n–DIFFUSION

By J. R. Philip*

[*Manuscript received September* 15, 1960]

Summary

Transfer processes in which an entity is transferred down a gradient of a concentration-like quantity satisfy the relation $\mathbf{q}=-\mathbf{A}.\mathbf{B}$, with $\mathbf{q}$ the flux density, $\mathbf{A}$ dependent on time, concentration and position, and $\mathbf{B}$ a function of the concentration gradient, $\nabla\theta$. In ordinary diffusion $\mathbf{B}=\nabla\theta$. This paper considers the more general transfer process, designated n-diffusion, for which $\mathbf{B}=|\nabla\theta|^{n-1}\nabla\theta$ $(n>0)$.

The paper deals with the simplest unsteady one-dimensional problem of n-diffusion (with $\mathbf{A}$ constant) into a semi-infinite region. The results are simply extended to the related problem in the (doubly) infinite region.

Solutions are found in terms of the incomplete beta-function, though for certain values of n solutions are expressible in terms of elementary functions. Infinite " tails " (analogous to that in 1-diffusion) occur for $0<n<1$, whilst the concentration profiles are finite for $n>1$. Distance of penetration into the region and cumulated flux vary as $(\text{time})^{1/(n+1)}$.

The present paper is intended as an introduction to later work on concentration- and space-dependent forms of n-diffusion which are immediately relevant to physical problems of interest.

I. Transfer Processes

Many branches of mathematical physics are concerned with transfer processes, in the sense that some entity is transferred down a gradient of a concentration-like quantity. For processes of this nature we have the general relation

$$\mathbf{q}=-\mathbf{A}.\mathbf{B}. \tag{1.1}$$

Here $\mathbf{q}$ is the vector flux density ; $\mathbf{A}$ is a function of time, concentration, and position (and is, in the general case, a tensor of rank 2) ; and $\mathbf{B}$ is a vector function of the concentration gradient.

Assuming conservation of the transferred entity and differentiating, we obtain

$$\partial\theta/\partial t=\nabla.(\mathbf{A}.\mathbf{B}). \tag{1.2}$$

Here θ is the concentration and t is the time.

(*a*) *Diffusion*

In this paper we use the word " diffusion " in a mathematical sense, with no implications as to the physics of the transfer process. For diffusion, as the term is generally employed,

$$\mathbf{B}=\nabla\theta. \tag{1.3}$$

We may categorize the various types of diffusion by the form of $\mathbf{A}$. $\mathbf{A}$ is constant for linear diffusion, whereas in time-, concentration-, and space-dependent

* Division of Plant Industry, C.S.I.R.O., Canberra.

diffusion **A** is a function of the appropriate variable. In more complicated types of diffusion **A** may vary with two, or with all three, of time, concentration, and space.

(*b*) n-*Diffusion*

In this paper we consider a generalized form of diffusion for which

$$\mathbf{B}=|\nabla\theta|^{n-1}\nabla\theta. \tag{1.4}^*$$

The various types of n-diffusion, as we shall call it, may be categorized according to the form of **A** in the same way as for 1-diffusion. In this paper we confine our attention to the case **A**=constant.† This study serves as the introduction to further work dealing with concentration-dependent n-diffusion and certain forms of space-dependent n-diffusion. These more complicated types of n-diffusion are immediately relevant to physical problems of interest, including unsteady vertical heat transfer from a horizontal surface by (turbulent) free convection (Priestley 1954), and unsteady turbulent flow of a liquid with a free surface over a plane. The latter does not seem to have been yet formulated in the literature as a problem in n-diffusion; Philip (1956) gives the parallel development for non-turbulent flow.

It is of some interest that n-diffusion has the most general form of **B** for which solutions of the simplest unsteady problems may be found by similarity methods. We apply these methods here to the case **A**=constant; but, as we propose to show in later work, they are effective also in concentration-dependent n-diffusion [$\mathbf{A}=f(\theta)$] and in space [x]-dependent n-diffusion with $\mathbf{A}\propto x^c$ ($c<n$). The methods apply also to the more general class of n-diffusion with $\mathbf{A}=f(\theta).x^c$.

II. n-Diffusion in a Semi-infinite Region

The one-dimensional form of the n-diffusion equation is

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial x}\left(A\left|\frac{\partial\theta}{\partial x}\right|^{n-1}\frac{\partial\theta}{\partial x}\right), \tag{2.1}$$

with A and n positive constants. x is the spatial dimension. We limit ourselves to the physically interesting case of positive n because the cases $n\leqslant 0$ involve irrelevant complications.

We consider the most elementary (and perhaps most important) unsteady transfer problem, namely, transfer in the semi-infinite region $x\geqslant 0$, subject to the conditions:

$$\left.\begin{aligned}\theta=\theta_0,\ \ t=0,\ \ x>0\,;\\ \theta=\theta_1,\ \ x=0,\ \ t\geqslant 0.\end{aligned}\right\} \tag{2.2}$$

* Of course, we could regard n-diffusion as characterized also by $\mathbf{B}=\nabla\theta$, but with **A** dependent on $\nabla\theta$ (as well as, in general, t, θ, and space); but the present treatment is perhaps more convenient.

