ON THE RELATION BETWEEN LUMINOSITY DISTANCE AND DOPPLER SHIFT IN RELATIVISTIC COSMOLOGY*

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Introduction
An approximate relation between luminosity distance $D$ and Doppler shift $\delta$ in cosmology is usually obtained by a succession of complicated expansions in series.

In the present paper it is shown that an expression for $D$, even to a higher approximation in $\delta$, can be obtained in a much simpler way.

The relation between luminosity distance $D$ and red-shift $\delta$ commonly used in relativistic cosmology is (McVittie 1956)+

$$D = \frac{c\delta}{h_1} \left(1 + \frac{h_1^2 + h_2^2}{2h_1^2} \delta \right), \quad \text{(1)}$$

where

$$h_1 = \frac{\dot{R}(t_0)}{R(t_0)} = \frac{\dot{R}_0}{R_0}, \quad \text{i.e. the Hubble constant}, \quad \text{(2)}$$

and

$$h_2 = \frac{\ddot{R}_0}{R_0}, \quad \text{(3)}$$

$R$ being a function of $t$.

The exact expression for $D$ is

$$D = \frac{R_0^2}{R} \frac{r}{1 + \alpha r^2/4}, \quad \text{(4)}$$

where $r$ is a function of $R$ given by the null-geodesic equation

$$c \int_{\dot{t}}^{t_0} \frac{dt}{\ddot{R}(t)} = \int_{0}^{r} \frac{dr}{1 + \alpha r^2/4}, \quad \text{(5)}$$

The constant $\alpha = +1$ for an elliptic space,

$=0$ for a flat space,

$=-1$ for a hyperbolic space.

The left-hand side of (5) could be written

$$c \int_{\dot{t}}^{t_0} \frac{dt}{\ddot{R}(t)} = c \int_{R_0}^{R} \frac{dR}{R \dot{R}},$$

and therefore

$$\int_{0}^{r} \frac{dr}{1 + \alpha r^2/4} = \psi, \quad \text{(6)}$$

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where
\[ \psi = c \int_{R}^{R_0} \frac{dR}{R \dot{R} \dot{R}'} \ldots \ldots \ldots \ldots \ldots (7) \]

If \( x = +1 \) it follows from (6) that
\[ r = 2 \tan \frac{1}{2} \psi, \]
and substituting this in (4) we have the following expression for the luminosity distance
\[ D = (R_0^2/R) \sin \psi. \ldots \ldots \ldots \ldots \ldots (8) \]

It is also easily seen that:
if \( x = 0 \),
\[ D = (R_0^2/R) \psi, \ldots \ldots \ldots \ldots \ldots (9) \]
if \( x = -1 \),
\[ D = (R_0^2/R) \sinh \psi. \ldots \ldots \ldots \ldots \ldots (10) \]

**Elliptic and Hyperbolic Spaces**

For these spaces
\[ D = (R_0^2/R) F(R), \ldots \ldots \ldots \ldots \ldots (11) \]
where
\[ F(R) = \frac{\sin \left( c \int_{R}^{R_0} \frac{dR}{R \dot{R} \dot{R}'} \right)}{\sinh \left( c \int_{R}^{R_0} \frac{dR}{R \dot{R} \dot{R}'} \right)} \ldots \ldots \ldots \ldots \ldots (12) \]

the \( \sin \) corresponding to the elliptic space and \( \sinh \) to the hyperbolic one.

