ESTIMATION OF THE MAXIMUM TEMPERATURE IN A RADIALLY CONSTRICTED GAS DISCHARGE BETWEEN ELECTRODES

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Summary

A steady-state deuterium discharge between two electrodes is considered and the free boundary surface of the plasma is assumed thermally insulated when pinched away from the walls of the discharge tube. Cooling is therefore by heat conduction to the electrodes, compared to which bremsstrahlung loss is shown to be negligible if the discharge is not too long. The main question examined is how much the maximum temperature T_m can be raised by constricting the cross section of the discharge near the centre.

The analysis is confined to substantially ionized deuterium, and curves based on the Saha equation are provided to show the minimum gas temperatures required for various particle densities. With neglect of thermoelectric effects, there exists a median plane, normal to the longitudinal axis of the discharge, about which the distributions of temperature and voltage are symmetrical. The analysis is carried through both by specializing the current density and heat flux vectors, $\mathbf{j}$ and $\mathbf{q}$, to cater for an isotropic plasma, and for a plasma made anisotropic by a strong, external magnetic field. Prior to detailed mathematical analyses, however, a simple continuity argument yields the important relationship, $\mathbf{q}+V\mathbf{j}=0$, where V is the electric potential, provided that everywhere within the discharge $\mathbf{q}$ is parallel to $\mathbf{j}$. This type of flow, termed for convenience longitudinal flow, is the main concern of the paper.

The detailed axi-symmetric analyses for slightly and greatly constricted discharges show that $T^2+(\sigma_0/K_0)V^2=T_m^2$, where σ_0 and K_0 are related to Spitzer's formulae for the electrical and thermal conductivities of a highly ionized gas, and T_m is the temperature on the median plane, where $V=0$. Use of an electric stream function Ψ and the electric potential V as curvilinear coordinates simplifies the plasma energy equation. An analytic solution of this equation if the temperature T is a function of Ψ only, shows that the heat flow can be everywhere perpendicular to the flow of electricity only when the streamlines are straight and parallel to the longitudinal axis of symmetry. Analytic solutions if T is a function of V, representing the longitudinal flow, are given for (1) straight streamlines parallel to the longitudinal axis of symmetry, (2) hyperbolic streamlines to represent a discharge constricted at the median plane. A curve giving the variation of T with distance along the linear discharge is included.

Upper-limit expressions for the central temperature T_m are obtained in terms of the total current carried by the longitudinally stabilized discharge and its characteristic dimensions, and a curve gives the dependence on constriction of T_m and of the resistance R between the electrodes for constant total discharge current. If a large radial constriction at the median plane is achieved by use of a strong guiding magnetic field which makes the conductivities anisotropic, a tensorial analysis is required, but leads to the same results for longitudinal flow. Where the thermal insulation and neglect of bremsstrahlung approximations apply, the direction of heat flow is not expected to depart significantly anywhere from that of the flow of electricity, and so the above curve should provide a useful guide to the increase of T_m and R due to constriction. For an area constriction of 400 : 1, T_m and R are increased by a factor of about 4.

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Characteristics relating T and the magnetic field H, when $\omega_e\tau_e=1$ (ω_e is the electron gyrofrequency, τ_e the electron collision time) are given for various values of the total particle density. Hence it is possible to find where the vector ($\omega_e\tau_e \ll 1$) and tensor ($\omega_e\tau_e \gg 1$) solutions are applicable.

I. Introduction

In the following analysis the flows of heat and electricity in a steady non-equilibrium state gas discharge between a pair of electrodes are examined, with the principal object of determining theoretically the maximum plasma temperature T_m. For simplicity, the treatment is confined to discharges of highly ionized gases having low bremsstrahlung loss and free boundary surfaces perfectly insulated thermally by a high vacuum. Non-constricted and constricted discharges are treated.

II. Application of the Saha Equation

The Saha equation (Saha 1920 ; Saha and Saha 1934 ; Allis 1956 ; Cobine 1958) applies to a gaseous system in thermal equilibrium, whereas we shall consider systems which depart from this condition by virtue of an excess of the electron temperature over the ion temperature, as discussed by Allis in the above

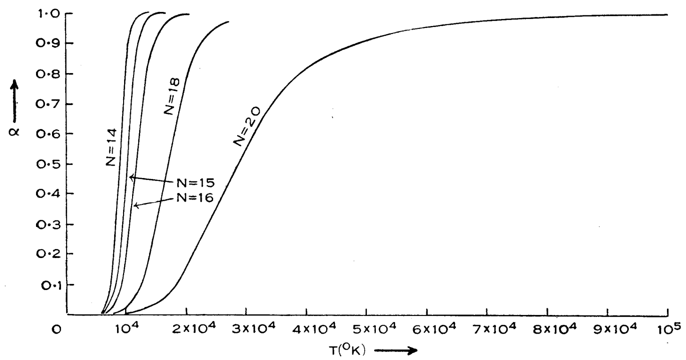

Fig. 1.—Saha curves for hydrogen. n_a atoms/cm³ ($n_a=10^N$); n_i singly ionized atoms/cm³ ; $\alpha=n_i/n_a$, degree of ionization ; T, gas temperature.

reference, and treated in greater detail by Alfvén (1950). However, as Allis points out, the temperature can never become less than that given by the Saha equation, and accordingly we apply it here to obtain some idea of the minimum temperatures at which a gas may be regarded as substantially ionized as the particle density varies over an appropriate range. Thus, applying Cobine's form of the Saha equation to a plasma, having equal concentrations of singly ionized atoms and electrons ($n_e=n_i$) we obtain

$$\log\{\alpha^2/(1-\alpha)\}=1\cdot5\log T-(N-15\cdot385)-5050V_i/T, \tag{2.1}$$

where, if n_a atoms/cm³ is the original concentration of the gas,

$\alpha=n_i/n_a$ is the degree of ionization,
T is the gas temperature in degrees Kelvin,
$N=\log n_a$,
V_i is the ionization potential of the gas in volts.

Here our main interest will be in the hydrogen isotope, deuterium, and so the temperature dependence of α for hydrogen has been calculated from equation (2.1) for N-values varying from 14 to 20, and presented in Figure 1.

For substantially ionized hydrogen gas ($\alpha \sim 0 \cdot 9$, say) the minimum temperature for a given value of N can now be readily obtained. It is of interest to note from each of these curves that this minimum temperature lies well below the temperature corresponding to the ionization potential of the gas concerned, particularly for the smaller values of N.

III. Preliminary Discussion of the Current Density and Heat Flux Vectors

In the absence of an external magnetic field **H**, the flows of electricity and heat in a system mutually interfere, and, as Callen (1948) has pointed out, the thermoelectric effects of Peltier and Seebeck may be viewed as the result of this interference. Further, if an external magnetic field is impressed on the system, thermomagnetic and galvanomagnetic effects (Ettingshausen, Hall, Nernst, and Righi-Leduc effects) appear. To account for these effects, we shall now follow the procedure adopted by Marshall in his comprehensive report (1957) and, for a region made anisotropic by the presence of a magnetic field, write the conduction current density vector, **j**, and the heat flux vector, **q**, as

$$\mathbf{j}=\underbrace{\sigma^{\mathrm{I}}\mathbf{E}^{\parallel}+\sigma^{\mathrm{II}}\mathbf{E}^{\perp}}_{\text{No special name}}+\underbrace{\sigma^{\mathrm{III}}\mathbf{h}\times\mathbf{E}^{\perp}}_{\text{Hall}}+\underbrace{\varphi^{\mathrm{I}}(\nabla T)^{\parallel}+\varphi^{\mathrm{II}}(\nabla T)^{\perp}}_{\text{Seebeck}}+\underbrace{\varphi^{\mathrm{III}}\mathbf{h}\times(\nabla T)^{\perp}}_{\text{Nernst}}, \qquad (3.1)$$

where $\mathbf{E}$ is the electric field,

$\mathbf{E}^{\parallel}$ is the component of $\mathbf{E}$ parallel to $\mathbf{H}$,

$\mathbf{E}^{\perp}$ is the component of $\mathbf{E}$ perpendicular to $\mathbf{H}$,

$\mathbf{h}=\mathbf{H}/H$ is a unit vector in the direction of $\mathbf{H}$,

∇T is the temperature gradient,

$(\nabla T)^{\parallel}$ is the component of ∇T parallel to $\mathbf{H}$,

$(\nabla T)^{\perp}$ is the component of ∇T perpendicular to $\mathbf{H}$,

σ^{I}, σ^{II}, and σ^{III} are coefficients of electrical conductivity,

φ^{I}, φ^{II}, and φ^{III} are thermal diffusion coefficients,

and

$$\mathbf{q}=-\lambda^{\mathrm{I}}(\nabla T)^{\parallel}-\lambda^{\mathrm{II}}(\nabla T)^{\perp}-\lambda^{\mathrm{III}}\mathbf{h}\times(\nabla T)^{\perp}+\Psi^{\mathrm{I}}\mathbf{j}^{\parallel}+\Psi^{\mathrm{II}}\mathbf{j}^{\perp}+\Psi^{\mathrm{III}}\mathbf{h}\times\mathbf{j}^{\perp}, \qquad (3.2)$$

where $\mathbf{j}^{\parallel}$ is the component of $\mathbf{j}$ parallel to $\mathbf{H}$,

$\mathbf{j}^{\perp}$ is the component of $\mathbf{j}$ perpendicular to $\mathbf{H}$,

λ^{I}, λ^{II}, and λ^{III} are thermal conductivity coefficients, Ψ^{I}, Ψ^{II}, and Ψ^{III} are coefficients giving the heat flux due to electric currents.

