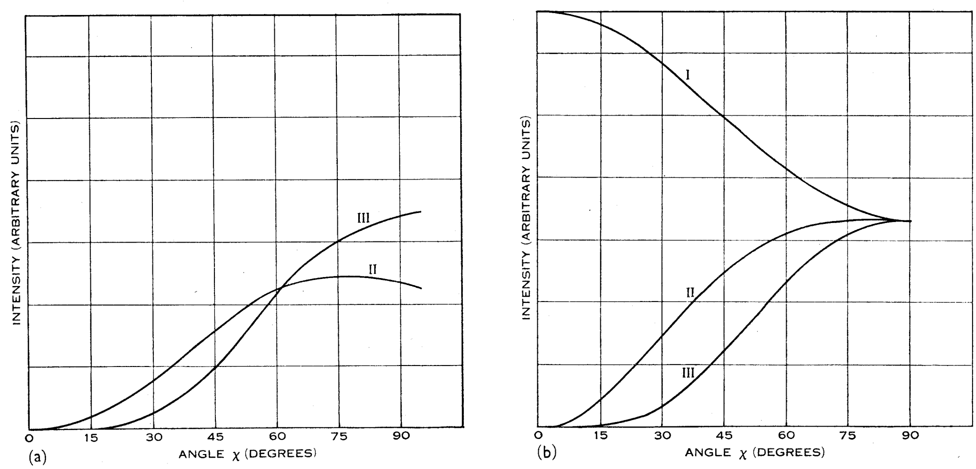

Fig. 2.—Corresponding emission polar diagrams for the extraordinary mode.

not only in media so dense that the plasma frequency is comparable with the gyro frequency but also in the limiting case of free space propagation in which the plasma frequency tends to zero, as shown in Appendix I.

To see the relevance of these results to gyro-type theories for the non-thermal solar radiation, let us first consider the bursts of spectral types II and

III. In these phenomena, the radiation at a given instant is concentrated into one or two frequency bands, the centre frequency of which decreases with time; very rapidly for the type III bursts, less rapidly for the type II bursts (Wild, Murray, and Rowe 1954). Recent observations have shown that in the case of type III bursts this decrease of frequency is associated with a rapid movement of the radiating disturbance (Wild, personal communication). In those cases in which two frequency bands are observed the data are consistent with the belief that the higher frequency band is simply the second harmonic of the lower. There is virtually no evidence that the third or higher harmonics are radiated; if these are present then their intensity, in the great majority of cases, must be at least one order of magnitude lower than that in the first two harmonics. The apparent dimensions of the disturbed regions in a type III burst are large, being up to 10 per cent. or even more of the area of the visible solar disk (Wild and Sheridan 1958). In general, these bursts are unpolarized; however, in quite a number of cases there is an appreciable component of circular polarization, and when this is so the sense of polarization is the same in both the high and the low frequency bands (Komesaroff 1958).

If these features are to be explained on the gyro theory, one has to imagine either that a burst of fast electrons is fired out from the Sun or that an electromagnetic disturbance travels outward giving rotational energy to a fraction of the local plasma electrons as it goes. On either picture the decrease in frequency with time would be attributed to a decrease in gyro frequency with increasing height above the solar corona. The energy in the fundamental band would have to be carried entirely by the ordinary mode, since the extraordinary mode cannot escape at the fundamental gyro frequency (Ryle 1950). On the other hand, one would expect the energy at the higher harmonics to be carried predominantly by the extraordinary mode for the reasons discussed in this paper. One might hope to explain the fact that the majority of bursts are not strongly polarized by the large size of the radiating regions together with the assumption that the sense of the magnetic field is different in different parts of the radiation front. However, when the bursts were polarized, one would expect the sense of polarization to be different in the higher and lower frequency bands and this is in disagreement with the few observations so far available (Komesaroff 1958). It is also very hard to see why the third harmonic should not be present and stronger than the first since, as we have shown, the power radiated at the fundamental frequency in the ordinary mode is less than that radiated at the third harmonic in the ordinary mode,* and very much less than that radiated at the third harmonic in the extraordinary mode. A similar objection is that the power radiated at the fundamental gyro frequency in the ordinary mode is very much smaller than the power radiated either in the extraordinary or in the ordinary mode at the second harmonic, and this is in disagreement with the observed ratio of the power radiated in the upper and lower frequency bands; at least

* The possibility that the absence of the third harmonic could be due to differential absorption can be eliminated, since the ordinary mode at the fundamental gyro frequency will be more heavily absorbed than either mode at the third harmonic of the gyro frequency, especially in the case when the plasma and gyro frequencies are of comparable magnitude.

in some cases. Once again the discrepancy is too great to be explained away by differential absorption.

Admittedly, we assumed in our quantitative analysis that the velocity v_0 of the fast radiating electrons was significantly less than c, but, if this were not the case, appreciable power would be radiated at quite a few harmonics of the gyro frequency, which is contrary to observation.

A final objection to the gyro theory of the rapid drift bursts is that the first harmonic can escape only when the plasma frequency is less than the gyro frequency. Since the type III bursts at least are often associated with quite small sunspots this requirement is not very likely to be met.