† We shall use the term " n-diffusion " to describe this simple case as well as to denote the whole class of phenomena defined by (1.4). Unfortunately, we can hardly apply the adjective " linear " to the simple case. We shall avoid ambiguity by specifically labelling all types of n-diffusion, other than that with **A** constant, as " concentration-dependent ", " space-dependent ", etc.

It is a simple matter to modify the methods and results of this paper so that they apply to the related problem in the infinite region with the governing conditions :

$$\left.\begin{array}{l} t=0,\ x>0,\ \theta=\theta_0;\ \ x<0,\ \theta=\theta_1; \\ t\geqslant 0,\ \displaystyle\int_{\theta_0}^{\theta_1} x\mathrm{d}\theta=0. \end{array}\right\} \tag{2.3}$$

The similarity substitutions

$$\vartheta=(\theta-\theta_0)/(\theta_1-\theta_0);\quad \varphi=x[|\theta_1-\theta_0|^{n-1}At]^{-1/(n+1)} \tag{2.4}$$

form the generalization in n-diffusion of the "Boltzmann transformation" (Boltzmann 1894) in 1-diffusion. Applying (2.4) to (2.1), (2.2), we obtain the ordinary equation

$$\frac{\varphi}{n+1}\frac{\mathrm{d}\vartheta}{\mathrm{d}\varphi}=\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(-\frac{\mathrm{d}\vartheta}{\mathrm{d}\varphi}\right)^n, \tag{2.5}$$

subject to the conditions

$$\varphi=0,\ \vartheta=1;\quad \varphi\to\infty,\ \vartheta\to 0. \tag{2.6}$$

When n is such that $\vartheta>0$ for all finite φ the second of conditions (2.6) immediately implies

$$\vartheta\to 0,\ \mathrm{d}\vartheta/\mathrm{d}\varphi\to 0. \tag{2.7}$$

But when n is such that $\vartheta=0$ at some finite φ (in fact, when $n>1$), we must consider further the behaviour of $\mathrm{d}\vartheta/\mathrm{d}\varphi$ near $\vartheta=0$. We therefore introduce the quantity $q(\vartheta)$, the flux density at the point with reduced concentration ϑ, and observe that

$$|q(\vartheta)|=A\left|\frac{\partial\theta}{\partial x}\right|^n=(A\,|\theta_1-\theta_0|^{2n}t^{-n})^{1/(n+1)}\left|\frac{\mathrm{d}\vartheta}{\mathrm{d}\varphi}\right|^n. \tag{2.8}$$

Now, since $q(0)=0$ and $A>0$ and $n>0$, we have, for $t>0$,

$$\vartheta=0,\ \mathrm{d}\vartheta/\mathrm{d}\varphi=0. \tag{2.9}$$

This result holds whether the least value of φ at which $\vartheta=0$, φ_0, is finite or not.

The substitution

$$\Psi=-\mathrm{d}\vartheta/\mathrm{d}\varphi \tag{2.10}$$

reduces (2.5) to

$$\varphi=-n(n+1)\Psi^{n-2}\mathrm{d}\Psi/\mathrm{d}\varphi. \tag{2.11}$$

This has the integrals

$$n=1;\quad \varphi^2=4\log[\Psi_1/\Psi]. \tag{2.12}$$

$$n\neq 1;\quad \varphi^2=\frac{2n(n+1)}{n-1}[\Psi_1^{n-1}-\Psi^{n-1}]. \tag{2.13}$$

Ψ_1 denotes the value of Ψ at $\varphi=0$ (i.e. at $\vartheta=1$).

A further integration of (2.12) and use of (2.6) yields

$$\vartheta = \operatorname{erfc} \varphi/2, \tag{2.14}$$

the well-known solution for $n=1$. In this case $\Psi_1 = \pi^{-\frac{1}{2}}$.

The case $n \neq 1$ needs further discussion. Referring to (2.11), we observe that, when φ and Ψ are both non-negative, $d\Psi/d\varphi$ is non-positive.* That is, Ψ is a non-negative monotonic function of φ, decreasing from its largest value, Ψ_1 at $\varphi=0$. It now follows from (2.13) that, for the case $0<n<1$, $\varphi \to \infty$ as $\Psi \to 0$ [i.e. as $\vartheta \to 0$: cf. (2.9)]; and that, for the case $n>1$, φ approaches the finite positive value $[\{2n(n+1)/(n-1)\}\Psi_1^{n-1}]^{\frac{1}{2}}$ as $\Psi \to 0$ (i.e. as $\vartheta \to 0$).

Integrating (2.13), and using the first of conditions (2.6), we obtain

$$n \neq 1\,; \quad \vartheta = 1 - \Psi_1 \int_0^{\varphi} \left(1 + \frac{(1-n)\varphi'^2}{2n(n+1)\Psi_1^{n-1}}\right)^{1/(n-1)} d\varphi'. \tag{2.15}$$

Ψ_1 can now be evaluated.