Using a Taylor expansion for \( F \), we have
\[ F(R) = F(R_0) + \left( \frac{dF}{dR} \right)_{R_0} (R - R_0) + \left( \frac{d^2F}{dR^2} \right)_{R_0} \frac{(R - R_0)^2}{2} + \left( \frac{d^3F}{dR^3} \right)_{R_0} \frac{(R - R_0)^3}{6} + \ldots, \ldots \ldots \ldots \ldots (13) \]
where, firstly, \( F(R_0) = 0 \), secondly,
\[ \frac{dF}{dR} = -c \cos \left( c \int_{R}^{R_0} \frac{dR}{R \dot{R} \dot{R}'} \right) \frac{1}{R \dot{R} \dot{R}'} \]
that is,
\[ \left( \frac{dF}{dR} \right)_{R_0} = \frac{-c}{R_0 \dot{R} \dot{R}'}, \ldots \ldots \ldots \ldots \ldots (14) \]
thirdly,
\[ \frac{d^2F}{dR^2} = \pm c^2 \sinh \left( c \int_{R}^{R_0} \frac{dR}{R \dot{R} \dot{R}'} \right) \frac{-1}{R^2 \dot{R}^2} - c \cos \left( c \int_{R}^{R_0} \frac{dR}{R \dot{R} \dot{R}'} \right) \frac{d}{d\dot{R} \dot{R}'} \left( \frac{1}{R \dot{R} \dot{R}'} \right). \ldots \ldots \ldots (15) \]
But
\[ \frac{d}{d\dot{R} \dot{R}' \dot{R}'} \left( \frac{1}{R \dot{R} \dot{R}'} \right) = \frac{d}{dt} \left( \frac{1}{R \dot{R} \dot{R}'} \right) \frac{dt}{d\dot{R} \dot{R}'} = - \frac{\dot{R}^2 + R \ddot{R}}{R^2 \ddot{R}^3}. \ldots \ldots \ldots (16) \]
Substituting this in (15),
\[ \frac{d^2F}{dR^2} = \pm c^2 \sinh \left( c \int_{R}^{R_0} \frac{dR}{R \dot{R} \dot{R}'} \right) \frac{1}{R^2 \dot{R}^2} + c \cos \left( c \int_{R}^{R_0} \frac{dR}{R \dot{R} \dot{R}'} \right) \frac{\dot{R}^2 + R \ddot{R}}{R^2 \ddot{R}^3}, \ldots \ldots \ldots (17) \]
and
\[
\left( \frac{d^2 F}{dR^2} \right)_{R_o} = c \frac{\dot{R}_0^2 + R_0 \ddot{R}_0}{R_0^3 \dot{R}_0^3}.
\] ............................................ (18)

Fourthly,
\[
\frac{d^3 F}{dR^3} = \mp e^3 \cos \left( e^3 \int R \frac{dR}{R \dot{R}} \right) \frac{-1}{R \dot{R}}^3
\]
\[
\mp e^2 \sin \left( e^3 \int R \frac{dR}{R \dot{R}} \right) (-2R^{-3} \dot{R}^{-1} - 2R^{-2} \ddot{R}^{-3} \dot{R}) \frac{1}{R}
\]
\[
\pm e^2 \sin \left( e^3 \int R \frac{dR}{R \dot{R}} \right) (R^2 + R \dot{R}^2)^{1/3}
\]
\[
\pm e^2 \cos \left( e^3 \int R \frac{dR}{R \dot{R}} \right) \left( - \frac{2R^3}{R \dot{R}^2} + \frac{\ddot{R}}{R \dot{R}^3} + \frac{3\ddot{R}^2}{R \dot{R}^4} \right) \frac{1}{R^3}
\]
\[
\text{and}
\]
\[
\left( \frac{d^3 F}{dR^3} \right)_{R_o} = \pm e^3 \frac{c^3}{R_0^3 \dot{R}_0^3} + e \left\{ \frac{c}{R_0} \left[ - \frac{2}{R_0} - \frac{2 \ddot{R}_0}{R_0^2 R_0^2} + \frac{\dddot{R}_0}{R_0^3 R_0} - \frac{3 \ddot{R}_0^2}{R_0^3 R_0^4} \right] \right\} \frac{(R - R_0)^3}{6}.
\] ............................................ (19)

Substituting (14), (18), and (19) in (13) we have
\[
F(R) = - \frac{c}{R_0 \dot{R}_0} (R - R_0) + e \left( \frac{\dot{R}_0^2 + R_0 \ddot{R}_0}{R_0^3 \dot{R}_0^3} \right) \frac{(R - R_0)^2}{2}
\]
\[
+ \left\{ \pm e^3 \frac{c}{R_0^3} + e \left[ \frac{c}{R_0} \left[ - \frac{2}{R_0} - \frac{2 \ddot{R}_0}{R_0^2 R_0^2} + \frac{3 \ddot{R}_0^2}{R_0^3 R_0^4} \right] \right] \frac{(R - R_0)^3}{6} \right\} \frac{(R - R_0)^3}{6}.
\] ............................................ (20)

