AVERAGE FORCES IN ELECTROMAGNETIC SYSTEMS

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Summary

The mechanical forces developed in electromagnetic systems are usually evaluated from surface integrals of the Maxwell stresses. For a large class of systems it is shown that the time average force can be expressed in terms of directly measurable circuit-theory parameters. The system must be linear and loss free, but there is no restriction on the frequency of excitation.

I. Introduction

We consider first a capacitor consisting of two parallel plates of separation x. When the capacitor is charged to a voltage V_0 there is an attractive force F_x between the plates. This attractive force may be obtained by finding the electrostatic field distribution and integrating the electric pressure on the plates. However, the force may also be obtained from energy conservation considerations giving

$$F_x=\tfrac{1}{2}V_0^2\partial C/\partial x, \tag{1}$$

where C is the electrical capacitance. F_x is now expressed in terms of the integral electrical parameter C.

For more complicated capacitors, equation (1) may still be used if x is taken as a generalized coordinate specifying the configuration of the capacitor. F_x is then interpreted as the generalized force corresponding to x by the procedure adopted in analytical mechanics.

If the capacitor is connected to an alternating voltage source, the instantaneous force may be calculated from equation (1). Of course, the frequency must be low so that the electric field is described adequately by the equations of electrostatics. The average force, $(F_x)_{\text{av}}$ is given by

$$(F_x)_{\text{av}}=\tfrac{1}{2}(V_0)^2_{\text{av}}\partial C/\partial x. \tag{2}$$

For sinusoidal voltages we may take,

$$V_0=\sqrt{2}.V_\omega \cos(\omega t+\varphi), \tag{3}$$

where V_ω is the r.m.s. voltage. We then have,

$$(F_x)_{\text{av}}=\tfrac{1}{2}V_\omega^2\partial C/\partial x. \tag{4}$$

Equation (4) is also expressible in terms of the susceptance $B(\omega)$ or reactance $X(\omega)$ of the capacitor,

$$(F_x)_{\text{av}}=\frac{1}{2\omega}\,V_\omega^2\,\frac{\partial B}{\partial x}, \tag{5}$$

$$(F_x)_{\text{av}}=\frac{1}{2\omega}\,I_\omega^2\,\frac{\partial X}{\partial x}, \tag{6}$$

where I_ω is the r.m.s. current flowing to the capacitor.

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A similar argument applies for the magnetic forces produced in an inductor L. The energy conservation equation corresponding to (1) is

$$F_x=\tfrac{1}{2}I_0^2\partial L/\partial x, \tag{7}$$

where I_0 is the steady current flowing through the inductor. For low frequency alternating currents, the average force is easily shown to be given by equations (5) and (6).

Equations like (5) and (6) expressing $(F_x)_{av}$ in terms of gross electrical parameters may be obtained for very general electromagnetic systems. In Section II a general result is obtained which relates $(F_x)_{av}$ to field quantities on a mathematical surface enclosing the system. In practical situations this fundamental relation may be reduced to simple equations like (5) and (6). This reduction is carried out in Section III. For a two-terminal element or network, equations (5) and (6) remain true *but the restriction to low frequencies is removed.* The element may in fact have dimensions larger than the wavelength. For multiterminal-pair networks and multiport microwave circuits, generalizations of equations (5) and (6) are obtained. Some possible applications and extensions of the theory are discussed in Section IV.

II. A General Expression for the Average Force

The electromagnetic system is assumed to be linear, loss free, and reciprocal. A closed surface S is taken enclosing the system. The system is assumed to be excited by sinusoidal electric and magnetic fields from outside S. These electric and magnetic fields are not independent; a knowledge of either the electric or magnetic field over S being sufficient to specify the fields inside the system. We suppose that the position or orientation of conductors or of polarizable material within S depends upon a generalized coordinate x. We shall suppose that a small change δx is carried out adiabatically; i.e. sufficiently slowly that all Fourier components of the fields of the system lie close to a single frequency, and the Fourier components of δx are all close to zero frequency. The assumption of differentiability, where required, is also made.

