THE CONSTANT FLUX PROBLEM IN NON-UNIFORM EXPONENTIAL MEDIA

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Summary

A description is given of radiative transfer, under conditions of constant flux, in a semi-infinite medium in which the attenuation coefficient increases exponentially with depth, and has small sinusoidal variations in a direction parallel to the surface.

The directional intensity at the surface is evaluated numerically in several cases, and the extrapolation to other cases is discussed.

I. INTRODUCTION

In astrophysical problems the equation of transfer has usually been applied to plane parallel semi-infinite media, uniform except for possible variation with depth. Recently, however, investigations of the granulation in the solar photosphere and of the structure of the umbra and penumbra of sunspots have focused attention on the need for solutions for non-uniform media. Giovanelli (1959)† investigated the problem in which the attenuation coefficient, scattering parameter, and source function are independent of depth, but exhibit small sinusoidal variations as functions of one coordinate parallel with the surface.

In any discussion of stellar atmospheres, however, it is necessary to consider the extent to which the increase of attenuation coefficient with depth affects the solution of the equation of transfer. In the present paper the solution of the constant flux problem is undertaken for a plane parallel semi-infinite medium in which the attenuation coefficient, in addition to small sinusoidal variation parallel to the surface, increases exponentially with depth.

The transfer equation is solved here in terms of the total intensity, which is then integrated numerically to obtain the directional intensity at the surface.

II. THE EQUATION OF TRANSFER

Consider an isotropic medium in which $\varkappa$ is the attenuation coefficient, $\varkappa\lambda$ the absorption coefficient, $\varkappa\bar{\omega}_0$ the coefficient of single scattering, and S, the source function, is the ratio of emission per unit volume and solid angle to the attenuation coefficient. For a non-uniform medium, $\varkappa$, λ, $\bar{\omega}_0$, and S are scalar functions of position. The intensity of radiation, I, is a function of position and direction, while the total intensity J, defined by

$$J=\int_{4\pi} I\mathrm{d}\Omega,$$

is a function of position only.

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† GIOVANELLI, R. G. (1959).—*Aust. J. Phys.* **12**: 164.

From the transfer equation, Giovanelli (1959) obtained an equation for J embodying an approximation similar to that of Eddington,

$$\nabla^2 J = \frac{1}{\varkappa}\nabla J.\nabla\varkappa + \varkappa^2(\lambda J - 4\pi S). \quad \ldots\ldots\ldots\ldots \quad (1)$$

If the medium is in radiative equilibrium, then

$$4\pi S = \lambda J,$$

and (1) simplifies to

$$\nabla^2 J = \frac{1}{\varkappa}\nabla J.\nabla\varkappa. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

The following discussion is confined to this conservative case.

III. Characteristics of the Medium

If the attenuation coefficient of a semi-infinite medium increases exponentially with depth below the surface and varies sinusoidally in one direction parallel to the surface, $\varkappa$ must have the form

$$\varkappa = \varkappa_0 e^{-\nu z}(1 + \alpha \cos lx), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

the z-axis being normal to the surface $z=0$ and positive outwards. The simple model of a stellar atmosphere, in which the attenuation coefficient tapers off exponentially to a negligibly small value, corresponds to the limiting case where $\varkappa_0$ tends to zero. A non-zero value of $\varkappa_0$, with $\varkappa=0$ for $z>0$, provides a simple model for an atmosphere whose attenuation coefficient initially increases rapidly with depth and then more slowly according to (3).

Since the horizontal structure size is determined by $2\pi/l$, and the scale height by ν^{-1}, the appearance of the medium will be found to depend largely on whether ν/l is large or small as compared with unity.

IV. Solution of the Equation of Transfer in the Conservative Case

To solve (2) for a medium in radiative equilibrium carrying constant mean flux, J is expressed in the form

$$J(x,z) = \sum_n j_n(z) \cos nlx.$$

Here $j_0(z)$ is the mean value of the total intensity at depth z, while $j_1(z)$ is the amplitude of the variations in $J(x,z)$ having the same periodicity as the structure of the medium. For $n \geqslant 2$ the functions $j_n(z)$ represent distortions in the intensity distribution, and if α is sufficiently small they may be disregarded. Then

$$J(x,z) \simeq j_0(z) + j_1(z) \cos lx, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