(a) *The Case* $0<n<1$

In this case the substitutions

$$\alpha = \tan^{-1}\left(\frac{(1-n)\Psi_1^{1-n}}{2n(1+n)}\right)^{\frac{1}{2}} \varphi, \tag{2.16}$$

$$m = 2n/(1-n), \tag{2.17}$$

reduce (2.15) to

$$\vartheta = 1 - \left(\frac{2n(1+n)}{(1-n)\Psi_1^{1-n}}\right)^{\frac{1}{2}} \Psi_1 \int_0^{\alpha} \cos^m \alpha' d\alpha'. \tag{2.18}$$

We introduce the identity

$$\int_0^{\frac{1}{2}\pi} \cos^m \alpha' d\alpha' = \tfrac{1}{2} B\left(\frac{m+1}{2}, \tfrac{1}{2}\right), \tag{2.19}$$

where $B(p, q)$ is the beta-function $\int_0^1 x'^{p-1}(1-x')^{q-1}dx'$. $B(p, q)$ is expressible in terms of gamma-functions, but it is more convenient here to retain beta-function forms.

Then it follows from (2.6), (2.18), and (2.19) that

$$\Psi_1 = \left(\frac{1-n}{2n(1+n)}\right)^{1/(1+n)} \cdot \left(\frac{2}{B\left(\frac{1+n}{2-2n}, \frac{1}{2}\right)}\right)^{2/(1+n)}. \tag{2.20}$$

* It follows at once that the family of $\varphi(\vartheta)$ curves do not possess points of inflexion.

(2.18) then reduces to the forms

$$\begin{aligned}\vartheta&=1-\int_0^{\alpha}\cos^m\alpha'\,\mathrm{d}\alpha'\bigg/\int_0^{\frac{1}{2}\pi}\cos^m\alpha'\,\mathrm{d}\alpha'\\&=1-\frac{2}{\mathrm{B}\left(\dfrac{m+1}{2},\frac{1}{2}\right)}\int_0^{\alpha}\cos^m\alpha'\,\mathrm{d}\alpha'\\&=\int_{\alpha}^{\frac{1}{2}\pi}\cos^m\alpha'\,\mathrm{d}\alpha'\bigg/\int_0^{\frac{1}{2}\pi}\cos^m\alpha'\,\mathrm{d}\alpha'\\&=\frac{2}{\mathrm{B}\left(\dfrac{m+1}{2},\frac{1}{2}\right)}\int_{\alpha}^{\frac{1}{2}\pi}\cos^m\alpha'\,\mathrm{d}\alpha'.\end{aligned}\tag{2.21}$$

Elimination of Ψ_1 from (2.16) yields

$$\alpha=\tan^{-1}\left[\frac{1-n}{2n(1+n)}\left\{\frac{2}{\mathrm{B}\left(\dfrac{1+n}{2-2n},\frac{1}{2}\right)}\right\}^{1-n}\right]^{1/(1+n)}\varphi=\tan^{-1}k_a\varphi.\tag{2.22}$$

We observe that, in this case, as φ goes from $0\to\infty$, α goes from $0\to\frac{1}{2}\pi$ and ϑ goes from $1\to 0$.

(*b*) *The Case* $n>1$

In this case the substitutions

$$\alpha=\sin^{-1}\left(\frac{(n-1)\Psi_1^{1-n}}{2n(n-1)}\right)^{\frac{1}{2}}\varphi,\tag{2.23}$$

$$m=(n+1)/(n-1),\tag{2.24}$$

reduce (2.15) once again to the form (2.18), but with the factor $(1-n)$ in the denominator of the expression to power $\frac{1}{2}$ replaced by $(n-1)$.

We may now use the value of φ_0 for $n>1$ to evaluate Ψ_1 for this case. The result is

$$\Psi_1=\left(\frac{n-1}{2n(n+1)}\right)^{1/(n+1)}\cdot\left(\frac{2}{\mathrm{B}\left(\dfrac{n}{n-1},\frac{1}{2}\right)}\right)^{2/(n+1)}.\tag{2.25}$$

The forms (2.21) hold here also, and the expression for α with Ψ_1 eliminated is

$$\alpha=\sin^{-1}\left[\frac{n-1}{2n(n+1)}\left\{\frac{2}{\mathrm{B}\left(\dfrac{n}{n-1},\frac{1}{2}\right)}\right\}^{1-n}\right]^{1/(n+1)}\varphi=\sin^{-1}k_b\varphi.\tag{2.26}$$

In this case, as φ goes from $0\to 1/k_b=\varphi_0$, α goes from $0\to\frac{1}{2}\pi$ and ϑ goes from $1\to 0$.

III. Solution in Terms of Elementary Functions

$\int_0^\alpha \cos^m \alpha' d\alpha'$ is expressible in terms of trigonometrical functions when m is integral; so that solutions for the cases

$$n=m/(2+m),\ m \text{ a positive integer}; \tag{3.1}$$

$$n=(m+1)/(m-1),\ m \text{ an integer greater than } 1; \tag{3.2}$$

can be found in terms of elementary functions.

The simplest solutions of each type are presented in Tables 1 and 2. It will be observed that, for each type, the functions entering the solution differ according as m is even or odd.