Since the system is loss free and reciprocal the only " true " currents which flow within S are the surface currents of ideal conductors. It is true that non-reciprocal materials with asymmetric conductivity tensors may have non-vanishing current densities without causing energy loss, but we exclude these. The equation of conservation of electric charge, namely,

$$\operatorname{div} \mathbf{J}+\partial\rho/\partial t=0, \tag{8}$$

then shows that the only " true " charges present are either surface charges on the conductors or static charges. We may suppose such static charges to be zero and take all field quantities with sinusoidal time dependence. Maxwell's equations are (in M.K.S. rationalized units):

$$\operatorname{curl} \mathbf{E}_\omega=-j\omega \mathbf{B}_\omega, \tag{9}$$

$$\operatorname{curl} \mathbf{H}_\omega=j\omega \mathbf{D}_\omega, \tag{10}$$

where $\mathbf{E}_\omega$, $\mathbf{H}_\omega$, $\mathbf{D}_\omega$, and $\mathbf{B}_\omega$ are complex field vectors and $j=\sqrt{(-1)}$. For convenience r.m.s. amplitudes are taken for these fields; for example,

$$E_\omega=\mathcal{R}.\mathcal{P}.(\sqrt{2}.\mathbf{E}_\omega e^{j(\omega t+\varphi)}). \tag{11}$$

In addition $\mathbf{D}_\omega$ and $\mathbf{E}_\omega$, $\mathbf{B}_\omega$ and $\mathbf{H}_\omega$ are related by

$$\mathbf{D}_\omega=[\varkappa_\omega].\mathbf{E}_\omega, \tag{12}$$

$$\mathbf{B}_\omega=[\mu_\omega].\mathbf{H}_\omega, \tag{13}$$

where $[\varkappa_\omega]$ and $[\mu_\omega]$ are the second rank tensors of permittivity and permeability. To preserve the linearity of the system, $[\varkappa_\omega]$ and $[\mu_\omega]$ are supposed independent of $\mathbf{E}_\omega$, $\mathbf{H}_\omega$. Further, to ensure that the system is loss free and reciprocal, they are taken real and symmetric.

The introduction of the tensors $[\varkappa_\omega]$ and $[\mu_\omega]$ rather than the more common scalars $\varkappa_\omega$ and μ_ω entails very little additional manipulation and establishes the theory for systems containing anisotropic materials.

Consider a surface integration of the complex Poynting vector $(\mathbf{E}_\omega \times \mathbf{H}_\omega^*)$ carried out over the surface S as the outer boundary, together with the surfaces S_1, S_2, . . . of any ideal conductors enclosed by S. In reckoning the total surface integrals consider the *inward* normal of S and the outward normals of S_1, S_2, . . . as positive, then

$$\int_{S,S_1,S_2\ldots} (\mathbf{E}_\omega \times \mathbf{H}_\omega^*).\mathrm{dA}=-\int_v \operatorname{div}\,(\mathbf{E}_\omega \times \mathbf{H}_\omega^*)\mathrm{d}v, \tag{14}$$

where v is the volume over which the fields extend. Since $\mathbf{E}_\omega$ is normal at the conducting surfaces S_1, S_2, . . ., the only contribution remaining is from S,

$$\int_S (\mathbf{E}_\omega \times \mathbf{H}_\omega^*).\mathrm{dA}=-\int_v \operatorname{div}\,(\mathbf{E}_\omega \times \mathbf{H}_\omega^*)\mathrm{d}v \tag{15}$$

$$=\int_v (\mathbf{E}_\omega.\mathrm{curl}\;\mathbf{H}_\omega^*-\mathbf{H}_\omega^*.\mathrm{curl}\;\mathbf{E}_\omega)\mathrm{d}v \tag{16}$$

$$=\mathrm{j}\omega\int_v (\mathbf{H}_\omega^*.\mathbf{B}_\omega-\mathbf{E}_\omega.\mathbf{D}_\omega^*)\mathrm{d}v \tag{17}$$

(by Maxwell's equations).