Using (3.1), (3.2) can be written

$$\mathbf{q}=-\underbrace{K^{\mathrm{I}}(\nabla T)^{\parallel}-K^{\mathrm{II}}(\nabla T)^{\perp}}_{\text{No special name}}-\underbrace{K^{\mathrm{III}}\mathbf{h}\times(\nabla T)^{\perp}}_{\text{Righi-Leduc}}+\underbrace{\xi^{\mathrm{I}}\mathbf{E}^{\parallel}+\xi^{\mathrm{II}}\mathbf{E}^{\perp}}_{\text{Peltier}}+\underbrace{\xi^{\mathrm{III}}\mathbf{h}\times\mathbf{E}^{\perp}}_{\text{Ettingshausen}}, \qquad (3.3)$$

where K^{I}, K^{II}, and K^{III} are thermal conductivity coefficients when electric currents are allowed to flow, given by

$$\left.\begin{aligned} K^{\mathrm{I}}&=\lambda^{\mathrm{I}}-\Psi^{\mathrm{I}}\varphi^{\mathrm{I}},\\ K^{\mathrm{II}}&=\lambda^{\mathrm{II}}-\Psi^{\mathrm{II}}\varphi^{\mathrm{II}}+\Psi^{\mathrm{III}}\varphi^{\mathrm{III}},\\ K^{\mathrm{III}}&=\lambda^{\mathrm{III}}-\Psi^{\mathrm{II}}\varphi^{\mathrm{III}}-\Psi^{\mathrm{III}}\varphi^{\mathrm{II}}, \end{aligned}\right\} \tag{3.4}$$

and ξ^{I}, ξ^{II}, and ξ^{III} are coefficients accounting for the contribution to the heat flux from the electric field $\mathbf{E}$, given by

$$\left.\begin{aligned} \xi^{\mathrm{I}}&=\Psi^{\mathrm{I}}\sigma^{\mathrm{I}},\\ \xi^{\mathrm{II}}&=\Psi^{\mathrm{II}}\sigma^{\mathrm{II}}-\Psi^{\mathrm{III}}\sigma^{\mathrm{III}},\\ \xi^{\mathrm{III}}&=\Psi^{\mathrm{II}}\sigma^{\mathrm{III}}+\Psi^{\mathrm{III}}\sigma^{\mathrm{II}}. \end{aligned}\right\} \tag{3.5}$$

The results (3.4) and (3.5) are not given explicitly by Marshall; they can alternatively be obtained from results given in Part 3 of his report (pp. 28–9) by neglecting the electron mass m_e relative to the ion mass m_i, and by noting that the K's here correspond to Marshall's θ's.

Further information on the terms representing the effects discovered by Ettingshausen, Hall, Nernst, Peltier, Righi and Leduc, and Seebeck in the expressions (3.1) and (3.3) for $\mathbf{j}$ and $\mathbf{q}$ is provided by Chapman and Cowling (1953), Hix and Alley (1958), Linhart (1960), and the above article by Callen.

An attempt to base the following analysis on equations (3.1) and (3.3) encounters mathematical difficulties. However, realistic solutions can be obtained to the present problem by considering two special forms of $\mathbf{j}$ and $\mathbf{q}$:

(i) $$\mathbf{j}=\sigma^{\mathrm{I}}\mathbf{E}^{\parallel}+\sigma^{\mathrm{II}}\mathbf{E}^{\perp}+\sigma^{\mathrm{III}}\mathbf{h}\times\mathbf{E}^{\perp} \tag{3.6}$$

and $$\mathbf{q}=-K^{\mathrm{I}}(\nabla T)^{\parallel}-K^{\mathrm{II}}(\nabla T)^{\perp}-K^{\mathrm{III}}\mathbf{h}\times(\nabla T)^{\perp}, \tag{3.7}$$

(ii) $$\mathbf{j}=\sigma^{\mathrm{I}}\mathbf{E}^{\parallel}+\sigma^{\mathrm{II}}\mathbf{E}^{\perp}+\varphi^{\mathrm{I}}(\nabla T)^{\parallel}+\varphi^{\mathrm{II}}(\nabla T)^{\perp} \tag{3.8}$$

and $$\mathbf{q}=-K^{\mathrm{I}}(\nabla T)^{\parallel}-K^{\mathrm{II}}(\nabla T)^{\perp}+\xi^{\mathrm{I}}\mathbf{E}^{\parallel}+\xi^{\mathrm{II}}\mathbf{E}^{\perp}. \tag{3.9}$$

This analysis is based on (3.6) and (3.7), which amount to (3.1) and (3.3) with the terms multiplied by φ and ξ omitted; i.e. we neglect the effect of thermal diffusion on $\mathbf{j}$ and the contribution of the electric field to $\mathbf{q}$. Limiting forms of (3.6) and (3.7) are obtained in the next section, for insertion later into the plasma energy equation, which is established in Section V.

Equations (3.8) and (3.9), which include thermoelectric effects, exclude thermomagnetic and galvanomagnetic effects, but nevertheless apply if the magnetic field is parallel to $\mathbf{E}$ and ∇T, will be used in a subsequent paper.

IV. $\mathbf{j}$ and $\mathbf{q}$ for Conditions of Isotropy and Extreme Anisotropy (Hall and Righi-Leduc Effects only included)

By associating with the unit vector $\mathbf{h}$ further unit vectors $\mathbf{f}$ and $\mathbf{g}$, to produce an orthogonal right-handed set, we can write (3.6) and (3.7) as

$$\mathbf{j}=\mathbf{f}(\sigma^{\mathrm{II}}E_f-\sigma^{\mathrm{III}}E_g)+\mathbf{g}(\sigma^{\mathrm{II}}E_g+\sigma^{\mathrm{III}}E_f)+\mathbf{h}\sigma^{\mathrm{I}}E^{\parallel}, \tag{4.1}$$

where $E_f=\mathbf{f}\cdot\mathbf{E}^{\perp}$ and $E_g=\mathbf{g}\cdot\mathbf{E}^{\perp}$, and

$$\mathbf{q}=\mathbf{f}[-K^{\mathrm{II}}(\nabla T)_f+K^{\mathrm{III}}(\nabla T)_g]+\mathbf{g}[-K^{\mathrm{II}}(\nabla T)_g-K^{\mathrm{III}}(\nabla T)_f]-\mathbf{h}K^{\mathrm{I}}(\nabla T)^{\parallel}, \tag{4.2}$$

where $(\nabla T)_f=\mathbf{f}\cdot(\nabla T)^{\perp}$ and $(\nabla T)_g=\mathbf{g}\cdot(\nabla T)^{\perp}$.

The tensorial nature of the electrical and thermal conductivity coefficients is perhaps at this stage best displayed by writing (4.1) and (4.2) using matrix notation. Thus, for (4.1)

$$\begin{bmatrix} j_f \\ j_g \\ j_h \end{bmatrix} = \begin{bmatrix} \sigma^{\mathrm{II}} & -\sigma^{\mathrm{III}} & 0 \\ \sigma^{\mathrm{III}} & \sigma^{\mathrm{II}} & 0 \\ 0 & 0 & \sigma^{\mathrm{I}} \end{bmatrix} \begin{bmatrix} E_f \\ E_g \\ E^{||} \end{bmatrix}, \tag{4.3}$$

so that in general the matrix form of the electrical conductivity tensor is

$$[\,\sigma\,] = \begin{bmatrix} \sigma^{\mathrm{II}} & -\sigma^{\mathrm{III}} & 0 \\ \sigma^{\mathrm{III}} & \sigma^{\mathrm{II}} & 0 \\ 0 & 0 & \sigma^{\mathrm{I}} \end{bmatrix}. \tag{4.4}$$

However, as Pease (1957) mentions, in the discharges we shall discuss the Hall current cannot flow. We therefore seek the modification to (4.4) for a zero Hall current.

Suppose that the Hall current becomes zero because of the appearance of a cancelling electric field, $\mathbf{E}_c^{\perp}$, in the direction of $\mathbf{h}\times\mathbf{E}^{\perp}$. Then, with total perpendicular electric field $(\mathbf{E}^{\perp}+\mathbf{E}_c^{\perp})$ we have from (3.6)

$$\mathbf{j}=\sigma^{\mathrm{I}}\mathbf{E}^{||}+(\sigma^{\mathrm{II}}\mathbf{E}^{\perp}+\sigma^{\mathrm{III}}\mathbf{h}\times\mathbf{E}_c^{\perp})+(\sigma^{\mathrm{II}}\mathbf{E}_c^{\perp}+\sigma^{\mathrm{III}}\mathbf{h}\times\mathbf{E}^{\perp}). \tag{4.5}$$

Putting the component of $\mathbf{j}$ parallel to $\mathbf{E}_c^{\perp}$ to zero gives

$$\mathbf{E}_c^{\perp}=-(\sigma^{\mathrm{III}}/\sigma^{\mathrm{II}})\mathbf{h}\times\mathbf{E}^{\perp}, \tag{4.6}$$

and so

$$\mathbf{j}=\mathbf{f}\sigma^{\perp}E_f+\mathbf{g}\sigma^{\perp}E_g+\mathbf{h}\sigma^{\mathrm{I}}E^{||}, \tag{4.7}$$

where we have introduced the perpendicular conductivity

$$\sigma^{\perp}=\sigma^{\mathrm{II}}[1+(\sigma^{\mathrm{III}}/\sigma^{\mathrm{II}})^2]. \tag{4.8}$$