Table 1

THE SIMPLEST SOLUTIONS OF TYPE (3.1)

m	n	ϑ	k_a
1	1/3	$1-k_a\varphi/(1+k_a^2\varphi^2)^{\frac{1}{2}}$	$\left(\frac{3}{4}\right)^{3/4}=0\cdot 806$
2	1/2	$1-\frac{2}{\pi}[\tan^{-1} k_a\varphi+k_a\varphi/(1+k_a^2\varphi^2)]$	$\left(\frac{2}{3\pi^{\frac{1}{2}}}\right)^{2/3}=0\cdot 521$
3	3/5	$1-\frac{1}{2}\frac{k_a\varphi(3+2k_a^2\varphi^2)}{(1+k_a^2\varphi^2)^{3/2}}$	$\left(\frac{5^5}{2^{17}.3^3}\right)^{1/8}=0\cdot 432$
4	2/3	$1-\frac{2}{3\pi}\left[3\tan^{-1} k_a\varphi+\frac{k_a\varphi(5+3k_a^2\varphi^2)}{(1+k_a^2\varphi^2)^2}\right]$	$\left(\frac{3^2}{2^2.5^3\pi}\right)^{1/5}=0\cdot 356$

Table 2

THE SIMPLEST SOLUTIONS OF TYPE (3.2)

m	n	ϑ	k_b
2	3	$1-\frac{2}{\pi}[\sin^{-1} k_b\varphi+k_b\varphi(1-k_b^2\varphi^2)^{\frac{1}{2}}]$	$\left(\frac{\pi^2}{2^6.3}\right)^{1/4}=0\cdot 476$
3	2	$1-\frac{1}{2}k_b\varphi(3-k_b^2\varphi^2)$	$\left(\frac{1}{2.3^2}\right)^{1/3}=0\cdot 382$
4	5/3	$1-\frac{2}{3\pi}[3\sin^{-1} k_b\varphi+k_b\varphi(5-2k_b^2\varphi^2)(1-k_b^2\varphi^2)^{\frac{1}{2}}]$	$\left(\frac{3^5.\pi^2}{2^{17}.5^3}\right)^{1/8}=0\cdot 332$
5	3/2	$1-\frac{k_b\varphi}{8}[15-10k_b^2\varphi^2+3k_b^4\varphi^4]$	$\left(\frac{2}{3.5}\right)^{3/5}=0\cdot 299$

These solutions in terms of elementary functions are of some interest. However, available tables (Pearson 1934) make solution in terms of incomplete beta-functions (see following Section IV) more useful for numerical purposes, even* in the cases treated in this Section.

IV. Solution in Terms of the Incomplete Beta-function

We observe that

$$\left.\begin{aligned}\int_\alpha^{\frac{1}{2}\pi}\cos^m\alpha'\mathrm{d}\alpha'\Big/\int_0^{\frac{1}{2}\pi}\cos^m\alpha'\mathrm{d}\alpha'=I_x\left(\frac{m+1}{2},\ \tfrac{1}{2}\right),\\ x=\cos^2\alpha.\end{aligned}\right\}\tag{4.1}$$

Here $I_x(p, q)$ is the incomplete beta-function ratio of Pearson (1934) defined by

$$I_x(p, q)=\int_0^x x'^{p-1}(1-x')^{q-1}\mathrm{d}x'\Big/\int_0^1 x'^{p-1}(1-x')^{q-1}\mathrm{d}x'.\tag{4.2}$$

Our solutions are thus expressible in terms of this function. (2.21) may be rewritten

$$\vartheta=I_x\left(\frac{m+1}{2},\ \tfrac{1}{2}\right),\tag{4.3}$$

with x specified as follows:

$$0<n<1\,;\quad x=(1+k_a^2\varphi^2)^{-1},\tag{4.4}$$

$$n>1\,;\quad x=1-k_b^2\varphi^2.\tag{4.5}$$

Pearson's tabulation of $I_x(p, q)$ is for $p=\frac{1}{2}(\frac{1}{2})11(1)50$. It follows that, without interpolation, Pearson provides values of $I_x\left(\frac{m+1}{2},\ \frac{1}{2}\right)$ only for $m=0(1)21(2)99$. Thus, curiously enough, the table gives (without interpolation) only solutions which could be found by the elementary methods of Section III.† The table, of course, allows solutions to be computed with much less labour than would be required otherwise.

V. Power Series and Asymptotic Forms of Solution

(a) *Power Series in* φ

The power series forms in φ provide information on the behaviour of the solution near $\vartheta=1$.

Case $0<n<1$. In this case (2.15) may be rewritten

$$\vartheta=1-\Psi_1\int_0^\varphi(1+k_a^2\varphi'^2)^{-1/(1-n)}\mathrm{d}\varphi'.\tag{5.1}$$

* In fact, as we see later, Pearson's tables are directly usable *only* in cases where the solution *is* expressible in terms of elementary functions.

† A P-fold increase in the number of immediately available solutions would be provided by a tabulation of $\int_0^\alpha\cos^m\alpha'\mathrm{d}\alpha'$ in the interval of $0\leqslant\alpha\leqslant\pi/2$ for $m=\frac{1}{P}\left(\frac{1}{P}\right)\frac{P-1}{P}$ $\left[\text{or, equivalently, by a tabulation of } I_x(p,\tfrac{1}{2}) \text{ for } p=\frac{P+1}{2P}\left(\frac{1}{2P}\right)\frac{2P-1}{2P}\right]$.