Now consider a variation of equation (17) generated by a change δx in the generalized coordinate x,

$$\delta\int_S (\mathbf{E}_\omega \times \mathbf{H}_\omega^*).\mathrm{dA}=\mathrm{j}\omega\delta\int_v (\mathbf{H}_\omega^*.\mathbf{B}_\omega-\mathbf{E}_\omega.\mathbf{D}_\omega^*)\mathrm{d}v$$

$$=\mathrm{j}\omega\int_v \delta(\mathbf{H}_\omega^*.\mathbf{B}_\omega-\mathbf{E}_\omega.\mathbf{D}_\omega^*)\mathrm{d}v+\mathrm{j}\omega\int_{\delta v} (\mathbf{H}_\omega^*.\mathbf{B}_\omega-\mathbf{E}_\omega.\mathbf{D}_\omega^*)\mathrm{d}v. \tag{18}$$

The second integral in equation (18) takes account of any change of the conducting surfaces. For example a paddle may rotate or a sliding piston may shift. Further, $\frac{1}{2}(\mathbf{E}_\omega.\mathbf{D}_\omega^*-\mathbf{H}_\omega^*.\mathbf{B}_\omega)$ is the average pressure acting normally on the surfaces into the volume v. Thus the work $\delta W_\omega^{(S)}$ done by the field components of angular frequency ω on the conducting surfaces during an adiabatic displacement δx is given by

$$\delta W_\omega^{(S)}=\tfrac{1}{2}\int_{\delta v} (\mathbf{H}_\omega^*.\mathbf{B}_\omega-\mathbf{E}_\omega.\mathbf{D}_\omega^*)\mathrm{d}v, \tag{19}$$

and equation (18) becomes

$$\delta\int_S (\mathbf{E}_\omega \times \mathbf{H}_\omega^*).\mathrm{d}\mathbf{A} = 2\mathrm{j}\omega\delta W_\omega^{(S)} + \mathrm{j}\omega\int_v \delta(\mathbf{H}_\omega^*.\mathbf{B}_\omega - \mathbf{E}_\omega.\mathbf{D}_\omega^*)\mathrm{d}v. \tag{20}$$

But from equations (12) and (13), together with the symmetry and reality of $[\varkappa_\omega]$ and $[\mu_\omega]$,

$$\begin{aligned}\delta(\mathbf{E}_\omega.\mathbf{D}_\omega^*) &= \delta\mathbf{E}_\omega.[\varkappa_\omega]^*.\mathbf{E}_\omega^* + \mathbf{E}_\omega.[\delta\varkappa_\omega]^*.\mathbf{E}_\omega^* + \mathbf{E}_\omega.[\varkappa_\omega]^*.\delta\mathbf{E}_\omega^* \\ &= -\mathbf{E}_\omega.[\delta\varkappa_\omega].\mathbf{E}_\omega^* + \mathbf{E}_\omega^*.\delta\mathbf{D}_\omega + \mathbf{E}_\omega.\delta\mathbf{D}_\omega^*,\end{aligned} \tag{21}$$

and

$$\begin{aligned}\delta(\mathbf{H}_\omega^*.\mathbf{B}_\omega) &= \delta\mathbf{H}_\omega^*.[\mu_\omega].\mathbf{H}_\omega + \mathbf{H}_\omega^*.[\delta\mu_\omega].\mathbf{H}_\omega + \mathbf{H}_\omega^*.[\mu_\omega].\delta\mathbf{H}_\omega \\ &= \mathbf{H}_\omega.[\delta\mu_\omega].\mathbf{H}_\omega^* + \delta\mathbf{H}_\omega^*.\mathbf{B}_\omega + \mathbf{B}_\omega^*.\delta\mathbf{H}_\omega.\end{aligned} \tag{22}$$

For mathematical convenience any surfaces of discontinuity on material boundaries are replaced by small regions of large, though finite, gradients of the material constants. This ensures the differentiability implied in the equations.