where j_1/j_0 is assumed small. Substituting (4) with (3) into (2) yields

$$\begin{aligned} j_0''(1+\alpha\cos lx) + (j_1'' - l^2 j_1)\{\cos lx + \tfrac{1}{2}\alpha(1+\cos 2lx)\} &= j_1.\tfrac{1}{2}\alpha l^2(1-\cos 2lx) \\ -\nu\{j_0' + \tfrac{1}{2}\alpha j_1' + (j_1' + \alpha j_0')\cos lx + \tfrac{1}{2}\alpha j_1' \cos 2lx\}, & \quad \ldots\ldots\ldots\ldots\ldots \quad (5) \end{aligned}$$

where $j_0'=\mathrm{d}j_0/\mathrm{d}z$ and $j''=\mathrm{d}^2j_0/\mathrm{d}z^2$. Equating the coefficients of $\cos nlx$ when $n=0$ and $n=1$ yields

$$(j_0''+\nu j_0')+\tfrac{1}{2}\alpha(j_1''+\nu j_1'-2l^2j_1)=0, \qquad \ldots\ldots\ldots\ldots (6)$$

and

$$\alpha(j_0''+\nu j_0')+(j_1''+\nu j_1'-l^2j_1)=0. \qquad \ldots\ldots\ldots\ldots (7)$$

The coefficients of $\cos 2lx$ correspond to the distortion terms and may be ignored for sufficiently small α.

Elimination of j_0 and use of the approximation $\alpha^2=0$ yield a differential equation for j_1, having the solution

$$j_1=C_1\mathrm{e}^{pz}+D_1\mathrm{e}^{qz}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

where $p=\tfrac{1}{2}\{-\nu+\sqrt{(\nu^2+4l^2)}\}$, $Q=-\tfrac{1}{2}\{\nu+\sqrt{(\nu^2+4l^2)}\}$, and C_1 and D_1 are constants of integration. Substitution in (**6**) or (**7**) yields the solution for j_0,

$$j_0=C_0+D_0\mathrm{e}^{-\nu z}+\tfrac{1}{2}\alpha(C_1\mathrm{e}^{pz}+D_1\mathrm{e}^{qz}), \qquad \ldots\ldots\ldots\ldots (9)$$

where C_0 and D_0 are also constants of integration. Thus

$$J(x,z)=C_0+D_0\mathrm{e}^{-\nu z}+(C_1\mathrm{e}^{pz}+D_1\mathrm{e}^{qz})(\tfrac{1}{2}\alpha+\cos lx). \qquad \ldots\ldots (10)$$

The constants of integration C_0, C_1, D_0, and D_1 are determined by the constant flux requirement and the boundary conditions. Deep within the medium the following conditions should hold.

(i) The mean intensity j_0 may increase linearly with optical depth but at no greater rate, as may be inferred by comparison with known solutions for uniform media.

(ii) The form of $J(x,z)$ should be self preserving, i.e. $\lim\limits_{z\to-\infty} j_1/j_0$ should exist and, from (**4**), should be small. As $z\to-\infty$, the mean optical depth behaves as $\mathrm{e}^{-\nu z}$. However, since $\lim\limits_{z\to-\infty} \mathrm{e}^{qz}/\mathrm{e}^{-\nu z}$ is infinite, condition (i) requires $D_1=0$.

Alternatively, from (**6**) and (**7**)

$$\begin{aligned}\lim_{z\to-\infty}\frac{j_1}{j_0}&=\lim_{z\to-\infty}\frac{D_1\mathrm{e}^{qz}}{\tfrac{1}{2}\alpha D_1\mathrm{e}^{qz}+D_0\mathrm{e}^{-\nu z}}\\ &=2/\alpha \ \text{ if } D_1\neq 0,\\ &=0 \quad\ \text{ if } D_1=0.\end{aligned}$$

Since α is small, condition (ii) also requires $D_1=0$.

At the surface $z=0$, the inward radiation flux is assumed zero. As usual in the treatment of the plane parallel case, this leads to the condition

$$\varkappa J(x,0)=-\tfrac{2}{3}\mathrm{d}J(x,0)/\mathrm{d}z.$$

Substituting for J at $z=0$, $\cos lx=\pm 1$, we obtain C_0 and C_1 in terms of D_0, whence finally

$$J(x,z)=D_0\left\{\frac{2}{3}\frac{\nu}{\varkappa_0}+\mathrm{e}^{-\nu z}-1-\frac{2}{3}\frac{\alpha\nu}{\varkappa_0+\frac{2}{3}p}\mathrm{e}^{pz}\cos lx\right\}. \qquad \ldots\ldots (11)$$

Here also terms of order α^2 have been ignored.