In the interval $0<\varphi<k_a^{-1}$ we may apply the binomial theorem and term-by-term integration, obtaining

$$\vartheta=1-\Psi_1\varphi\left[1-\frac{1}{3}\cdot\frac{k_a^2\varphi^2}{1-n}+\frac{2-n}{2!5}\cdot\left(\frac{k_a^2\varphi^2}{1-n}\right)^2-\frac{(2-n)(3-2n)}{3!7}\cdot\left(\frac{k_a^2\varphi^2}{1-n}\right)^3+\cdots\right]. \tag{5.2}$$

The corresponding result in the case $n>1$ is

$$\vartheta=1-\Psi_1\varphi\left[1-\frac{1}{3}\cdot\frac{k_b^2\varphi^2}{n-1}-\frac{n-2}{2!5}\cdot\left(\frac{k_b^2\varphi^2}{n-1}\right)^2-\frac{(n-2)(2n-3)}{3!7}\cdot\left(\frac{k_b^2\varphi^2}{n-1}\right)^3-\cdots\right]. \tag{5.3}$$

We observe that series (5.3) is finite whenever $1/(n-1)$ is a positive integer (i.e. whenever the integer m of (3.2) is odd). Also, since $k_b\varphi<1$ for $\varphi<\varphi_0$, series (5.3) converges throughout the interval $0\leqslant\varphi<\varphi_0$.

(b) *Asymptotic Form for ϑ small* ($0<n<1$)

Forms of solution discussed to this point do not readily provide a full picture of behaviour near $\vartheta=0$. Accordingly, the asymptotic form developed here for $0<n<1$ is of some interest.

We rewrite (2.15) as

$$\vartheta=\Psi_1\int_\varphi^\infty (k_a\varphi')^{-2/(1-n)}[1+(k_a\varphi')^{-2}]^{-1/(1-n)}\mathrm{d}\varphi'. \tag{5.4}$$

Then, applying the binomial theorem and term-by-term integration, we have

$$\vartheta=(1-n)\Psi_1k_a{}^{-2/(1-n)}\varphi^{-(1+n)/(1-n)}\left[\frac{1}{1+n}-\frac{1}{(3-n)(1-n)(k_a\varphi)^2}\right.$$
$$\left.+\frac{2-n}{2!(5-3n)(1-n)^2(k_a\varphi)^4}-\frac{(2-n)(3-n)}{3!(7-5n)(1-n)^3(k_a\varphi)^6}+\cdots\right]. \tag{5.5}$$

With some reductions involving (2.20) and (4.4), (5.5) yields the approximations, good for ϑ small (φ large),

$$\left.\begin{aligned}\vartheta&\approx\left[2n\left(\frac{1+n}{1-n}\right)^n\right]^{1/(1-n)}\varphi^{-(1+n)/(1-n)},\\ \varphi&\approx\left[2n\left(\frac{1+n}{1-n}\right)^n\right]^{1/(1+n)}\cdot\vartheta^{-(1-n)/(1+n)}.\end{aligned}\right\} \tag{5.6}$$

Although series (5.5) is conveniently (and properly) described as asymptotic, it converges in the interval $\varphi>k_a^{-1}$.

(c) *Power Series in* $(1-k_b\varphi)$ ($n>1$)

On the other hand, in the case $n>1$, behaviour near $\vartheta=0$ may be examined by means of the power series in $(1-k_b\varphi)$. Putting

$$\varepsilon=1-k_b\varphi, \tag{5.7}$$

we may here reduce (2.15) to

$$\vartheta=\frac{2^{1/(n-1)}\Psi_1}{k_b}\int_0^{\varepsilon}\varepsilon'^{1/(n-1)}(1-\tfrac{1}{2}\varepsilon')^{1/(n-1)}d\varepsilon'. \tag{5.8}$$

Since $\varepsilon\leqslant 1$, we may apply the binomial theorem and term-by-term integration, obtaining

$$\vartheta=2^{1/(n-1)}(n-1)\cdot\frac{\Psi_1}{k_b}\cdot\varepsilon^{n/(n-1)}\left[\frac{1}{n}-\frac{1}{2n-1}\cdot\frac{\varepsilon}{2(n-1)}\right.$$
$$\left.-\frac{n-2}{2!(3n-2)}\cdot\left(\frac{\varepsilon}{2(n-1)}\right)^2-\frac{(n-2)(2n-3)}{3!(4n-3)}\left(\frac{\varepsilon}{2(n-1)}\right)^3-\cdots\right]. \tag{5.9}$$

With some reductions involving (2.25) and (4.5), (5.9) yields the approximations, good for ϑ small,

$$\left.\begin{aligned}\vartheta&\approx\frac{(2\varepsilon)^{n/(n-1)}}{\dfrac{n}{n-1}\left[\mathrm{B}\left(\dfrac{n}{n-1},\tfrac{1}{2}\right)\right]},\\ \varphi&\approx\varphi_0\left[1-\tfrac{1}{2}\left[\frac{n}{n-1}\left[\mathrm{B}\left(\frac{n}{n-1},\tfrac{1}{2}\right)\right]\vartheta\right]^{(n-1)/n}\right].\end{aligned}\right\} \tag{5.10}$$

It will be noted that series (5.9) is convergent throughout the interval $0\leqslant\varepsilon\leqslant 1$.