Equation (20) becomes

$$\begin{aligned}\delta\int_S (\mathbf{E}_\omega \times \mathbf{H}_\omega^*).\mathrm{d}\mathbf{A} = 2\mathrm{j}\omega\delta W_\omega^{(S)} &+ \mathrm{j}\omega\int_v (\mathbf{E}_\omega.[\delta\varkappa_\omega].\mathbf{E}_\omega^* + \mathbf{H}_\omega.[\delta\mu_\omega].\mathbf{H}_\omega^*)\mathrm{d}v \\ &+ \mathrm{j}\omega\int[(\delta\mathbf{H}_\omega^*.\mathbf{B}_\omega - \mathbf{E}_\omega.\delta\mathbf{D}_\omega^*) + (\mathbf{B}_\omega^*.\delta\mathbf{H}_\omega - \mathbf{E}_\omega^*.\delta\mathbf{D}_\omega)]\mathrm{d}v.\end{aligned} \tag{23}$$

But by Maxwell's equations the last integral in equation (23) is

$$\begin{aligned}&\int_v [(\mathbf{E}_\omega.\mathrm{curl}\,\delta\mathbf{H}_\omega^* - \delta\mathbf{H}_\omega^*.\mathrm{curl}\,\mathbf{E}_\omega) + (\delta\mathbf{H}_\omega.\mathrm{curl}\,\mathbf{E}_\omega^* - \mathbf{E}_\omega^*.\mathrm{curl}\,\delta\mathbf{H}_\omega)]\mathrm{d}v \\ &= -\int_v [\mathrm{div}\,(\mathbf{E}_\omega \times \delta\mathbf{H}_\omega^*) - \mathrm{div}\,(\mathbf{E}_\omega^* \times \delta\mathbf{H}_\omega)]\mathrm{d}v \\ &= \int_{S,S_1,S_2\ldots} [(\mathbf{E}_\omega \times \delta\mathbf{H}_\omega^*) - (\mathbf{E}_\omega^* \times \delta\mathbf{H}_\omega)].\mathrm{d}\mathbf{A} \\ &= \int_S [(\mathbf{E}_\omega \times \delta\mathbf{H}_\omega^*) - (\mathbf{E}_\omega^* \times \delta\mathbf{H}_\omega)].\mathrm{d}\mathbf{A},\end{aligned} \tag{24}$$

since elsewhere $\mathbf{E}_\omega$ is normal to the conducting surfaces.

Equation (24) may then be written

$$\int_S (\delta\mathbf{E}_\omega \times \mathbf{H}_\omega^* + \mathbf{E}_\omega^* \times \delta\mathbf{H}_\omega).\mathrm{d}\mathbf{A} = 2\mathrm{j}\omega\delta W_\omega^{(S)} + \mathrm{j}\omega\int_v (\mathbf{E}_\omega.[\delta\varkappa_\omega].\mathbf{E}_\omega^* + \mathbf{H}_\omega.[\delta\mu_\omega].\mathbf{H}_\omega^*)\mathrm{d}v. \tag{25}$$

To deal with the final term in equation (25) we consider the Maxwell stress tensor T_{ij}. In terms of instantaneous field vectors,

$$T_{ij} = E_iD_j - \tfrac{1}{2}\delta_{ij}\mathbf{E}.\mathbf{D} + H_iB_j - \tfrac{1}{2}\delta_{ij}\mathbf{H}.\mathbf{B}. \tag{26}$$

This tensor gives a force density,

$$k_i = \sum_{j=1}^{3} \frac{\partial T_{ij}}{\partial r_j}. \tag{27}$$

For a region free of " true " currents and charges, this becomes

$$k_i = -\tfrac{1}{2}\mathbf{E}.\frac{\partial[\varkappa]}{\partial r_i}.\mathbf{E} - \tfrac{1}{2}\mathbf{H}.\frac{\partial[\mu]}{\partial r_i}.\mathbf{H} + \frac{\partial}{\partial t}(\mathbf{D} \times \mathbf{B}), \tag{28}$$

where the Maxwell equations and the symmetry of $[\varkappa]$ and $[\mu]$ have been used. For sinusoidal fields the average of the momentum reaction term $\partial(\mathbf{D}\times\mathbf{B})/\partial t$ is zero and

$$(k_i)_{\mathrm{av}}=-\tfrac{1}{2}\mathbf{E}_\omega.\frac{\partial[\varkappa_\omega]}{\partial r_i}.\mathbf{E}_\omega^*-\tfrac{1}{2}\mathbf{H}_\omega.\frac{\partial[\mu_\omega]}{\partial r_i}.\mathbf{H}_\omega^*. \tag{29}$$

Further, when the displacement δx is carried out adiabatically this average force may be used for computing the work done by the field as the material is moved.