We see now that when the Hall current cannot flow, the matrix of the conductivity tensor varies from the form (4.4) to the diagonal form

$$[\,\sigma\,] = \begin{bmatrix} \sigma^{\perp} & 0 & 0 \\ 0 & \sigma^{\perp} & 0 \\ 0 & 0 & \sigma^{\mathrm{I}} \end{bmatrix}. \tag{4.9}$$

For (4.2),

$$\begin{bmatrix} q_f \\ q_g \\ q_h \end{bmatrix} = -\begin{bmatrix} K^{\mathrm{II}} & -K^{\mathrm{III}} & 0 \\ K^{\mathrm{III}} & K^{\mathrm{II}} & 0 \\ 0 & 0 & K^{\mathrm{I}} \end{bmatrix} \begin{bmatrix} (\nabla T)_f \\ (\nabla T)_g \\ (\nabla T)^{||} \end{bmatrix}, \tag{4.10}$$

giving the matrix form of the thermal conductivity tensor as

$$[\,K\,] = \begin{bmatrix} K^{\mathrm{II}} & -K^{\mathrm{III}} & 0 \\ K^{\mathrm{III}} & K^{\mathrm{II}} & 0 \\ 0 & 0 & K^{\mathrm{I}} \end{bmatrix}. \tag{4.11}$$

To identify the forms of $[\sigma]$ and $[K]$ under isotropic conditions, and conditions of extreme anisotropy, we first use Marshall's report (pp. 66, 69) to construct Table 1, where, with the symbol $\simeq$ showing that the usual plasma approximations ($m_i \gg m_e$, $n_i \approx n_e \approx \frac{1}{2}n$, $e_e = -e_i = -e$) have been made, the electron gyrofrequency is

$$\omega_e \simeq eH/m_e c, \tag{4.12}$$

or

$$\omega_e \simeq 1 \cdot 758 \times 10^7 H \text{ s}^{-1} \text{ gauss}^{-1}, \tag{4.13}$$

and the electron collision time is

$$\tau_e \simeq \frac{3}{4\sqrt{(2\pi)}} \frac{m_e^{\frac{1}{2}} k^{3/2} T^{3/2}}{n_e e^4 \ln \lambda}, \tag{4.14}$$

or

$$\tau_e \simeq 0 \cdot 551 \frac{T^{3/2}}{n \ln \lambda} \text{ s cm}^{-3} \text{ deg}^{-3/2}, \tag{4.15}$$

with $\lambda = \overline{h/p_0}$, where $h = (kT/4\pi n_e e^2)^{1/2}$ is the Debye shielding distance, $p_0 = e^2/3kT$ an impact parameter defined by Spitzer (1956), and k is Boltzmann's constant.

Here we have written Marshall's τ in terms of Spitzer's $\ln \lambda$, using $\Psi \approx 2 \ln \lambda$. It is worth noting that Spitzer's t_c (p. 78) for electrons interacting with themselves is closely equal to Marshall's τ. Ln λ has a slow dependence on n_e and T, and a value in the region of 10 (values of $\ln \lambda$ are given by Spitzer in Table 5.1, and values of Ψ are given by Marshall, Appendix B).

Since $\omega_e \tau_e = (\frac{2}{3})^{1/2} \lambda_e / \rho_e$, where λ_e is the electron mean free path and ρ_e the electron gyroradius (Alfvén, p. 49), we see that for a relatively dense gas in the presence of negligible magnetic field,

$$\omega_e \tau_e \ll 1, \tag{4.16}$$

while for a tenuous gas in the presence of strong **H**,

$$\omega_e \tau_e \gg 1. \tag{4.17}$$

We now determine the forms of **j** and **q** for the conditions described by (4.16) and (4.17).

(*a*) **j** *and* **q** *for Isotropic Conditions* ($\omega_e \tau_e \ll 1$)

TABLE 1

COMPONENTS OF THE ELECTRICAL AND THERMAL CONDUCTIVITY TENSORS

Components of $[\sigma]$	Components of $[K]$
$\sigma^{\mathrm{I}} = 1 \cdot 931 \dfrac{n_e e^2 \tau_e}{m_e}$	$K^{\mathrm{I}} = 7 \cdot 18 \dfrac{n_e k^2 T \tau_e}{m_e}$
$\sigma^{\mathrm{II}} = \dfrac{n_e e^2 \tau_e}{m_e} \dfrac{\omega_e^2 \tau_e^2 + 1 \cdot 802}{\omega_e^4 \tau_e^4 + 6 \cdot 282 \omega_e^2 \tau_e^2 + 0 \cdot 933}$	$K^{\mathrm{II}} = \dfrac{2 n_e k^2 T \tau_e}{m_e} \left\{ \dfrac{0 \cdot 458 \omega_e^2 \tau_e^2 + 3 \cdot 01}{\omega_e^4 \tau_e^4 + 6 \cdot 282 \omega_e^2 \tau_e^2 + 0 \cdot 933} + \dfrac{4 \cdot 458}{\omega_e^2 \tau_e^2 + 12 \cdot 716} \right\}$
$\sigma^{\mathrm{III}} = \dfrac{n_e e^2 \tau_e}{m_e} \dfrac{-\omega_e \tau_e (\omega_e^2 \tau_e^2 + 4 \cdot 382)}{\omega_e^4 \tau_e^4 + 6 \cdot 282 \omega_e^2 \tau_e^2 + 0 \cdot 933}$	$K^{\mathrm{III}} = \dfrac{2 \cdot 5 n_e k^2 T \tau_e}{m_e} \left\{ \dfrac{\omega_e \tau_e}{\omega_e^2 \tau_e^2 + 12 \cdot 716} - \dfrac{\omega_e \tau_e (\omega_e^2 \tau_e^2 + 6 \cdot 2)}{\omega_e^4 \tau_e^4 + 6 \cdot 282 \omega_e^2 \tau_e^2 + 0 \cdot 933} \right\}$

When $\omega_e\tau_e \ll 1$, Table 1 gives

$$\left.\begin{aligned} &\sigma^{\mathrm{I}}=1\cdot931 n_e e^2\tau_e/m_e,\\ &\sigma^{\mathrm{II}}\approx\sigma^{\mathrm{I}},\\ &\sigma^{\mathrm{III}}\approx-\sigma^{\mathrm{I}}\,2\cdot432\,\omega_e\tau_e\sim 0.\end{aligned}\right\} \tag{4.18}$$

Hence (4.8) gives $\sigma^{\perp}\approx\sigma^{\mathrm{I}}$, and (4.9) becomes

$$[\,\sigma\,] = \begin{bmatrix} \sigma^{\mathrm{I}} & 0 & 0 \\ 0 & \sigma^{\mathrm{I}} & 0 \\ 0 & 0 & \sigma^{\mathrm{I}} \end{bmatrix}. \tag{4.19}$$

The electrical conductivity is thus a scalar quantity, $\sigma=\sigma^{\mathrm{I}}$, when the magnetic field is negligible and the gas is sufficiently dense that collisions are dominant.

Similarly, from Table 1

$$\left.\begin{aligned} &K^{\mathrm{I}}=7\cdot18 n_e k^2 T\tau_e/m_e,\\ &K^{\mathrm{II}}\approx K^{\mathrm{I}},\\ &K^{\mathrm{III}}\approx-K^{\mathrm{I}}\,2\cdot286\,\omega_e\tau_e\sim 0,\end{aligned}\right\} \tag{4.20}$$

and accordingly (4.11) may be written

$$[\,K\,] = \begin{bmatrix} K^{\mathrm{I}} & 0 & 0 \\ 0 & K^{\mathrm{I}} & 0 \\ 0 & 0 & K^{\mathrm{I}} \end{bmatrix}, \tag{4.21}$$

so that the thermal conductivity is a scalar, $K=K^{\mathrm{I}}$.

Using (4.19) and (4.21), and the notation of (3.6) and (3.7), equations (4.3) and (4.10) reduce to the isotropic forms

$$\mathbf{j}=\sigma\mathbf{E}, \tag{4.22}$$

the simple form of Ohm's law, and

$$\mathbf{q}=-K\nabla T, \tag{4.23}$$

which, referring to the definition of K following (3.3), is a form of Fourier's law (Weatherburn 1957).

(b) **j** *and* **q** *for Conditions of Extreme Anisotropy* ($\omega_e\tau_e \gg 1$)

When $\omega_e\tau_e \gg 1$, Table 1 gives

$$\left.\begin{aligned} &\sigma^{\mathrm{I}}=1\cdot931\frac{n_e e^2\tau_e}{m_e},\\ &\sigma^{\mathrm{II}}\approx\sigma^{\mathrm{I}}\frac{1}{1\cdot931\,\omega_e^2\tau_e^2}\sim 0,\\ &\sigma^{\mathrm{III}}\approx-\sigma^{\mathrm{I}}\frac{1}{1\cdot931\,\omega_e\tau_e}\sim 0.\end{aligned}\right\} \tag{4.24}$$

From (4.8), $\sigma^{\perp}\approx\frac{1}{2}\sigma^{\mathrm{I}}$, and so (4.9) becomes

$$[\,\sigma\,] = \begin{bmatrix} \frac{1}{2}\sigma^{\mathrm{I}} & 0 & 0 \\ 0 & \frac{1}{2}\sigma^{\mathrm{I}} & 0 \\ 0 & 0 & \sigma^{\mathrm{I}} \end{bmatrix}. \tag{4.25}$$

The electrical conductivity is therefore a simple tensor when the magnetic field is strong and the gas tenuous.