VI. Some General Properties of n-Diffusion

In this section we discuss certain general features of n-diffusion in the semi-infinite region (subject to conditions (2.2)) which emerge from this study.

(a) *Profile Character*

We consider how the character of the concentration profiles, in the reduced form $\varphi(\vartheta)$, varies with n.

(i) *Depth of Penetration* ; *Order of Profiles near* $\vartheta=0$.—An obvious feature of the profiles is the sharp distinction between the case $0<n\leqslant 1$, for which φ_0 is infinite, and the case $n>1$, for which φ_0 is finite. We have, for the latter case,

$$\varphi_0=\frac{1}{k_b}=\left[\frac{2n(n+1)}{n-1}\left\{\frac{2}{\mathrm{B}\left(\dfrac{n}{n-1},\tfrac{1}{2}\right)}\right\}^{n-1}\right]^{1/(n+1)}. \tag{6.1}$$

φ_0 is a monotonic function of n, decreasing as n increases for $n>1$.

When $(n-1)$ is small, but positive, we have the approximation

$$\varphi_0\approx 2(n-1)^{-\frac{1}{2}}. \tag{6.2}$$

An approximation for n large is

$$\varphi_0\approx(2n)^{1/(n+1)}. \tag{6.3}$$

Also

$$\lim_{n\to\infty}\varphi_0=1. \tag{6.4}$$

Figure 1 shows $\varphi_0(n)$, together with approximations (6.2) and (6.3).

In the case $0>n\geqslant 1$, the rate of approach of φ to infinity as $\vartheta\to 0$ is of some interest. It is useful to introduce the notation $\varphi^{(r)}$ to denote the φ function for $n=n_r$.

We then have from (5.6) that,

$$\left.\begin{array}{c}\text{if } 0<n_1<n_2\leqslant 1,\\ \varphi^{(1)}\to\infty \text{ as } \vartheta\to 0 \text{ more rapidly than does } \varphi^{(2)}.\end{array}\right\} \tag{6.5}$$

(6.5) and the monotonic decreasing property of φ_0 for $n>1$ supply an ordering relation for the family of $\varphi(\vartheta)$ curves close to $\vartheta=0$. This is that,

$$\left.\begin{array}{c}\text{if } 0<n_1<n_2 \text{ and } \vartheta \text{ is sufficiently small,}\\ \varphi^{(1)}(\vartheta)>\varphi^{(2)}(\vartheta).\end{array}\right\} \tag{6.6}$$

(ii) *Concentration Gradient at* $x=0$; *Order of Profiles near* $\vartheta=1$.—Equations (2.20) and (2.25), and the known result for $n=1$, provide the relation between Ψ'_1 (i.e. minus the concentration gradient at $x=0$ in reduced form) and n. It is readily shown, both for $\nu>0$ and $\nu<0$, that

$$\text{for } n=1+\nu,\ \lim_{\nu\to 0}\Psi'_1=\pi^{-\frac{1}{2}}. \tag{6.7}$$

It is thus confirmed that $\Psi'_1(n)$ is continuous through $0<n<1$, $n=1$, and $n>1$. We note that

$$\lim_{n\to 0}\Psi'_1=\infty\,;\quad \lim_{n\to\infty}\Psi'_1=1\,; \tag{6.8}$$

and that, when n is small, we have the approximation

$$\Psi'_1\approx 2/\pi^2 n. \tag{6.9}$$

Figure 2 shows $\Psi'_1(n)$, together with approximation (6.9).

An important property of $\Psi'_1(n)$ is the fact that it possesses a minimum value, approximately $0\cdot 55794$, at $n=n_*\approx 1\cdot 29$. The existence of this minimum leads to the following relations ordering the family of $\varphi(\vartheta)$ curves close to $\vartheta=1$.

$$\left.\begin{array}{c}\text{If } 0<n_1<n_2\leqslant n_* \text{ and } (1-\vartheta) \text{ is sufficiently small,}\\ \varphi^{(1)}(\vartheta)<\varphi^{(2)}(\vartheta).\end{array}\right\} \tag{6.10}$$

$$\left.\begin{array}{c}\text{If } n_*\leqslant n_1<n_2 \text{ and } (1-\vartheta) \text{ is sufficiently small,}\\ \varphi^{(1)}(\vartheta)>\varphi^{(2)}(\vartheta).\end{array}\right\} \tag{6.11}$$

Ordering relations (6.6) and (6.10) are in opposite senses, so that any two members of the set of profiles $0<n\leqslant n_*$ intersect once and once only in the interval $0<\vartheta<1$. This set of profiles contains the sub-set $0<n\leqslant 1$ with φ_0 infinite and the sub-set $1<n\leqslant n_*$ with φ_0 finite. Figure 3 shows typical members of the set, the "finite φ_0" sub-set being represented by the profile $n=5/4$.