In addition a torque density $\mathbf{G}$ results from the Maxwell tensor for anisotropic materials,

$$\mathbf{G}=\mathbf{D}\times\mathbf{E}+\mathbf{B}\times\mathbf{H}. \tag{30}$$

For sinusoidal fields,

$$(\mathbf{G})_{\mathrm{av}}=\mathscr{R}.\mathscr{P}.(\mathbf{D}_\omega\times\mathbf{E}_\omega^*+\mathbf{B}_\omega\times\mathbf{H}_\omega^*). \tag{31}$$

The total work done during an adiabatic displacement δx is

$$\delta W_\omega^{(P)}=\int_v(\mathbf{k})_{\mathrm{av}}.\delta\mathbf{r}\mathrm{d}v+\int_v(\mathbf{G})_{\mathrm{av}}.\delta\boldsymbol{\theta}\mathrm{d}v, \tag{32}$$

where $\delta\mathbf{r}$ and $\delta\boldsymbol{\theta}$ are the local displacements and rotations of polarizable material generated by δx.

Using (30) and (31) $\delta W_\omega^{(P)}$ may be written

$$\delta W_\omega^{(P}=\tfrac{1}{2}\int_v\mathbf{E}_\omega.[\delta\varkappa_\omega].\mathbf{E}_\omega^*\mathrm{d}v+\tfrac{1}{2}\int_v\mathbf{H}_\omega.[\delta\mu_\omega].\mathbf{H}_\omega^*\mathrm{d}v, \tag{33}$$

where $[\delta\varkappa_\omega]$ and $[\delta\mu_\omega]$ are changes at fixed points in the system due to both translation and rotation of polarizable material. Material movement is considered as the result of rigid body rotations and translations involving no strain. Under these conditions other proposed stress tensors such as that used by Livens (1929) lead to equation (33), and the choice of the Maxwell tensor is purely one of convenience.

Equation (25) then becomes

$$\int_S(\delta\mathbf{E}_\omega\times\mathbf{H}_\omega^*+\mathbf{E}_\omega^*\times\delta\mathbf{H}_\omega).\mathrm{d}A=2\mathrm{j}\omega(\delta W_\omega^{(S)}+\delta W_\omega^{(P)}) \tag{34}$$

$$=2\mathrm{j}\omega\delta W, \tag{35}$$

where δW is the total work done by the electromagnetic forces during the displacement δx.

δW may be equated to $(F_x)_{\mathrm{av}}\delta x$ where $(F_x)_{\mathrm{av}}$ is the average generalized force resulting from the electromagnetic fields giving,

$$2\mathrm{j}\omega(F_x)_{\mathrm{av}}\delta x=\int_S(\delta\mathbf{E}_\omega\times\mathbf{H}_\omega^*+\mathbf{E}_\omega^*\times\delta\mathbf{H}_\omega).\mathrm{d}\mathbf{A}, \tag{36}$$

which is the fundamental result.

III. Reduction of the Surface Integral

The electric and magnetic fields on S are not independent. A theorem of electromagnetism (Stratton 1941, Sec. 9.2) asserts that a knowledge of either $(\mathbf{E}_\omega\times\mathbf{n})$ or $(\mathbf{H}_\omega\times\mathbf{n})$ over S ($\mathbf{n}$ a unit normal) is sufficient to determine the fields of the whole system. In the surface integral of equation (36) only tangential components of $\mathbf{E}_\omega$ and $\mathbf{H}_\omega$ are involved. We may think of the system as being excited by fictitious surface currents and magnetic currents, and surface charges and magnetic charges (Stratton 1941, Sec. 8.14). By this device the variation δ may be performed with either $(\mathbf{E}_\omega \times \mathbf{n})$ or $(\mathbf{H}_\omega \times \mathbf{n})$ constant to obtain useful specializations of equation (36). The change $\delta \mathbf{H}_\omega$ or $\delta \mathbf{E}_\omega$ then results completely from changes within the system. We may write

$$(F_x)_{\mathrm{av}} = \frac{1}{2\mathrm{j}\omega} \int_S \mathbf{E}_\omega^* \times \left(\frac{\partial \mathbf{H}_\omega}{\partial x}\right)_{(\mathbf{E}_\omega \times \mathbf{n})} . \mathrm{d}\mathbf{A}, \tag{37}$$