The flux per unit area across any layer parallel to the surface is

$$-\frac{1}{3\varkappa}\frac{\mathrm{d}J(x,z)}{\mathrm{d}z}.$$

Deep within the medium this has the value

$$\frac{\nu D_0}{3\varkappa_0(1+\alpha\cos lx)}$$

and, if the net flux is independent of $\varkappa_0$ and ν, (**11**) takes the form

$$J(x,z)=A\left\{\tfrac{2}{3}+\frac{\varkappa_0}{\nu}(\mathrm{e}^{-\nu z}-1)-\frac{\alpha}{3/2+p/\varkappa_0}\mathrm{e}^{pz}\cos lx\right\}, \quad \ldots. \quad (\mathbf{12})$$

where the constant A replaces $D_0\nu/\varkappa_0$ and is determined solely by the mean value of the net flux. In particular, the contrast at the surface, $j_1(0)/j_0(0)$, given by (**12**), is

$$j_1/j_0=-\alpha\varkappa_0/(\varkappa_0+\tfrac{2}{3}p). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{13})$$

Thus an intensity maximum is associated with an attenuation minimum.

It is of interest to note that in the limit as $\nu\to 0$ (i.e. if the medium is uniform with depth), then $p\to l$ and

$$\lim_{\nu\to 0} J(x,z)=A\left\{\tfrac{2}{3}-\varkappa_0 z-\frac{\alpha}{3/2+l/\varkappa_0}\mathrm{e}^{lz}\cos lx\right\}.$$

This is similar in form to Giovanelli's solution for a medium which is uniform with depth.

V. The Directional Intensity at the Surface

If θ is the angle between the emergent radiation and the z-axis, and φ is the azimuthal angle referred to the x-axis, the directional intensity $I(x,z,\theta,\varphi)$ at the surface ($z=0$) is found by integrating $J(x,z)$ along a path cutting the surface at x, in a direction given by θ and φ. The position coordinate s of any point on the path measures the distance from the surface to that point along the path, and is taken as positive below the surface. This yields the relations

$$\left.\begin{aligned} z&=-s\cos\theta,\\ \Delta x&=-s\sin\theta\cos\varphi.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{14})$$

In a volume of the path ds at s, the radiation scattered or emitted into unit solid angle in the direction of the path is $(1/4\pi)\varkappa(s)J(s)\mathrm{d}s$, where from (**3**) and (**14**)

$$\varkappa(s)=\varkappa_0\mathrm{e}^{\nu s\cos\theta}[1+\alpha\cos l(x-cs)], \quad \ldots\ldots\ldots \quad (\mathbf{15})$$

c representing $\sin\theta\cos\varphi$. Let $\Phi_s(t)$ be the fraction of the radiation which has suffered no further scattering or absorption at a distance t from the surface. Then in a further distance $-\mathrm{d}t$,

$$\mathrm{d}\Phi_s(t)=-\Phi_s(t)\varkappa(t)(-\mathrm{d}t).$$

After integrating this equation the fraction remaining at the surface, $\Phi_s(0)$, is found to be

$$\Phi_s(0)=\exp\left(\frac{\varkappa_0}{\nu\cos\theta}[\{1+\alpha\cos\psi\cos(lx+\psi)\}-e^{\nu s\cos\theta}\{1+\alpha\cos\psi\cos(lx-lcs+\psi)\}]\right), \quad \ldots\ldots\ldots\ldots (\mathbf{16})$$

where $\tan\psi=(l/\nu)\tan\theta$. Thus the directional intensity at the surface is

$$I(x,0,\theta,\varphi)=\frac{1}{4\pi}\int_0^\infty \Phi_s(0)\varkappa(s)J(s)\mathrm{d}s, \quad \ldots\ldots\ldots (\mathbf{17})$$

where

$$J(s)=A\left\{\tfrac{2}{3}+\frac{\varkappa_0}{\nu}(e^{\nu s\cos\theta}-1)-\frac{\alpha}{3/2+p/\varkappa_0}e^{-ps\cos\theta}\cos l(x-cs)\right\} \ldots (\mathbf{18})$$

from (**14**) and (**11**).

It will be noted from (**17**) and (**11**) that the profiles of $I(x,0,\theta,\varphi)$ and $J(x,z)$ depend only on the values chosen for the ratios $\varkappa_0/\nu$, ν/l and on the value of α. Although the structure size of each profile depends on the actual values of l and ν, every position coordinate in (**17**) and (**11**) is multiplied by l or ν (or p, where $p=\frac{1}{2}\{-\nu+\sqrt{(\nu^2+4l^2)}\}$). Thus by specifying only the ratios ν/l and $\varkappa_0/l$, and evaluating I at various values of lx, (**17**) may be applied to media of any structure size merely by using the appropriate value of l.