Similarly, from Table 1

$$\left.\begin{aligned} K^{\mathrm{I}} &= 7\cdot 18\frac{n_e k^2 T\tau_e}{m_e}, \\ K^{\mathrm{II}} &\approx K^{\mathrm{I}}\frac{1\cdot 369}{\omega_e^2\tau_e^2}, \\ K^{\mathrm{III}} &\approx 0, \end{aligned}\right\} \tag{4.26}$$

and so (4.11) may be written

$$[\;K\;] = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & K^{\mathrm{I}} \end{bmatrix}, \tag{4.27}$$

with $\varepsilon = K^{\mathrm{I}}(1\cdot 369/\omega_e^2\tau_e^2) \ll 1$; for the gyrotropic plasma considered here the thermal conductivity is thus a simple tensor.

Using (4.25) and (4.27), and the notation of (3.6) and (3.7), equations (4.3) and (4.10) reduce to

$$\mathbf{j} = \sigma\mathbf{E}^{||} + \tfrac{1}{2}\sigma\mathbf{E}^{\perp}, \tag{4.28}$$

and

$$\mathbf{q} = -K(\nabla T)^{||}. \tag{4.29}$$

Equations (4.28) and (4.29) represent **j** and **q** for extreme anisotropy, when the conduction is predominantly in the direction of **H**.

Clearly, the analysis that follows must fall into two main parts. The first, corresponding to (*a*) above, essentially employs vector methods; the second, corresponding to (*b*) above, will be of a simple tensorial nature. To summarize the principles on which the analyses will be based:

(i) Expressions for the thermal and electrical conductivities for fully ionized deuterium will be employed, and Figure 1 can be used to show the temperature and concentration conditions where this is permissible, and

(ii) The cases for $\omega_e\tau_e \ll 1$, $\omega_e\tau_e \gg 1$ will be examined separately, and Figure 2 has been included to show the dividing line, $\omega_e\tau_e = 1$, between them. This figure, which relates H and T for given $n = 2n_e$, and is based on $\omega_e\tau_e \approx 10^7(HT^{3/2}/n \ln \lambda)$ gauss^{-1} cm^{-3} deg$^{-3/2}$ obtained from (4.13) and (4.15), can be used to discover where the vector and tensor solutions are applicable within the indicated practical ranges of temperature and total concentration, which were chosen after consideration of Figure 1.

V. Establishment of the Plasma Energy Equation

With reference to, say, Weatherburn (pp. 49–50) the energy equation in terms of **j** and **q** can here be written as

$$\operatorname{div} \mathbf{q} + P = \mathbf{j}\cdot\mathbf{E}, \tag{5.1}$$

where

$$P = 1\cdot 42 \times 10^{-34} n_i^2 T^{1/2} \text{ W cm}^3 \text{ deg}^{-1/2} \tag{5.2}$$

is the loss due to bremsstrahlung for a fully ionized gas of atomic number $Z=1$ (Spitzer, p. 90).

We recall that for the steady-state condition

$$\mathbf{E}=-\text{grad } V, \tag{5.3}$$

where V is the electric potential.

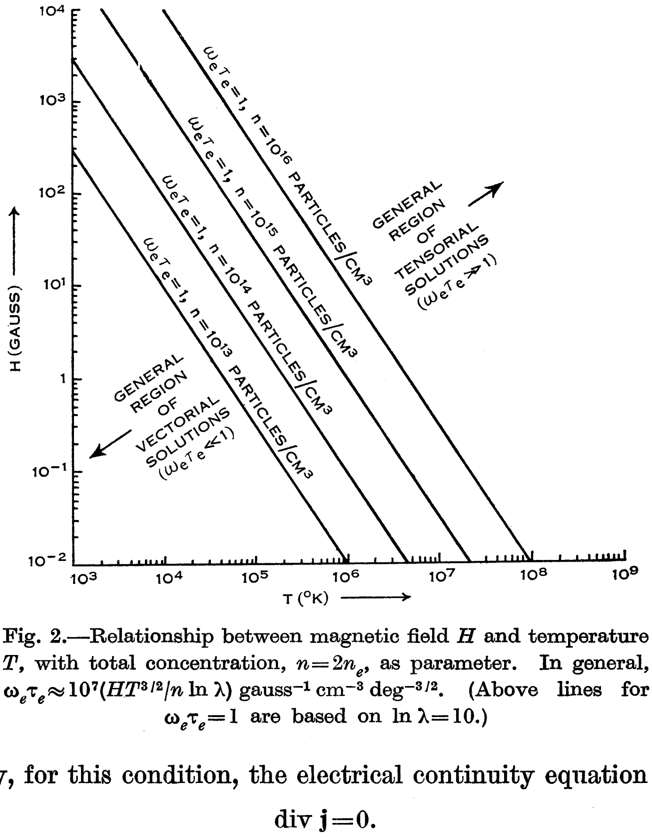

Fig. 2.—Relationship between magnetic field H and temperature T, with total concentration, $n=2n_e$, as parameter. In general, $\omega_e\tau_e \approx 10^7(HT^{3/2}/n \ln \lambda)$ gauss^{-1} cm^{-3} deg$^{-3/2}$. (Above lines for $\omega_e\tau_e=1$ are based on $\ln \lambda=10$.)

Similarly, for this condition, the electrical continuity equation reduces to

$$\text{div } \mathbf{j}=0. \tag{5.4}$$

For fully ionized deuterium, we can from Table 1 and equation (4.14) obtain expressions showing explicitly the temperature dependence of σ and K. However, it is of interest to compare σ and K for an actual gas with their values in a Lorentz gas, and so we choose to follow Spitzer and Härm (1953), by writing

$$\sigma=2\left(\frac{2}{\pi}\right)^{3/2}\frac{k^{3/2}T^{3/2}}{m_e^{1/2}e^2 \ln \lambda}\gamma_E, \tag{5.5}$$

and

$$K=20\left(\frac{2}{\pi}\right)^{3/2}\frac{k^{7/2}T^{5/2}}{m_e^{1/2}e^4 \ln \lambda}\delta_T, \tag{5.6}$$

where the charge on the electron, e, has been taken in e.s.u. Here σ and K for an actual gas have been expressed in terms of their values in a Lorentz gas by means of the transport coefficients γ_E and δ_T, given by Spitzer and Härm as

$\gamma_E=0\cdot5816$, $\delta_T=0\cdot2252$. (Spitzer and Härm's σ and K agree closely with those of Marshall.)

Writing (5.5) as

$$\sigma=\sigma_0(T^{3/2}/\ln\lambda), \tag{5.7}$$

and (5.6) as

$$K=K_0(T^{5/2}/\ln\lambda), \tag{5.8}$$

we obtain by inserting numerical values

$$\sigma_0=1\cdot53\times10^{-4}\ \text{ohm}^{-1}\ \text{cm}^{-1}\ \text{deg}^{-3/2}, \tag{5.9}$$

and

$$K_0=4\cdot396\times10^{-12}\ \text{joule s}^{-1}\ \text{cm}^{-1}\ \text{deg}^{-7/2}. \tag{5.10}$$

In general, expression (5.2) for the bremsstrahlung loss has a dependence on Z^3, so that the loss due to free-free transitions can assume significant proportions if the plasma becomes contaminated by high-Z elements; such contamination may well occur if the temperature at each electrode is allowed to become too high. However, if the length of the discharge is of order L, and the central temperature is T_m, then when $Z=1$ we find from (4.23) and (5.8)

$$\operatorname{div}\mathbf{q}\sim-\frac{K_0}{L^2}\frac{T_m^{7/2}}{\ln\lambda}. \tag{5.11}$$

Hence, using (5.2) and (5.10),

$$\frac{|\operatorname{div}\mathbf{q}|}{P}\sim3\times10^{21}\frac{T_m^3}{L^2n_i^2}\ \text{cm}^{-4}\ \text{deg}^{-3}. \tag{5.12}$$

For $T_m=10^5$ °K, $n_i=10^{15}$ ions/cm^3, $L=20$ cm, (5.12) gives $|\operatorname{div}\mathbf{q}|/P\sim7\cdot5\times10^3$. We therefore conclude that for conditions of interest, neglect of the bremsstrahlung loss compared with the heat conduction loss to the electrodes in equation (5.1) does not appear to involve significant error provided the discharge is not too long. With the reasonable assumption that perfect thermal insulation exists at the plasma boundary surface when it is pinched away from the walls of the discharge tube during the containment period, we conclude that Joule heating of the plasma is balanced by heat conduction to the electrodes while the steady-state condition prevails: the plasma energy equation is now

$$\operatorname{div}\mathbf{q}=\mathbf{j}\cdot\mathbf{E}. \tag{5.13}$$

Using equations (5.3) and (5.4), equation (5.13) can be written

$$\operatorname{div}\mathbf{M}=0, \tag{5.14}$$

where

$$\mathbf{M}=\mathbf{q}+V\mathbf{j}. \tag{5.15}$$

VI. Solutions of the Plasma Energy Equation

The analysis of a general, constricted discharge having symmetry about a median plane is conveniently carried through by introduction of an orthogonal curvilinear coordinate system, which can be specialized later to deal with the cases of non-constriction, and constriction corresponding to various practical conditions. However, before considering these cases, we can gain physical insight into the problem by considering a curved stream tube of a discharge constricted in the median plane and having smoothly increasing cross section towards the electrodes.

(a) Curved Stream Tube of a Constricted Discharge

Figure 3 shows a curved stream tube having symmetry about a median plane. Considering the semi-tube above the median plane, we see that on the median plane the total heat flux entering the tube at A is Q_0, and the total electric current entering at A is I. The electric potential on the median plane is assumed constant at V_0. Emerging from the tube at B is a total heat flux Q. Equation (5.4) and the definition of a stream tube (see, for example, Milne-Thomson 1955) ensure that the total electric current leaving the tube at B is I. The electric potential on the end of the tube at B is assumed constant at $V<V_0$.