or

$$(F_x)_{\mathrm{av}} = \frac{1}{2\mathrm{j}\omega} \int_S \left(\frac{\partial \mathbf{E}_\omega}{\partial x}\right)_{(\mathbf{H}_\omega \times \mathbf{n})} \times \mathbf{H}_\omega^* . \mathrm{d}\mathbf{A}, \tag{38}$$

where $\mathbf{H}_\omega$ is a linear functional of $(\mathbf{E}_\omega \times \mathbf{n})$ and $\mathbf{E}_\omega$ is a linear functional of $(\mathbf{H}_\omega \times \mathbf{n})$, over S. The functionals are determined by the details of the system and depend on x.

The results may also be expressed in terms of the complex power P_ω which is defined as the integral of the complex Poynting vector over the surface, i.e.

$$P_\omega = \int_S \mathbf{E}_\omega \times \mathbf{H}_\omega^* . \mathrm{d}\mathbf{A}, \tag{39}$$

yielding

$$(F_x)_{\mathrm{av}} = \frac{1}{2\mathrm{j}\omega} \left(\frac{\partial P_\omega^*}{\partial x}\right)_{(\mathbf{E}_\omega \times \mathbf{n})}, \tag{40}$$

or

$$(F_x)_{\mathrm{av}} = \frac{1}{2\mathrm{j}\omega} \left(\frac{\partial P_\omega}{\partial x}\right)_{(\mathbf{H}_\omega \times \mathbf{n})}. \tag{41}$$

For many practical systems equations (40) and (41) enable us to make a complete reduction to circuit theory parameters. First, consider a system for which ordinary a.c. circuit theory is applicable. We require that the fields on S are quasi-stationary and that the excitation is provided by n pairs of conductors. Let V_K be the complex r.m.s. voltage of angular frequency ω applied to the Kth terminal-pair and I_K the resulting complex r.m.s. current flowing to the system from this pair. The fields on S are the quasi-stationary fields resulting from the potential differences V_K and the currents I_K. By a well-known result for the flow of energy into the system,

$$P_\omega = \sum_{K=1}^{n} V_K I_K^*. \tag{42}$$

The electric field amplitude over S is maintained constant by

$$V_K = \text{constant}, \quad K = 1, \ldots, n,$$

and equation (40) becomes

$$(F_x)_{\mathrm{av}} = \frac{1}{2\mathrm{j}\omega} \left(\frac{\partial P_\omega^*}{\partial x}\right)_{V_K}. \tag{43}$$

Similarly, the magnetic field condition in equation (41) is satisfied by

$$I_K = \text{constant}, \quad K=1, \ldots, n,$$

and equation (41) becomes

$$(F_x)_{\text{av}} = \frac{1}{2j\omega}\left(\frac{\partial P_\omega}{\partial x}\right)_{I_K}. \tag{44}$$

However, the V_K and I_K are related linearly by admittance and impedance matrices (Guillemin 1953) $Y_{KL}(\omega)$, $Z_{KL}(\omega)$,

$$I_K = \sum_{L=1}^{n} Y_{KL} V_L, \tag{45}$$

$$V_K = \sum_{L=1}^{n} Z_{KL} I_L. \tag{46}$$

For the loss-free reciprocal networks considered, Z_{KL}, Y_{KL} are purely imaginary and symmetrical.

Substitution into equations (43) and (44) yields

$$(F_x)_{\text{av}} = \frac{1}{2j\omega}\sum_{K=1}^{n}\sum_{M=1}^{n} V_K^* \frac{\partial Y_{KM}}{\partial x} V_M, \tag{47}$$

and

$$(F_x)_{\text{av}} = \frac{1}{2j\omega}\sum_{K=1}^{n}\sum_{M=1}^{n} I_K^* \frac{\partial Z_{KM}}{\partial x} I_M. \tag{48}$$

Equations (47) and (48) give the force in terms of derivatives of the admittance or impedance parameters of the system. They are generalizations of equations (5) and (6). For a two-terminal system equations (47) and (48) become

$$(F_x)_{\text{av}} = \frac{1}{2j\omega} V V^* \frac{\partial Y}{\partial x} = \frac{1}{2\omega} |V|^2 \frac{\partial B}{\partial x}, \tag{49}$$

$$(F_x)_{\text{av}} = \frac{1}{2j\omega} I I^* \frac{\partial Z}{\partial x} = \frac{1}{2\omega} |I|^2 \frac{\partial X}{\partial x}, \tag{50}$$

which are just equations (5) and (6).