Ordering relations (6.6) and (6.11) are in the same sense, so that the set of profiles $n\geqslant n_*$ are non-intersecting in the interval $0<\vartheta<1$. φ_0 is finite for this set of profiles, which are illustrated in Figure 4 ; the profile for $n=1$ is shown also for comparison.

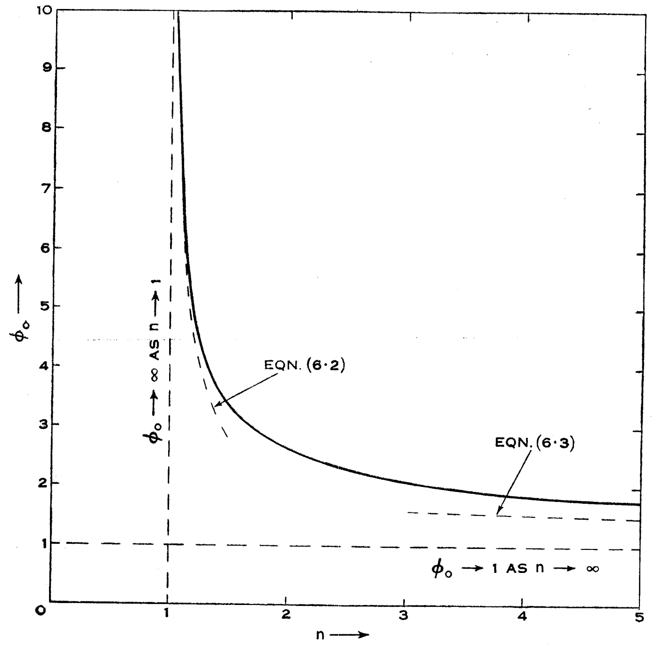

Fig. 1.—n-diffusion in the semi-infinite region. Variation with n of depth of penetration in the reduced form φ_0.

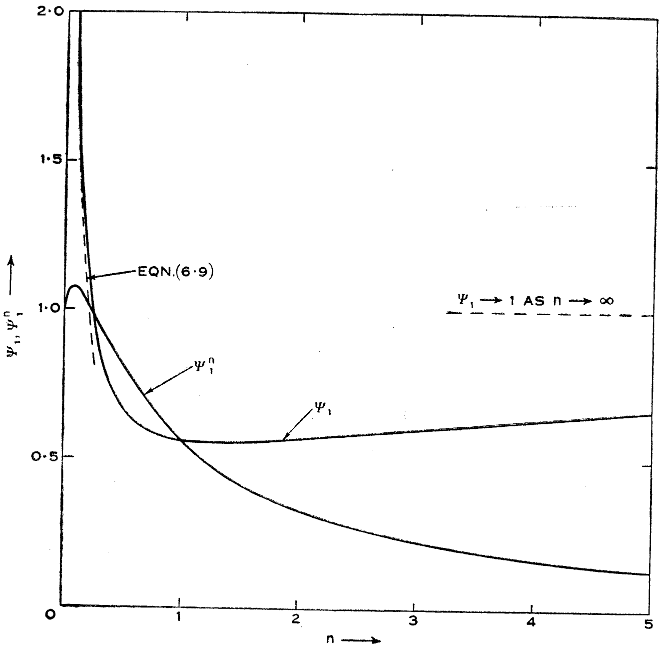

Fig. 2.—n-diffusion in the semi-infinite region. Variation with n of concentration gradient at surface $x=0$ in the reduced form Ψ_1. Also shown is Ψ_1^n, which represents flux density at $x=0$ in reduced form.

It will be understood that intersection relations between the two sets are somewhat complicated and that a representation of the whole family of profiles on the one graph would appear, at first glance, rather disorderly.

(iii) *Relationships connecting* Ψ_1 *and* k_a *and* k_b.—We note the following relationships :

$$0<n<1\,;\quad \Psi_1/k_a=2/\mathrm{B}\left(\frac{1+n}{2-2n},\,\tfrac{1}{2}\right), \tag{6.12}$$

$$n>1\,;\quad \Psi_1/k_b=\Psi_1\varphi_0=2/\mathrm{B}\left(\frac{n}{n-1},\,\tfrac{1}{2}\right). \tag{6.13}$$