It is possible that some of these objections would be removed or at least mitigated by a more exact analysis in which one allows for the finite temperature of the background plasma and for Doppler effects in the radiation from the gyrating electrons. However, at the present time the weight of evidence is still strongly against the gyro theory, at least as far as the types II and III bursts are concerned.

The position is perhaps more hopeful in the case of the type I disturbances, which are strongly associated with large sunspots and which consist of short-duration bursts, with a narrow frequency spread, superimposed upon a continuous disturbance of much longer duration and with a wide frequency spread. Both bursts and continuum are strongly circularly polarized and it is conceivable that the continuum disturbance may actually consist of a very large number of small bursts occurring over a wide range of heights in the solar corona. There is some rather weak evidence of correlation between bursts at different frequencies, but there is definitely no sign of a 2 to 1 frequency correlation as in types II and III.

Much of the data is compatible with the suggestion that the type I disturbances are due to second harmonic gyro radiation. However, one objection to this is that such an explanation would lead one to expect that the radiation would be in the extraordinary mode: the evidence on this point is too scanty to be conclusive, but it suggests that the radiation is in the ordinary mode, at least in the majority of cases (Payne-Scott and Little 1952). If this is confirmed, it would seem that the gyro hypothesis could be saved only if (*a*) the absorption in the extraordinary mode were greater by about three orders of magnitude than that in the ordinary mode at the same frequency* or if (*b*) the plasma frequency at the point of generation were so high, at least in the majority of cases, that the second harmonic could not escape in the extraordinary mode. It may be noted that, in a zero temperature plasma, the conditions that the ordinary mode at the mth harmonic should escape and that the extraordinary mode should be trapped is

$$\omega_0 < m\omega_H < \sqrt{\{m/(m-1)\}}\omega_0.$$

* This would be quite impossible, in any conceivable model of the solar corona, if absorption were due simply to free-free transitions in the background plasma. However, as has been pointed out by Gross (1951), a finite temperature plasma permeated with a magnetic field may possess stop bands at the harmonics of the gyro frequency and it is not inconceivable that this phenomenon could discriminate against the extraordinary mode.

Here ω_0, ω_H are the plasma and gyro angular frequencies of the plasma, and we have assumed $v_0^2/c^2 \ll 1$. It is not implausible that this condition might be met, when $m=2$, above a large sunspot.

Of course, it is not sufficient merely to prove that waves with the right sense of circular polarization could both be generated and escape; one also has to explain quantitatively the observed energy flux. Now in the larger type I disturbances the effective black-body temperature of the source may be as high as 3×10^{10} °K or even higher (Wild and Sheridan 1958), and the electrons of a gas at this temperature would have an average energy of $\sim 3 \times 10^6$ eV. However, the kinetic energy of the gyrating electrons must be very much less than this or the bursts would not consist of isolated narrow bands, nor would the emergent radiation be so strongly circularly polarized. To obtain a radiation flux with a black-body temperature much greater than that appropriate to the average kinetic energy of a non-equilibrium electron gas implies a process of radiation transfer very different from that usually encountered in stellar atmospheres, but as pointed out elsewhere (Twiss, in preparation) it is conceivable that conditions could arise in the disturbed solar corona where the absorption coefficient was negative so that the medium behaved like an amplifier. If this latter effect were really present one could certainly explain the observed radiation flux. However, a detailed discussion of this possibility involves a theory for the case where a large number of gyrating electrons is present, and this is beyond the scope of the present paper.

VIII. Acknowledgments

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Appendix I

Resolution of the Radiated Field into Ordinary and Extraordinary Modes in the Limit of Vanishing Electron Density

Even when the density of the background plasma electrons is vanishingly small, the magneto-ionic modes still have defined polarization so that the radiation emitted by an electron gyrating "*in vacuo*" under the action of an external magnetic field may still be resolved into ordinary and extraordinary modes.

In spherical coordinates (r, χ, φ) the magneto-ionic modes in the limiting case of zero plasma frequency obey the constraint (Mitra 1947)

$$E_\varphi/E_\chi = -H_\chi/H_\varphi = (-\mathrm{i}/2\cos\chi)[(\omega_H/\omega_m)\sin^2\chi \pm \{(\omega_H/\omega_m)^2\sin^4\chi + 4\cos^2\chi\}^{\frac{1}{2}}], \qquad \text{(A1)}$$

where ω_H is the angular gyro frequency of a background plasma electron,
ω_m is the mth harmonic of the angular gyro frequency of the fast gyrating electron,

and where the upper sign refers to the extraordinary, the lower sign to the ordinary mode.

For the radiation at the fundamental gyro frequency of the fast electron

$$\omega_H/\omega_m = \omega_H/\omega_1 = (1-\beta^2)^{-\frac{1}{2}} \simeq 1 + \tfrac{1}{2}\beta^2$$

to the first order in $\beta^2 = v_0^2/c^2$, where v_0 is the velocity of the fast electron.