However, the fields within the two-terminal network need not be of low frequency and the network may have any degree of complexity. Equations (47) and (48) were obtained in an earlier paper (Smith 1960) in a restricted form, applicable to low frequency lumped-element circuits only.

The force may be expressed by equations (47) and (48) for another common practical system. A microwave circuit (Montgomery, Dicke, and Purcell 1948) with a number of entering waveguides may be described by " voltages " and " currents ", V_K and I_K. The fields in the incoming waveguides are expanded in terms of normal mode fields. Each V_K is a coefficient of a normal mode electric field distribution in one of the guides and each I_K is a coefficient of a normal mode magnetic field distribution. The normal mode fields usually correspond to propagating modes but do not need to.

The V_K, I_K of the whole system are related linearly by admittance and impedance matrices as in equations (45) and (46). Further, the orthogonality of the modes and the choice of magnitudes enables the complex power P_ω to be written

$$P_\omega = \int_S (\mathbf{E}_\omega \times \mathbf{H}_\omega^*).\mathrm{d}\mathbf{A} = \sum_K V_K I_K^*, \tag{51}$$

the surface integration extending over the interiors of the waveguides. Equations (40) and (41) then lead to equations (43), (44) and (47), (48).

IV. Applications and Extensions

The results of Section III may be used as a foundation for the calibration of certain a.c. electrical measuring instruments. The results are relevant for instruments which are essentially loss free and dependent on the Maxwell stresses for providing an indicating force. The electrostatic voltmeter, quadrant electrometer, or current dynamometer are simple practical examples. A class of microwave power measuring instruments which rely on radiation pressures (e.g. Cullen and Stephenson 1952 ; Cullen, Rogal, and Okamura 1958) provides further examples. Equations (47) and (48) are applicable to these systems. A measurement of the admittance or impedance derivatives, together with the force produced in the instrument, provides a calibration of the instrument. If the derivatives are measured in terms of an absolute impedance or admittance standard, this calibration is absolute. The establishment of an impedance standard is a separate problem. In effect, it involves the calculation of a field distribution, but the standard is designed so that the calculations are simple. In the high frequency domain the characteristic impedance of an accurately constructed waveguide or transmission line usually constitutes the standard. In the low frequency region the standard is commonly derived from a calculable mutual inductance or capacitance.

The average force has been obtained for sinusoidally varying fields, but the extension to general time dependence follows immediately from a Fourier resolution of the exciting fields. The force is everywhere expressed as a quadratic function of field quantities. By Parseval's theory, the contributions to the average force from individual Fourier components are additive.

We have considered loss-free reciprocal systems only. The restriction to reciprocal systems is not of great importance in practice. However, the restriction to loss-free systems could be more serious. In some cases a means has been found for approximating the average force in lossy lumped-element systems (Smith 1960).

V. References

Cullen, A. L., Rogal, B., and Okamura, S. (1958).—*Trans. Inst. Radio Engrs.* **MTT-6** : 133.

Cullen, A. L., and Stephenson, I. M. (1952).—*Proc. Instn. Elect. Engrs.* IV **99** : 294.

Guillemin, E. A. (1953).—" Introductory Circuit Theory." (Wiley : New York.)

Livens, G. H. (1929).—" Theory of Electricity." (Cambridge Univ. Press.)

Montgomery, C. G., Dicke, R. H., and Purcell, E. M. (1948).—" Principles of Microwave Circuits." (McGraw-Hill : New York.)

Smith, W. E. (1960).—A theory of steady forces in variable-parameter networks. Instn. Elect. Engrs. Monograph No. 366M. *Proc. Instn. Elect. Engrs.* C **107** : 228.

Stratton, J. A. (1941).—" Electromagnetic Theory." (McGraw-Hill : New York.)