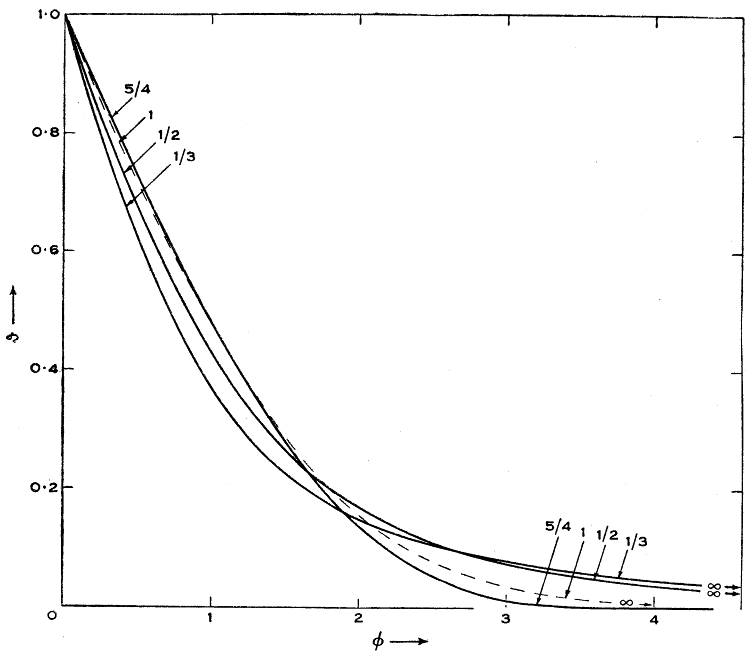

Fig. 3.—n-diffusion in the semi-infinite region. Concentration profiles in the reduced form $\vartheta(\varphi)$ for " intersecting " set $n\leqslant n_*$. Numerals on curves denote values of n. Infinite profiles are identified by the infinity sign. Profile for $n=1$ shown broken for clarity.

(*b*) *Variation with Time*

The form of the time-dependence of the phenomenon follows immediately from (2.4). We have, in particular, the following results :

$$x(\vartheta)\propto t^{1/(n+1)}, \tag{6.14}$$

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}x}(\vartheta)\propto t^{-1/(n+1)}, \tag{6.15}$$

$$q_1\propto t^{-n/(n+1)}, \tag{6.16}$$

$$\int_0^t q_1\mathrm{d}t\propto\int_0^1 \varphi\mathrm{d}\vartheta\propto t^{1/(n+1)}, \tag{6.17}$$

q_1 denotes the value of the flux density q at $x=0$ ($\vartheta=1$).

(*c*) *Flux Density and Integrated Flux Density*

We have the following expressions for the flux density and its time integral:

$$q_1 = A^{1/(n+1)} |\, \theta_1 - \theta_0 \,|^{2n/(n+1)} . t^{-n/(n+1)} . \Psi_1^n, \tag{6.18}$$

$$\int_0^t q_1 \mathrm{d}t = (n+1) A^{1/(n+1)} |\, \theta_1 - \theta_0 \,|^{2n/(n+1)} . t^{1/(n+1)} . \Psi_1^n. \tag{6.19}$$

Ψ_1^n is a measure, in reduced form, of flux density and of total transfer to time t. Some interest therefore attaches to the dependence of Ψ_1^n on n. This

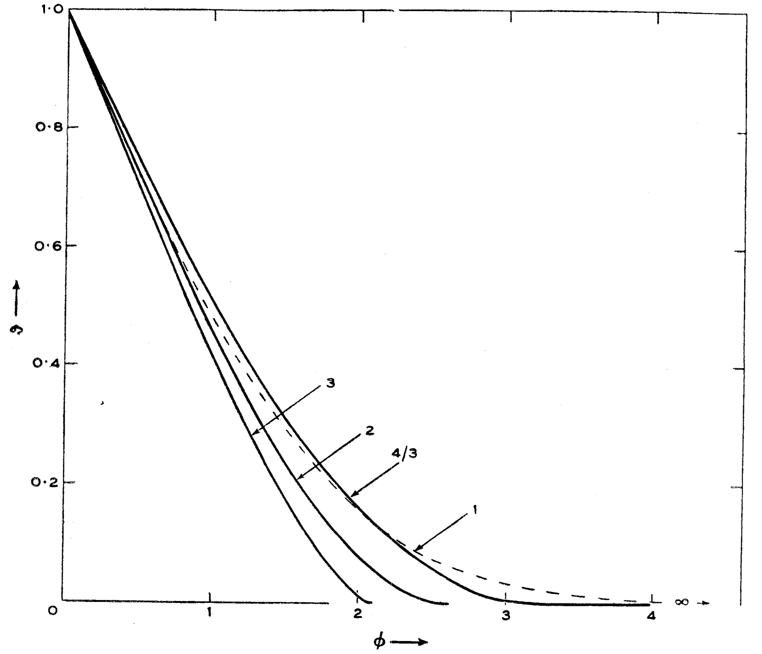

Fig. 4.—As Figure 3, but for "non-intersecting set" $n \geqslant n_*$. Profile for $n=1$ also shown for comparison (broken curve).

is illustrated in Figure 2. We note that Ψ_1^n has a maximum value approximately equal to $e^{2/\pi^2 e}$ $[\approx 1 \cdot 08]$ at $n \approx 2/\pi^2 e [\approx 0 \cdot 075]$; and that $\lim_{n \to 0} \Psi_1^n = 1$ and $\lim_{n \to \infty} \Psi_1^n = 0$.

VII. References

Boltzmann, L. (1894).—*Ann. Phys., Lpz.* **53**: 595.

Pearson, K. (1934).—"Tables of the Incomplete Beta-Function." (Cambridge Univ. Press.)

Philip, J. R. (1956).—*Aust. J. Phys.* **9**: 570.

Priestley, C. H. B. (1954).—*Aust. J. Phys.* **7**: 176.