Accordingly, at the first harmonic, we have

$$(E_\varphi/E_\chi)_e = [(E_\varphi/E_\chi)_o]^{-1} \simeq -\mathrm{i}\cos\chi\{1 + \tfrac{1}{2}\beta^2\sin^2\chi/(1+\cos^2\chi)\}, \qquad \text{(A2)}$$

where the subscripts e and o refer to the extraordinary and to the ordinary mode respectively.

The polarization ellipses for the two modes are similar, but their major axes are mutually perpendicular.

Now for the radiated field (Schott 1912) we have

$$\left(\frac{E_\varphi}{E_\chi}\right)_{\text{rad}} = \frac{-\mathrm{i}}{\cos\chi} \cdot \frac{J_{m-1}(m\beta\sin\chi) - J_{m+1}(m\beta\sin\chi)}{J_{m-1}(m\beta\sin\chi) + J_{m+1}(m\beta\sin\chi)}. \qquad \text{(A3)}$$

Hence for the fundamental mode, $m=1$, we have, to the first order in β^2,

$$(E_\varphi/E_\chi)_{\text{rad}} = (-\mathrm{i}/\cos\chi)(1 - \tfrac{1}{4}\beta^2\sin^2\chi). \qquad \text{(A4)}$$

Comparison between equations (**A2**) and (**A4**) shows that, although the radiated power is carried predominantly by the extraordinary mode, the ordinary mode is also excited. To determine the proportion of ordinary mode excited at a given angle χ to the axial magnetic field, let $\mathrm{i}\rho^{\frac{1}{2}}$ be the ratio of the major axis of the polarization ellipse of the ordinary mode to that of the extraordinary mode. Then we have

$$\left.\begin{aligned} &E_{\chi e} = E_0\exp(\mathrm{i}\omega t), \quad E_{\varphi e} = (-\mathrm{i}/\cos\chi)\{1+\tfrac{1}{2}\beta^2\sin^2\chi/(1+\cos^2\chi)\}E_0\exp(\mathrm{i}\omega t), \\ &E_{\chi o} = (-\mathrm{i}/\cos\chi)\{1+\tfrac{1}{2}\beta^2\sin^2\chi/(1+\cos^2\chi)\}\mathrm{i}\rho_1^{\frac{1}{2}}E_0\exp(\mathrm{i}\omega t), \quad E_{\varphi o} = \mathrm{i}\rho_1^{\frac{1}{2}}E_0\exp(\mathrm{i}\omega t), \end{aligned}\right\} \qquad \text{(A5)}$$

where E_0 is a constant of proportionality.

Hence

$$\frac{\rho_1^{\frac{1}{2}} \sec \chi\{1+\frac{1}{2}\beta^2 \sin^2 \chi/(1+\cos^2 \chi)\}}{1-\rho_1^{\frac{1}{2}} \sec \chi\{1+\frac{1}{2}\beta^2 \sin^2 \chi/(1+\cos^2 \chi)\}}=\frac{-i}{\cos \chi}(1-\tfrac{1}{4}\beta^2 \sin^2 \chi),$$

so that

$$\rho_1=\tfrac{1}{16}\beta^4[\sin^4 \chi \cos^2 \chi(3+\cos^2 \chi)^2/(1+\cos^2 \chi)^4], \quad \ldots . \quad \textbf{(A6)}$$

and the ratio of the *power* radiated in the ordinary mode to that in the extraordinary mode at the fundamental gyro frequency of the fast electron is therefore proportional to β^4 as in the case, considered in the text, in which the plasma frequency associated with the background electrons is comparable with the gyro frequency.

For harmonics higher than the first, it is sufficient to work to zero order in β^2 since, for the magneto-ionic modes at the mth harmonic, $m>1$,

$$E_\varphi/E_\chi \simeq -i\{(\tfrac{1}{2}m) \sin \chi \tan \chi \pm[1+\{(\tfrac{1}{2}m) \sin \chi \tan \chi\}^2]^{\frac{1}{2}}\}, \quad \ldots \quad \textbf{(A7)}$$

where, as before, the positive and negative choices of sign in equation (**A7**), refer to the extraordinary and to the ordinary mode respectively.

For the radiated field, on the other hand, we have from equation (**A3**)

$$(E_\varphi/E_\chi)_{\mathrm{rad}}=-i \sec \chi,$$

if we neglect terms $0(\beta^2)$.

Hence

$$\frac{\rho_m^{\frac{1}{2}}-\{(\sin \chi \tan \chi)/2m+[1+\{(\sin \chi \tan \chi)/2m\}^2]^{\frac{1}{2}}\}}{1+\rho_m^{\frac{1}{2}}\{(\sin \chi \tan \chi)/2m+[1+\{(\sin \chi \tan \chi)/2m\}^2]^{\frac{1}{2}}\}}=-\sec \chi,$$

and

$$\rho_m=\frac{1-\cos \chi\{(\sin \chi \tan \chi)/2m+[1+\{(\sin \chi \tan \chi)/2m\}^2]^{\frac{1}{2}}\}}{\cos \chi+\{(\sin \chi \tan \chi)/2m+[1+\{(\sin \chi \tan \chi)/2m\}^2]^{\frac{1}{2}}\}}, \quad \ldots \quad \textbf{(A8)}$$

which is independent of β^2.