VI. Calculations and Discussion

In the special case of the normal intensity at the surface, $\theta=\varphi=0$ and (**17**) becomes

$$I(x,0,0,0)=\frac{A}{4\pi}\left(\tfrac{2}{3}+\frac{1}{1+\alpha\cos lx}\right)-\frac{A}{4\pi}\cdot\frac{\alpha\cos lx}{3/2+p/\varkappa_0}\int_0^\infty e^{-\tau}\left(\frac{\nu}{\varkappa'}\tau+1\right)^{-p/\nu}\mathrm{d}\tau, \quad \ldots\ldots\ldots\ldots (\mathbf{19})$$

where $\varkappa'=\varkappa_0(1+\alpha\cos lx)$ and $\tau(s)=(\varkappa'/\nu)(e^{\nu s}-1)$. The integral in (**19**) may be related to the incomplete gamma function. In three cases approximate forms are obtained.

(i) If $\nu/l\gg1$, $-p/\nu\simeq-(l/\nu)^2\simeq0$, and

$$I\simeq\frac{5}{12}\,\frac{A}{\pi}(1-\alpha\cos lx). \quad \ldots\ldots\ldots\ldots\ldots (\mathbf{20})$$

(ii) If $\nu/l\ll1$, $-p/\nu\simeq-l/\nu\simeq-\infty$, and

$$I\simeq\frac{5}{12}\,\frac{A}{\pi}\left(1-\frac{3}{5}\alpha\cos lx\right). \quad \ldots\ldots\ldots\ldots (\mathbf{21})$$

(iii) If $\varkappa_0/l\ll1$, $\alpha/(3/2+p/\varkappa_0)\ll1$, and

$$I\simeq\frac{5}{12}\,\frac{A}{\pi}\left(1-\frac{3}{5}\alpha\cos lx\right). \quad \ldots\ldots\ldots\ldots (\mathbf{22})$$

It is useful to denote the ratio of the fractional variation in the surface intensity, $\Delta I/I_0$, to the fractional x-variation in attenuation coefficient, α, by the symbol β. Thus β describes the relative contrast at the surface. In case (i) above, $\beta=1\cdot0$, while in (ii) and (iii), $\beta=0\cdot6$. The case (iii) ($\varkappa_0/l\ll1$) is important in astrophysics and it may seem surprising at first that β has the value $0\cdot6$ for all non-zero values of ν/l (the case $\nu/l\ll1$ and $\varkappa_0/l\ll1$ is trivial). However, it will be shown that, when the radiation emerges at an angle to the normal, the relative contrast decreases as ν/l decreases.

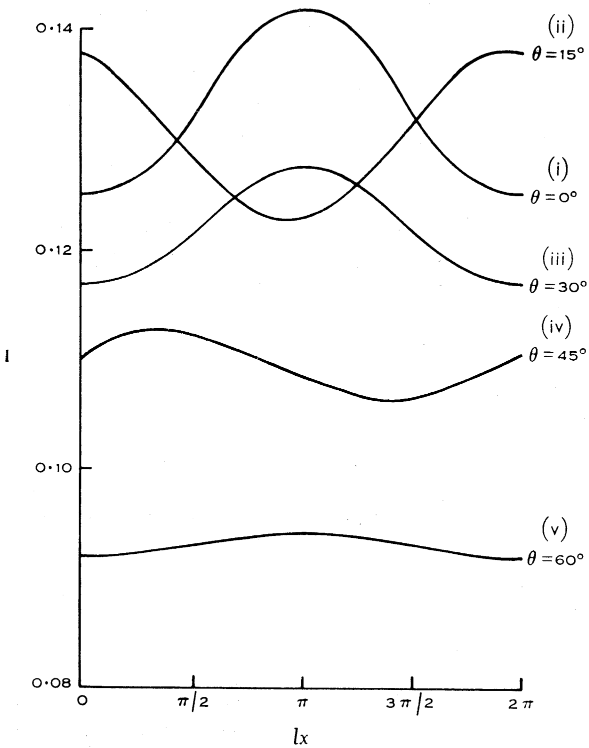

Fig. 1.—The directional intensity at the surface, $I(x,0,\theta,\varphi)$, against lx for $\varphi=0$ and (i) $\theta=0$, (ii) $\theta=15°$, (iii) $\theta=30°$, (iv) $\theta=45°$, (v) $\theta=60°$. In all cases $\varkappa_0/l=10^{-5}$, $\nu/l=1$, and $\alpha=0\cdot1$.