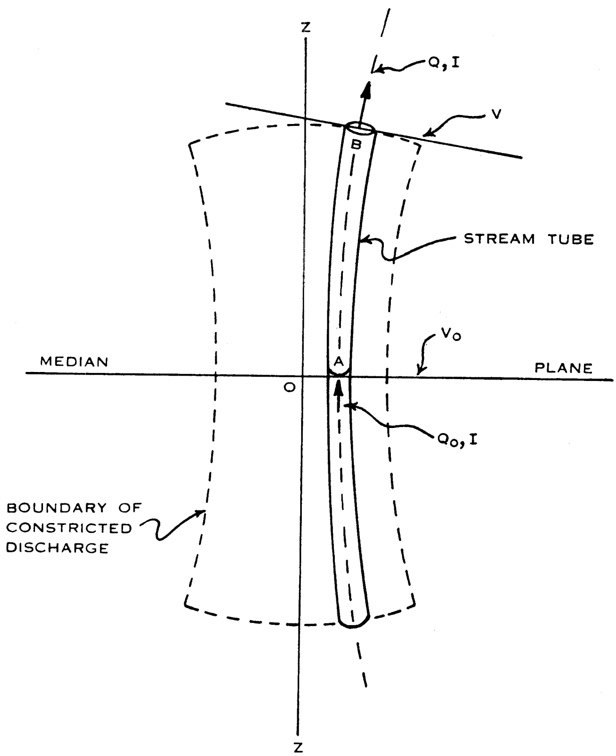

Fig. 3.—Curved stream tube of a constricted discharge.

For the steady-state condition the Joule heating, $(V_0-V)I$, must be balanced by the net heat outflow, $Q-Q_0$, and so

$$(V_0-V)I=Q-Q_0. \tag{6.1.1}$$

In the absence of thermoelectric effects it can be reasonably assumed that on the median plane $V_0=Q_0=0$, and (6.1.1) then reduces to

$$\mathbf{q}+V\mathbf{j}=0, \tag{6.1.2}$$

where we have returned to current and heat flux densities.

Formally, we can proceed as follows. Referring again to Figure 3, we can, in the absence of thermoelectric effects, assume a symmetry which leads to

$V_0=0$, $\partial T/\partial z=0$ on the median plane. Hence, if it is further assumed that on the median plane the temperature is everywhere constant, and equal to the maximum temperature T_m, then, with reference to equation (5.15), we see that $\mathbf{M}$ vanishes on the median plane. Thus, bearing in mind that the divergence of $\mathbf{M}$ is also zero (*vide* equation (5.14)), it follows that at every point within the tube, and within any similar tube contained by the constricted discharge, $\mathbf{M}=0$, or, using (5.15),

$$\mathbf{q}+V\mathbf{j}=0. \tag{6.1.3}$$

Clearly, this important result does not depend on the shape of the stream tubes, provided there is a median plane and that $\mathbf{q}$ and $\mathbf{j}$ at each point have the same direction, and since this type of heat and electricity flow is of main interest here, we give it the special name of longitudinal flow. Where the discharge has no external magnetic field acting upon it, and is non-constricted, the flow is expected to be approximately longitudinal owing to the assumed thermal insulation of the boundary surface mentioned in Section V. Where a strong external guiding magnetic field is used to constrict the discharge, both electric current and heat flow along the magnetic field lines, and this approximation holds even more closely.

Using (4.22) and (4.23) for $\mathbf{j}$ and $\mathbf{q}$ when $\omega_e\tau_e \ll 1$, together with (5.7) and (5.8), we obtain from (6.1.3)

$$T^2+(\sigma_0/K_0)V^2=T_m^2, \tag{6.1.4}$$

which shows that for longitudinal flow T is a function of V only. Inserting in (6.1.4) the values given by (5.9) and (5.10),

$$(T_m^2-T^2)^{1/2}=5899V \text{ deg V}^{-1}. \tag{6.1.5}$$

Similarly, for longitudinal flow the relationship between T and V when $\omega_e\tau_e \gg 1$ will be obtained in Section VI (*e*). As might be expected, it turns out to be the same as found here.

(*b*) *Use of the Stream Function* Ψ *and the Electric Potential* V *as Orthogonal Curvilinear Coordinates* ($\omega_e\tau_e \ll 1$)

It is mathematically convenient to concentrate first on axi-symmetric discharges for which $\omega_e\tau_e \ll 1$, accepting the fact that generally magnetic fields may not be very large.

In the case of steady axi-symmetric flow of an incompressible fluid, for which div $\mathbf{v}=0$, where $\mathbf{v}$ is the velocity of a fluid particle, it is helpful to analyse in terms of a stream function (Stokes 1842).

Similarly, since from (5.4) the vector $\mathbf{j}$ is solenoidal, it is found advantageous here to introduce an electric stream function Ψ, such that

$$I(P)=2\pi\Psi(P), \tag{6.2.1}$$

where $I(P)$ is the total electric current through a surface generated by rotation of a line NP, not necessarily an equipotential, about the symmetry axis OZ

in Figure 4. If P moves along a line LQ such that $\Psi(P)=\text{const.}$, LQ is by definition a current stream line, everywhere tangent to $\mathbf{j}$. Rotation of LQ about OZ results in a stream surface $\Psi(P)=(1/2\pi)I(P)=\text{const.}$ Clearly, the stream function vanishes on the axis of symmetry. By integration,

$$I(P)=2\pi\int_0^{s(P)} \rho j_n \mathrm{d}s, \tag{6.2.2}$$

where, with reference to Figure 4, ρ is a cylindrical coordinate, s is arc length, measured along NP from the axis of symmetry, j_n is the component of $\mathbf{j}$ normal to ds.

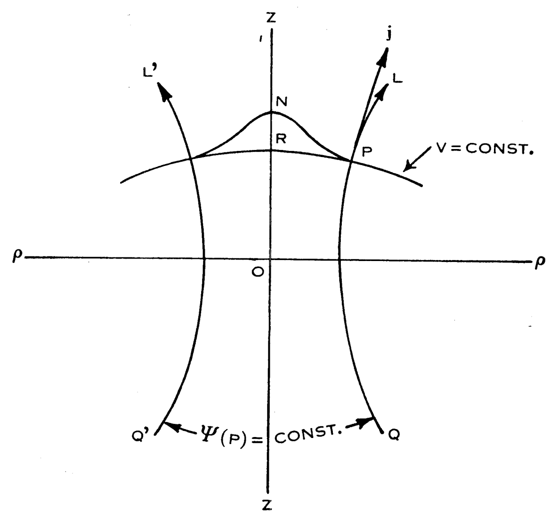

Fig. 4.—V and Ψ as orthogonal curvilinear coordinates (axi-symmetric flow).

From (6.2.1) and (6.2.2),

$$j_n=\frac{1}{\rho}\,\frac{\partial \Psi}{\partial s}, \tag{6.2.3}$$

which confirms that, when the electric stream function is constant and the element of arc lies along a stream line, there is no flow of electricity across the corresponding stream surface. If in (6.2.3) the element of arc is taken to lie along an equipotential line, such as RP in Figure 4, j_n becomes j, the magnitude of $\mathbf{j}$, and

$$I(P)=2\pi\int_0^{s(P)} \rho j \mathrm{d}s \quad (V=\text{const.}). \tag{6.2.4}$$

Since axi-symmetric flow is being examined, it is possible to write the energy equation (5.13) in a form which proves to be initially useful by introducing as curvilinear coordinates Ψ, φ, and V. Noting from (4.22) that $\mathbf{E}$ has the direction of $\mathbf{j}$, we see from (5.3) that the equipotential surfaces $V=\text{const.}$, generated by rotation of lines such as RP through $\varphi=2\pi$ about OZ, are orthogonal to the streamlines and stream surfaces $\Psi=$const. With E as the magnitude of $\mathbf{E}$, we note from Figure 4 that the scale factors are

$$\left.\begin{aligned} h_1&=\left|\frac{\partial s}{\partial \Psi}\right|=\frac{1}{\rho j},\\ h_2&=\left|\frac{\partial s}{\partial \varphi}\right|=\rho,\\ h_3&=\left|\frac{\partial s}{\partial V}\right|=\frac{1}{E},\end{aligned}\right\} \tag{6.2.5}$$

and so grad T has components

$$\operatorname{grad} T=\left(\rho j\frac{\partial T}{\partial \Psi},\ 0,\ E\frac{\partial T}{\partial V}\right). \tag{6.2.6}$$

Referring to Weatherburn (p. 16) and equations (4.22), (4.23), (5.7), (5.8), (6.2.5), and (6.2.6), we can write the plasma energy equation (5.13) in terms of Ψ and V (the physical variables are not dependent on φ because the flow is axi-symmetric) as follows :

$$\frac{1}{5}\frac{K_0\sigma_0}{(\ln\lambda)^2}\frac{\partial}{\partial\Psi}\left(\rho^2\frac{\partial(T^5)}{\partial\Psi}\right)+\frac{1}{2}\frac{K_0}{\sigma_0}\frac{\partial}{\partial V}\left(\frac{\partial(T^2)}{\partial V}\right)=-1. \tag{6.2.7}$$

Two families of analytic solutions of this non-linear equation have been found. One, in Section VI (*d*) on, completes the discussion for longitudinal flow ; the other, which for completeness is briefly presented in Section VI (*c*), represents flow with $\mathbf{q}$ perpendicular to $\mathbf{j}$. Otherwise, for given boundary conditions numerical solutions of (5.13), and hence (6.2.7), could be obtained.