Substituting this in (11) and remembering that
\[
1 - R_0/R = - \delta,
\]
and
\[
R = R_0/(1+\delta) = R_0(1 - \delta + \delta^2 - \delta^3 . . .),
\]
we have
\[
D = - \frac{cR_0^2}{R} \left( - \delta \right) + \left( \frac{R_0^2 + R_0 \ddot{R}_0}{R_0^3 \dot{R}_0} \right) R_0 \left( 1 - \delta + \delta^2 - \delta^3 . . . \right) (\delta^2)
\]
\[
+ \left\{ \pm \frac{e^3}{R_0} + e \left[ \frac{2}{R_0} - \frac{2 \ddot{R}_0}{R_0^2 R_0^2} - \frac{3 \ddot{R}_0^2}{R_0^3 R_0^4} \right] \frac{(\delta^3)}{6} \right\} \frac{(1 - \delta + \delta^2 . . .)^2}{R_0^2}
\]
or, after a few reductions,
\[
D = c \frac{\delta}{k_1} + \frac{k_1^2 + k_2^2}{2h_1^2}
\]
\[
= \left\{ \frac{c}{6} \frac{k_1^2 + h_2^2}{h_1^2} + \frac{c}{6k_1} \left[ \frac{h_3^2}{h_1^2} - \frac{3}{h_1^2} \frac{h_4^2}{h_1^4} \right] \right\} \pm \frac{c^2}{6k_1^2h_1^4} \delta^2.
\] ............................................ (21)
This is the luminosity distance–red-shift relation, correct to the third order in \( \delta \). The plus sign in the last term is used for elliptic spaces and the minus sign for hyperbolic ones.

**Flat Spaces**

In this case the luminosity distance is given by

\[
D = \frac{R_0^2}{c} \int_R^{R_0} \frac{dR}{cR_0 R_1} \quad \text{i.e.} \quad F = \frac{c}{R_0} \int_R^{R_0} \frac{dR}{R_0 R_1}.
\]  

Then, firstly,

\[
\left( \frac{dF}{dR} \right)_{R_0} = -\frac{c}{R_0 R_1},
\]

secondly,

\[
\left( \frac{d^2F}{dR^2} \right)_{R_0} = \frac{\tilde{R}_0^2 + \tilde{R}_0 \tilde{R}_1}{\tilde{R}_0^3 \tilde{R}_1^3},
\]

thirdly,

\[
\left( \frac{d^3F}{dR^3} \right)_{R_0} = \frac{c}{\tilde{R}_0} \left( -\frac{2 \tilde{R}_0}{\tilde{R}_1^2} - \frac{2 \tilde{R}_1}{\tilde{R}_0^2 \tilde{R}_1} - \frac{3 \tilde{R}_1^2}{\tilde{R}_0^3 \tilde{R}_1^3} + \frac{2 \tilde{R}_0}{\tilde{R}_0^3 \tilde{R}_1^3} \right).
\]

It is evident, by comparison of (14), (18), (19) and (23), (24), (25) that the luminosity distance–red-shift relation is

\[
D = \frac{c}{h_1} \delta \left( 1 + \frac{h_1^2 + h_2^2}{2h_1^2} - \frac{1}{6} \left( \frac{h_1^2 + h_2^2}{h_1^2} + \left( \frac{h_3^2 + h_2^2}{h_1^2} - \frac{3 h_2^2}{h_1^2} \right) + \frac{\alpha c^2}{R_0^2 h_1^3} \right) \right),
\]

where \( \alpha = +1 \) for an elliptic space, 
\( \alpha = 0 \) for a flat space, 
\( \alpha = -1 \) for a hyperbolic space, 

and

\[
h_3 = \frac{\tilde{R}_0}{R_0}.
\]

The first two terms in (26) give

\[
D = \frac{c}{h_1} \delta + \frac{h_1^2 + h_2^2 \delta^2}{2h_1^3},
\]

which is the formula commonly used in relativistic cosmology and which does not distinguish between the three types of spaces.