For all cases involving non-zero values of θ and φ, $I(x,0,\theta,\varphi)$ must be evaluated numerically for various values of the coordinates x, θ, and φ, and of the parameters of the medium $\varkappa_0/l$, ν/l, and α. In Figures 1, 2, and 3, the results are shown for a typical case in which $\varkappa_0/l=10^{-5}$, $\nu/l=1$, and $\alpha=0\cdot1$. The directional intensity $I(x,0,\theta,\varphi)$ is plotted against lx for various values of θ and φ. The curves obtained are obviously periodic, with wavelength $2\pi/l$. The characteristic features, the average intensity I_0, the relative contrast β, and the difference in phase between $I(\theta)$ and $I(0)$, ξ, are recorded in Table 1 for the curves of Figure 1. Also tabulated is the " darkening ", $I_0(\theta)/I_0(0)$, which is compared with Eddington's approximate relation $I_0(\theta)/I_0(0)=(2+3\cos\theta)/5$.

The contrast decreases with θ, as expected. The phase difference between $I(\theta)$ and $I(0)$ arises from the location of the source below the surface; it may easily be shown to be consistent with a simple model in which the sources of radia-

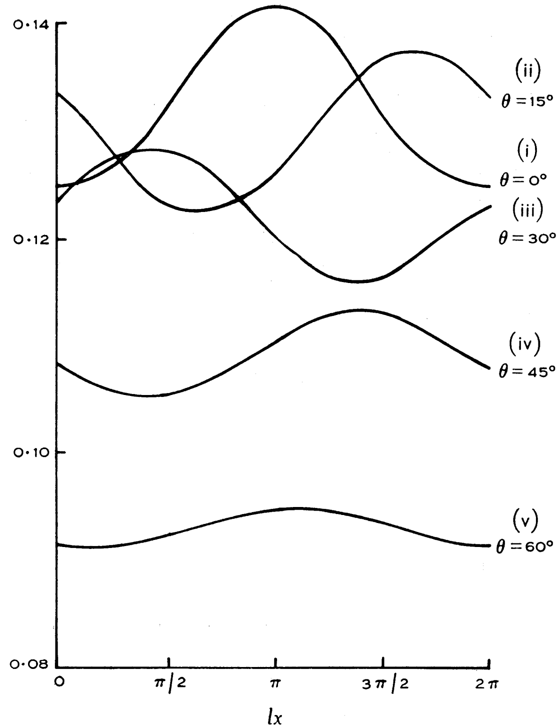

Fig. 2.—As for Figure 1, but $\varphi=45°$.

tion are distributed over a plane parallel to the surface at about unit optical depth, varying sinusoidally in one direction. As $\varphi \to 90°$, $\xi \to 0$, as expected. The value of $\varkappa_0/l$ chosen in this case satisfies the condition for a continuously tapering

TABLE 1

FEATURES OF FIGURE 1

θ	0°	15°	30°	45°	60°
$I_0(\theta)$	0·1333	0·1305	0·1222	0·1095	0·0930
β	0·615	0·590	0·442	0·292	0·124
$\xi/2\pi$	0	0·45	1·01	1·71	2·50
$I_0(\theta)/I_0(0)$	1	0·979	0·921	0·821	0·697
$(2+3\cos\theta)/5$	1	0·980	0·920	0·824	0·700

medium. As expected, almost identical curves are found for $\varkappa_0/l=10^{-4}$ and 10^{-6}.

To facilitate the extrapolation of the curves of Figures 1, 2, and 3 to other values of ν/l, many other cases have been evaluated. In Figure 4, β is plotted against γ, where $\gamma = \log_{10} \nu/l$, for $\varkappa_0/l = 10^{-5}$, $\alpha = 0 \cdot 1$, and various values of θ. For all but large values of θ, the relative contrast β is approximately $0 \cdot 6$ when