(*c*) *Solution of the Energy Equation when the Heat Flow is everywhere Perpendicular to the Flow of Electricity* ($\omega_e\tau_e\ll 1$)

In this case T is a function of Ψ only, (6.2.6) becomes

$$\operatorname{grad} T=(\rho j\partial T/\partial\Psi,\ 0,\ 0),$$

and (6.2.7) reduces to

$$\frac{1}{5}\frac{K_0\sigma_0}{(\ln\lambda)^2}\frac{\partial}{\partial\Psi}\left(\rho^2\frac{\partial(T^5)}{\partial\Psi}\right)=-1. \tag{6.3.1}$$

Integration gives

$$T^5(\Psi)-T^5(0)=-5\frac{(\ln\lambda)^2}{K_0\sigma_0}\int_0^{\Psi}\frac{\Psi'}{\rho^2}\mathrm{d}\Psi'. \tag{6.3.2}$$

In general ρ is a function of Ψ and V. However, inspection of (6.3.2) shows that ρ can only be a function of Ψ, otherwise the equation is inconsistent. Hence, by inversion

$$\Psi=\Psi(\rho). \tag{6.3.3}$$

The physical interpretation of (6.3.3) is simply that the current stream lines are straight, and parallel to OZ. The heat flow is perpendicular to the electric stream lines, and so is purely radial. This requires heat flow across the free boundary surface of the plasma. Since we are concerned with heat flow to the electrodes, this solution is not considered further in this paper. However, it is still possible to have purely radial temperature gradients if bremsstrahlung is included, and for this case Pease (1957) has obtained the stationary-state radial distribution functions and current-voltage characteristics of a bremsstrahlung-cooled, axially symmetric, pinched discharge generated in a long, straight tube, in which end effects can be neglected.

(*d*) *Solution of the Energy Equation for Longitudinal Flow* ($\omega_e\tau_e \ll 1$)

As was seen in Section VI (*a*), the flow for T a function of V only is longitudinal. For this important type of flow, (6.2.6) becomes grad $T=(0, 0, E\partial T/\partial V)$, and the energy equation (6.2.7) reduces to

$$\frac{1}{2}\frac{K_0}{\sigma_0}\frac{\partial}{\partial V}\left(\frac{\partial(T^2)}{\partial V}\right)=-1. \tag{6.4.1}$$

Integration gives

$$T^2+(\sigma_0/K_0)V^2=T_m^2, \tag{6.4.2}$$

in agreement with (6.1.4). Rearranging,

$$V=\pm(K_0/\sigma_0)^{1/2}(T_m^2-T^2)^{1/2}. \tag{6.4.3}$$

It is convenient to take $V>0$ for $z<0$, so that the flow of electric current is upward in Figure 4. Thus, in (6.4.3) and following equations the upper sign refers to $z<0$ and the lower sign to $z>0$.

Introducing now more general orthogonal curvilinear coordinates w, v, and u, we assume henceforth that V and T are functions of u only. It thus follows that the only non-zero component of $\mathbf{j}$ is j_u, and from equations (4.22), (5.3), (5.7), and (6.4.3),

$$j_u=\pm\frac{(K_0\sigma_0)^{1/2}}{\ln\lambda}\frac{T^{5/2}}{(T_m^2-T^2)^{1/2}}\frac{1}{h_3}\frac{\partial T}{\partial u}, \tag{6.4.4}$$

or

$$j_u=(1/h_3)g(u). \tag{6.4.5}$$

From the continuity equation (5.4), we obtain for axial symmetry

$$j_u=f(w)/h_1h_2. \qquad (6.4.6$$

Combination of (6.4.4) and (6.4.6) results in the useful equation

$$f(w)\int_0^u\frac{h_3}{h_1h_2}\mathrm{d}u'=\pm\frac{(K_0\sigma_0)^{1/2}}{\ln\lambda}\int_{T_m}^{T}\frac{T'^{5/2}\mathrm{d}T'}{(T_m^2-T'^2)^{1/2}}, \tag{6.4.7}$$

which yields an immediate solution if the variables in $h_3/(h_1h_2)$ are separable.

We shall now consider two axi-symmetric discharges having boundary surfaces of prescribed shape.

Case 1.—*Streamlines parallel to the Axis of Symmetry*

In this case we consider a discharge between parallel, circular electrodes of radius ρ_2, separated by a distance $2z_e$ ($z_e>0$). The plasma is contained within a cylindrical boundary surface $\rho=\rho_1$, $\rho_1<\rho_2$. The cross section in the ρ–z plane is shown in Figure 5 (*a*).

Using cylindrical coordinates, with $w=\rho$, $v=\varphi$, and $u=z$, we have $h_1=h_3=1$, $h_2=\rho$, and $V=V(z)$, $T=T(z)$, $\Psi=\Psi(\rho)$. In the ρ–z plane the streamlines are parallel to OZ and the equipotential lines are normal to OZ. Using the above scale factors, (6.4.5) and (6.4.6) yield

$$j_z=g(z)=f(\rho)/\rho=\text{const.} \tag{6.4.8}$$

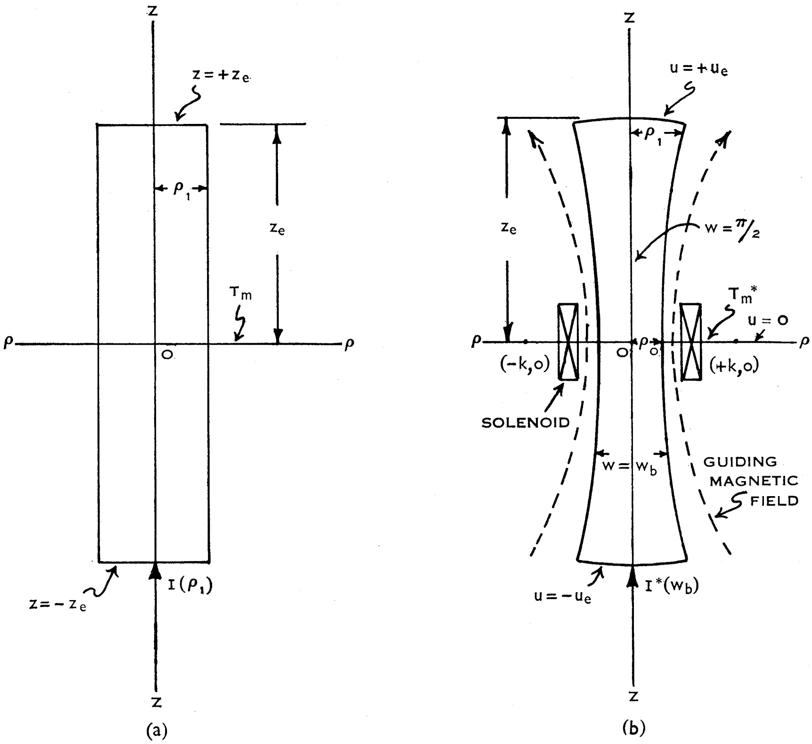

Fig. 5.—Discharge cross sections in the ρ-z plane. (a) Streamlines parallel to OZ; (b) streamlines curved.

Using this result, (6.2.4) gives

$$I(\rho)=\pi\rho^2 j_z, \tag{6.4.9}$$

and (6.2.1)

$$\Psi(\rho)=\frac{1}{2}\rho^2 j_z. \tag{6.4.10}$$

Introducing a variable $\theta=\cos^{-1}(T/T_m)$, and again using the above scale factors, together with the constant $j_z=f(\rho)/\rho$, (6.4.7) becomes in this case

$$z=\mp\frac{(K_0\sigma_0)^{1/2}}{\ln\lambda}\,\frac{T_m^{5/2}}{j_z}\int_0^{\theta}\cos^{5/2}\theta'd\theta'. \tag{6.4.11}$$

With $z_0>0$, let $2z_0$ be the length which the discharge of central temperature T_m would have if the electrodes were at zero temperature. Then, using Wallis's integral $W_{5/2}=\int_0^{\pi/2}\cos^{5/2}\theta d\theta=\frac{1}{2}\pi^{1/2}\frac{\Gamma(7/4)}{\Gamma(9/4)}\approx 0\cdot 72$ (Franklin 1944; Dwight 1957 for a table of the gamma function), (6.4.11) gives

$$\frac{z}{z_0}=\mp\frac{1}{0\cdot 72}\int_0^{\theta}\cos^{5/2}\theta'\mathrm{d}\theta'. \tag{6.4.12}$$

Numerical integration of (6.4.12) results in Figure 6, which shows T/T_m plotted against z/z_0 for this discharge.