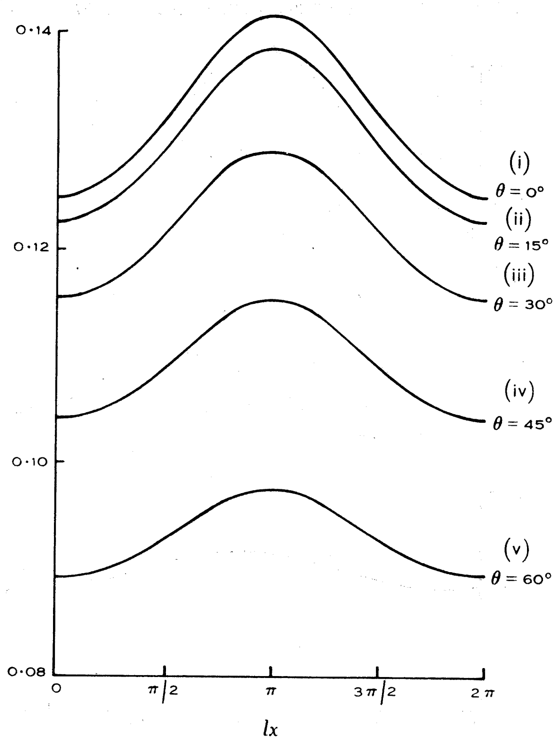

Fig. 3.—As for Figure 1, but $\varphi = 90°$.

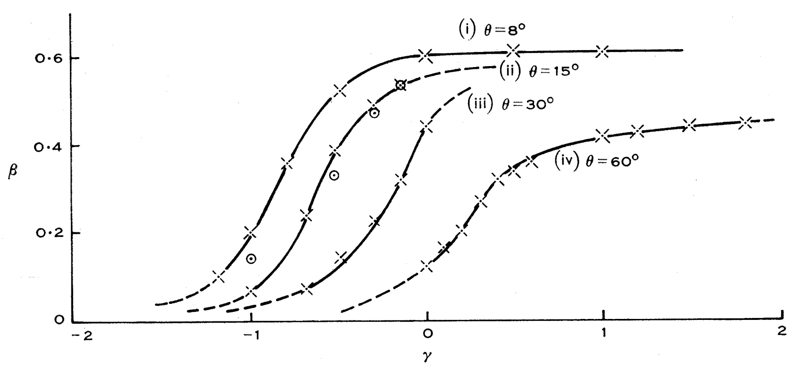

Fig. 4.—The graphs of β against γ, where $\gamma = \log_{10} \nu/l$ for $\varphi = 0°$ and (i) $\theta = 8°$, (ii) $\theta = 15°$, (iii) $\theta = 30°$, (iv) $\theta = 60°$. In these cases, $\alpha = 0 \cdot 1$ and $\varkappa_0/l = 10^{-5}$. The points plotted thus, $\odot$, are values of β calculated for $\alpha = 0 \cdot 2$, $\theta = 15°$, and $\varkappa_0/l = 10^{-5}$.

ν/l is large (i.e. small scale heights and coarse structures), but decreases to zero as ν/l decreases; the greater the value of θ, the greater the rate of decrease. Points have also been plotted for $\theta = 15°$ and $\alpha = 0 \cdot 2$. The values of β undergo little change provided $\beta > 0 \cdot 2$. Thus the amplitude of the variations in intensity is directly proportional to the fractional variation in attenuation coefficient, α, provided α is small.

VII. Conclusions

The emission of radiation from the surface ($z=0$) of a semi-infinite medium, below which the attenuation coefficient takes the form (**3**), has been examined in the conservative case (radiative equilibrium) assuming constant flux across planes parallel to the surface.

The total intensity is given by (**12**) and the variation in total intensity across the surface by (**13**). These expressions reduce to the form given by Giovanelli (1959) in the special case of a medium uniform with depth.

The behaviour of the directional intensity across the surface may be described by the relative contrast β ($\beta=(1/\alpha)(\Delta I/I_0)$). The extent to which the increase in attenuation coefficient with depth affects β may be summarized thus:

(i) For radiation normal to the surface, $0 \cdot 6 < \beta < 1$, depending on the ratios $\varkappa_0/l$ and ν/l. In the important case of a medium which tapers off continuously ($\varkappa_0/l \ll 1$), $\beta = 0 \cdot 6$ for all non-zero values of ν/l.

(ii) The behaviour of radiation emerging obliquely is shown for a special case in Figures 1–3 and Table 1. Figure 4 shows that for a medium tapering off continuously, $0 < \beta < 0 \cdot 6$. For sufficiently coarse structures and small scale heights ($\nu/l \gg 1$), $\beta = 0 \cdot 6$ for all but large angles of emergence, while in all cases, the relative contrast falls off as the angle of emergence increases.

VIII. Acknowledgments

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