If the electrodes are at some finite temperature $T(\mp z_e)=T_e=T_m\cos\theta_e$, the corresponding ratio z_e/z_0 may be read off Figure 6, and so z_0 is determined in terms of the known electrode separation $2z_e$. Then Figure 6 gives the variation of temperature, normalized to T_m, with respect to distance along the discharge,

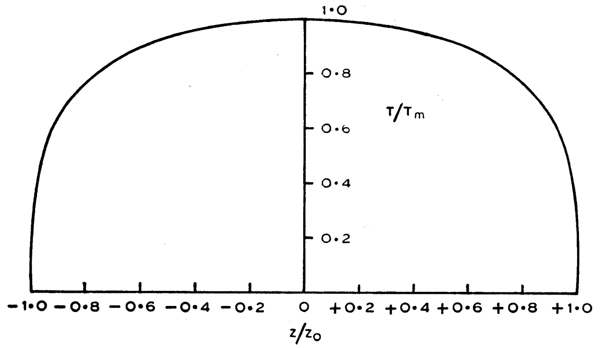

Fig. 6.—Temperature *v.* distance characteristic for non-constricted cylindrical discharge of circular cross section.

normalized to z_0. Actually, z_e and z_0 differ by less than 5% if $T_e<\frac{1}{2}T_m$. Since the condition of interest is $T_e\ll T_m$, we can thus obtain some useful approximate results when the temperature of the electrodes is non-zero. The central temperature is derived from (6.4.11) as

$$T_m=\left(\frac{z_e j_z\ln\lambda}{(K_0\sigma_0)^{1/2}\int_0^{\theta_e}\cos^{5/2}\theta\mathrm{d}\theta}\right)^{2/5}\approx\left(\frac{z_e j_z\ln\lambda}{0\cdot 72(K_0\sigma_0)^{1/2}}\right)^{2/5}. \tag{6.4.13}$$

Inserting into this result the numerical values given by (5.9) and (5.10), and taking $\ln\lambda=10$,

$$T_m\approx 1962\left(\frac{z_e I(\rho_1)}{\rho_1^2}\right)^{2/5}\mathrm{A}^{-2/5}\,\mathrm{cm}^{2/5}\,\mathrm{deg}, \tag{6.4.14}$$

where $I(\rho_1)=\pi\rho_1^2 j_z$, the total current carried by the discharge, has been introduced by means of (6.4.9).

Similarly, (6.4.3) gives

$$T_m\approx 5899V_e\ \mathrm{V}^{-1}\,\mathrm{deg}, \tag{6.4.15}$$

and so, from (6.4.14) and (6.4.15),

$$V_e\approx 0\cdot 333\left(\frac{z_e I(\rho_1)}{\rho_1^2}\right)^{2/5}\mathrm{A}^{-2/5}\,\mathrm{cm}^{2/5}\,\mathrm{V}, \tag{6.4.16}$$

where $V_e=V(-z_e)=-V(+z_e)$.

From (6.4.14) it is seen that for given z_e and $I(\rho_1)$, T_m increases as ρ_1 decreases, as would be physically expected.

This case has been studied by Kaufman and Furth (1958) and Haines (1960).

Case 2.—Streamlines Curved

Suppose that the "linear" discharge of Case 1 is modified by the effect of a magnetic field, produced, say, by a short solenoid with axis OZ, situated symmetrically about $z=0$. Here we assume the constriction to be produced by fields compatible with $\omega_e\tau_e \ll 1$, and expect the streamlines to be shaped as in Figure 5 (*b*). To obtain a solution which should reasonably represent the practical situation, let us approximate the current streamlines by hyperbolae. Mathematically we suppose that the coordinates u and w (with $V=V(u)$ and $\Psi=\Psi(w)$ as before) are related to ρ and z by the conformal transformation

$$\rho+iz=k\cosh(u+iw),\quad k \text{ defined below.} \tag{6.4.17}$$

Expansion gives

$$\rho=k\cosh u\cos w, \tag{6.4.18}$$

and

$$z=k\sinh u\sin w, \tag{6.4.19}$$

and so

$$\frac{\rho^2}{k^2\cosh^2 u}+\frac{z^2}{k^2\sinh^2 u}=1, \tag{6.4.20}$$

$$\frac{\rho^2}{k^2\cos^2 w}-\frac{z^2}{k^2\sin^2 w}=1. \tag{6.4.21}$$

In the ρ–z plane the curves $u=$const., $w=$const. form a family of confocal ellipses and hyperbolae, with common foci at $(\pm k, 0)$. Hence the equipotential surfaces $V(u)=$const. are ellipsoids, and the streamlines $\Psi(w)=$const. are hyperbolae. For this discharge the boundary surface is a hyperboloid of one sheet, where we take $w_b>0$, and $w=\frac{1}{2}\pi$ on the axis OZ. Since T and V are functions of u, the heat and electricity flow along the hyperbolic streamlines: the flow is longitudinal, as required.

For the conformal transformation (6.4.17) the scale factors are

$$h_1=h_3=k(\sinh^2 u+\sin^2 w)^{1/2}, \tag{6.4.22}$$

and

$$h_2=k\cosh u\cos w. \tag{6.4.23}$$

The total current through any surface of constant potential and temperature intersecting the free boundary surface of the discharge is obtained from (6.2.4) as

$$I^*(w_b)=2\pi\int_0^{s(b)} j_u h_2 \mathrm{d}s. \tag{6.4.24}\dagger$$

Thus, using (6.4.6) and $\mathrm{d}s=-h_1\mathrm{d}w$, this becomes

$$I^*(w_b)=2\pi\int_{w_b}^{\frac{1}{2}\pi} f(w)\mathrm{d}w. \tag{6.4.25}$$

† The star superscript is used when necessary to indicate a Case 2 quantity.

Combination of (6.4.5), (6.4.6), (6.4.22), and (6.4.23) leads to

$$g(u)\cosh u=f(w)/k\cos w=A=\text{const.},\tag{6.4.26}$$

a result which permits integration of (6.4.25) to give

$$I^*(w_b)=2\pi kA(1-\sin w_b).\tag{6.4.27}$$

Inspection of Figure 5 (*b*) gives, with the aid of (6.4.18) and (6.4.19),

$$\left.\begin{aligned}\rho_0&=k\cos w_b,\\ \rho_1&=k\cosh u_e\cos w_b,\\ z_e&=k\sinh u_e,\end{aligned}\right\}\tag{6.4.28}$$

where $u=\mp u_e$ gives the electrodes.

Equations (6.4.28) yield

$$\sinh u_e=[(\rho_1/\rho_0)^2-1]^{1/2},\tag{6.4.29}$$

$$z_e=k[(\rho_1/\rho_0)^2-1]^{1/2},\tag{6.4.30}$$

$$\sin w_b=[1-(\rho_1/z_e)^2(\rho_1/\rho_0)^{-2}\{(\rho_1/\rho_0)^2-1\}]^{1/2}.\tag{6.4.31}$$

In this case use of (6.4.7), (6.4.22), (6.4.23), and the variable $\theta=\cos^{-1}(T/T_m^*)$ gives

$$\int_0^u\frac{du'}{\cosh u'}=\frac{\mp(K_0\sigma_0)^{1/2}T_m^{*\,5/2}}{\ln\lambda\{f(w)/k\cos w\}}\int_0^\theta\cos^{5/2}\theta'd\theta'.\tag{6.4.32}$$

Integrating the left-hand side, and using (6.4.26) and (6.4.27)

$$\tan^{-1}(\sinh u)=\frac{\mp 2\pi k(K_0\sigma_0)^{1/2}(1-\sin w_b)T_m^{*\,5/2}}{\ln\lambda I^*(w_b)}\int_0^\theta\cos^{5/2}\theta'd\theta',\tag{6.4.33}$$

or

$$T_m^*=\left(\frac{\mp\ln\lambda I^*(w_b)\tan^{-1}(\sinh u)}{2\pi k(K_0\sigma_0)^{1/2}(1-\sin w_b)\int_0^\theta\cos^{5/2}\theta'd\theta'}\right)^{2/5}.\tag{6.4.34}$$

By considering $u=\mp u_e$ at the electrodes, where $T_e=T(\mp u_e)=T_m^*\cos\theta_e$, we can use (6.4.29) to (6.4.31) to write (6.4.34) in the form

$$T_m^*=\left(\frac{\ln\lambda I^*(w_b)(\nu^2-1)^{1/2}\tan^{-1}(\nu^2-1)^{1/2}}{2\pi z_e(K_0\sigma_0)^{1/2}\{1-[1-(\rho_1/z_e)^2\nu^{-2}(\nu^2-1)]^{1/2}\}\int_0^{\theta_e}\cos^{5/2}\theta d\theta}\right)^{2/5},\tag{6.4.35}$$

where $\nu=\rho_1/\rho_0$ is defined as the radial compression ratio. As ν approaches unity, (6.4.35) reduces to the exact form of (6.4.13), Case 1. Again, for $T_e\ll T_m^*$,

$$T_m^*\approx\left(\frac{\ln\lambda\, I^*(w_b)(\nu^2-1)^{1/2}\tan^{-1}(\nu^2-1)^{1/2}}{(2)(0\cdot 72)\pi z_e(K_0\sigma_0)^{1/2}\{1-[1-(\rho_1/z_e)^2\nu^{-2}(\nu^2-1)]^{1/2}\}}\right)^{2/5}.\tag{6.4.36}$$

Again referring to (5.9) and (5.10), and taking $\ln\lambda=10$, (6.4.36) becomes

$$T_m^*\approx 1487\left(\frac{z_eI^*(w_b)}{\rho_1^2}\right)^{2/5}\left(\frac{(\rho_1/z_e)^2(\nu^2-1)^{1/2}\tan^{-1}(\nu^2-1)^{1/2}}{1-[1-(\rho_1/z_e)^2\nu^{-2}(\nu^2-1)]^{1/2}}\right)^{2/5}\text{A}^{-2/5}\,\text{cm}^{2/5}\,\text{deg}.\tag{6.4.37}$$

This is a cumbersome expression, and it is fortunate that we can take advantage of the inequality $(\rho_1/z_e)^2 \lesssim 0\cdot 04 \ll 1$, which is in accord with typical physical conditions, and which permits simplification of (6.4.37) to the form

$$T_m^* \approx 1962\left(\frac{z_e I^*(w_b)}{\rho_1^2}\right)^{2/5}\left(\frac{\nu^2 \tan^{-1}(\nu^2-1)^{1/2}}{(\nu^2-1)^{1/2}}\right)^{2/5} \mathrm{A}^{-2/5}\,\mathrm{cm}^{2/5}\,\mathrm{deg}. \tag{6.4.38}$$

In this case (5.9), (5.10), and (6.4.3) give

$$T_m^* \approx 5899 V_e^* \;\mathrm{V}^{-1}\,\mathrm{deg}, \tag{6.4.39}$$

when $T_e \ll T_m^*$, and so, from (6.4.38) and (6.4.39),

$$V_e^* \approx 0\cdot 333\left(\frac{z_e I^*(w_b)}{\rho_1^2}\right)^{2/5}\left(\frac{\nu^2 \tan^{-1}(\nu^2-1)^{1/2}}{(\nu^2-1)^{1/2}}\right)^{2/5} \mathrm{A}^{-2/5}\,\mathrm{cm}^{2/5}\,\mathrm{V}, \tag{6.4.40}$$

where $V_e^* = V(-u_e) = -V(+u_e)$.

However, it should be recalled at this stage that the above results apply when $\omega_e\tau_e \ll 1$ and the discharge constriction is not pronounced, so that ν is small. We now examine the case of a well-constricted discharge.

(e) *Solution of the Energy Equation for Longitudinal Flow* ($\omega_e\tau_e \gg 1$)

Here it is supposed that a strong external guiding magnetic field is used to produce a well-constricted discharge in which both electric current and heat flow along the magnetic field lines, so that the magnetic field lines generate flux tubes which coincide with the current stream tubes of the type shown in Figure 3, and the flow is longitudinal. With $\omega_e\tau_e \gg 1$, (5.15) can be written by means of (4.28) and (4.29) as

$$\mathbf{M} = -K(\nabla T)^{\|} + (\sigma \mathbf{E}^{\|} + \tfrac{1}{2}\sigma \mathbf{E}^{\perp})V. \tag{6.5.1}$$

Using the formal analysis of Section VI (*a*) as a guide, we assume $\mathbf{M}=0$ and see what this implies later. Then (6.5.1) gives

$$\mathbf{E}^{\perp} = 0, \quad (\sigma V \neq 0), \tag{6.5.2}$$

and

$$\sigma V \mathbf{E}^{\|} = K(\nabla T)^{\|}, \tag{6.5.3}$$

or, writing $\mathbf{E}^{\|} = -(\nabla V)^{\|}$, and using (5.7) and (5.8)

$$(\nabla[T^2 + (\sigma_0/K_0)V^2])^{\|} = 0. \tag{6.5.4}$$

Integrating (6.5.4), and applying the result to the axis of the discharge, we obtain the familiar form,

$$T^2 + (\sigma_0/K_0)V^2 = T_m^2, \tag{6.5.5}$$

from the symmetry about the median plane.

The assumption $\mathbf{M}=0$ here leads to $\mathbf{E}=\mathbf{E}^{\|}$ (from (6.5.2)), but imposes no particular condition on $(\nabla T)^{\perp}$. Nevertheless, $(\nabla T)^{\perp}$ is likely to be zero in view of our original approximations of perfect thermal insulation at the free boundary surface, and neglect of bremsstrahlung loss from the plasma, so that $\nabla T = (\nabla T)^{\|}$. Indeed, if the electron mean free path and gyroradius are not small compared with the average diameter of the discharge (and this may well be the case unless the gas is dense and the total confining magnetic field extremely strong), no mechanism exists that can give rise to a radial temperature gradient.

Since $\mathbf{E}=\mathbf{E}^{\|}$, it follows that for this solution the total electric field must be shaped to follow the streamlines defined by the magnetic field. Then

$$\mathbf{j}=\sigma\mathbf{E}, \tag{6.5.6}$$

$$\mathbf{q}=-K\nabla T, \tag{6.5.7}$$

and the solution proceeds as in Section VI (d), Case 2, where $\omega_e\tau_e \ll 1$. In fact, for strictly longitudinal flow, with $\mathbf{q}$ and $\mathbf{j}$ parallel to $\mathbf{H}$ at every point, the results obtained in Section VI (d), such as (6.4.38) and (6.4.40), must hold for all values of $\omega_e\tau_e$ (that is, for all values of total H), and hence for all values of the radial compression ratio ν. However, it is to be noted that strictly longitudinal flow for intermediate values of $\omega_e\tau_e$ may be difficult to achieve in practice.

The case $\omega_e\tau_e \gg 1$ clearly can arise if the self-magnetic field of the discharge becomes large, but, remembering Teller's intuitively-based criterion for determining the stability of a plasma confined by a magnetic field (Bishop 1960), we see that this configuration is unstable, and hence of no interest to us.

However, when there is an external guiding magnetic field, large compared with the self-magnetic field of the discharge, so that the resultant magnetic field is predominantly in the direction of the external field, the case $\omega_e\tau_e \gg 1$ will arise, and Teller's criterion suggests a guiding magnetic field shaped as in Figure 5 (b) for a stable configuration. The ratio of external to self-magnetic field for stability of the discharge is at present being examined.

VII. Comparison of Maximum Temperatures and Discussion of Results

Since we have shown that bremsstrahlung loss from the plasma can be neglected for cases of interest (Section V), the expressions derived in Section VI (d) for T and T^* give upper limits for the central temperature.

Forming now the ratio T^*/T by means of (6.4.14) and (6.4.38), we obtain

$$\frac{T^*}{T}=\varepsilon=\left(\frac{\nu^2\tan^{-1}(\nu^2-1)^{1/2}}{(\nu^2-1)^{1/2}}\right)^{2/5}, \tag{7.1}$$

if $I(\rho_1)=I^*(w_b)$, and the discharges have common ρ_1 and z_e, as in Figures 5 (a) and 5 (b). From the derivation of (6.4.38), it will be understood that (7.1) applies only when $(\rho_1/z_e)^2 \ll 1$.

The dependence of the temperature ratio ε on the compression ratio ν for practical cases of interest can therefore be obtained from (7.1), which is shown plotted in Figure 7 for $1<\nu<20$.

Further, if the discharge resistance between electrodes is R, then from (6.4.16) and (6.4.40) we see that for conditions as above,

$$\varepsilon=R^*/R. \tag{7.2}$$

This result should prove useful when experimental determinations of ε are attempted.

For $\nu=20$, which corresponds to a ratio of 400 for the cross-sectional areas in the median plane before and after constriction, we note from Figure 7 that

$\varepsilon \approx 4$. This low value of ε for large constriction of the discharge can be understood physically by referring to equations (5.7) and (5.8). The electrical conductivity σ increases as $T^{3/2}$; the thermal conductivity K as $T^{5/2}$. Since the Joule heating of the plasma, j^2/σ, is inversely proportional to σ, and the conduction of heat to the electrodes is proportional to K, the increase in central temperature that might be expected from severe constriction of a discharge is offset by the reduced efficiency of Joule heating and the increased efficiency of heat conduction to the electrodes, and the small values of ε given by Figure 7 result as a compromise.

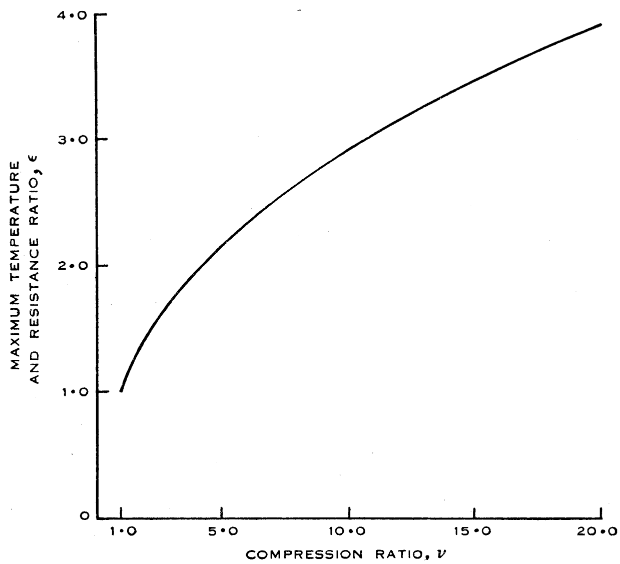

Fig. 7.—Maximum temperature and resistance ratio, ε, *v.* radial compression ratio, ν.

We have seen that for $\omega_e\tau_e \ll 1$, the plasma is isotropic, and that for $\omega_e\tau_e \gg 1$, the plasma possesses extreme anisotropy. For intermediate values of $\omega_e\tau_e$, the plasma will have a state lying between these extremes. Since, in a given case, the total central magnetic field, H_m, and central temperature can be determined, it follows that the numerical value of $\omega_e\tau_e$ (and thus an indication of the state of the plasma) can be obtained from Figure 2 if, in addition, the total concentration, n, is known. Using T_m^*, which reduces to T_m in the non-constricted case, an estimate of n can be obtained by assuming, for non-constricted and constricted discharges, that the kinetic pressure nkT_m^* of the plasma, is comparable to the magnetic pressure $H_m^2/8\pi$ of the total confining field. If here we take H_m in gauss and T_m^* in degrees Kelvin, this yields

$$n = 2 \cdot 88 \times 10^{14} H_m^2 / T_m^* \text{ particles cm}^{-3}, \tag{7.3}$$

which can then be used with the values of H_m and T_m^* to obtain an order of magnitude estimate of $\omega_e\tau_e$ from Figure 2.

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