THE INFLUENCE OF THERMOELECTRIC EFFECTS ON THE MAXIMUM TEMPERATURE IN A RADially CONSTRICTED GAS DISCHARGE BETWEEN ELECTRODES

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Summary

In an earlier paper the author provided a method for estimating the maximum temperature in a steady-state, centrally constricted, highly ionized deuterium discharge between electrodes. The analysis applied to discharges not too long, so that bremsstrahlung loss could be neglected compared to the main heat loss by conduction to the electrodes, and thermoelectric effects were not included.

Here the analysis is generalized to include thermoelectric effects, and carried through for strictly longitudinal flow, for which at every point within the discharge the heat flux vector \( q \) and the current density vector \( j \) are parallel to the magnetic field \( H \).

Again a simple continuity argument shows that \( q + Vj = 0 \), where \( V \) is the electric potential, but now the equipotential surface on which \( q = V = 0 \) is displaced from midway between the electrodes nearly to the cathode. In the linear case the maximum temperature is displaced somewhat from the midway position towards the anode. A similar remark applies to the constricted discharge. The important influence of inclusion of thermoelectric effects is that the maximum temperature is increased by approximately 14\% for about the same applied voltage producing a given current in a particular discharge geometry. The characteristic relating the maximum temperature and resistance ratio \( \varepsilon \) and the radial compression ratio \( \gamma \) obtained in the earlier paper is not changed by thermoelectric effects. Comparison of voltage and also temperature versus distance characteristics for linear and constricted discharges without and with thermoelectric effects is given by means of graphs.

I. INTRODUCTION

In an earlier paper (Seymour 1961), referred to hereafter as S, a method was given for estimating the maximum temperature in a steady non-equilibrium state, centrally constricted, substantially ionized deuterium discharge between electrodes. The free boundary surface of the plasma was assumed thermally insulated when isolated from the walls of the discharge tube, and cooling was therefore by heat conduction to the electrodes, compared to which the bremsstrahlung loss was shown to be negligible if the discharge was not too long. Neglect of thermoelectric and other effects led to symmetry of the distributions of temperature and voltage about a median plane normal to the longitudinal axis of the discharge, on which the plasma temperature was assumed constant at its maximum value \( T_m \).

With \( \omega_e \) the electron gyrofrequency and \( \tau_e \) the electron collision time, the estimation of maximum temperature in discharges having straight and hyperbolic

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current streamlines was made via a vector analysis for $\omega_e \tau_e \ll 1$, and a tensor analysis for $\omega_e \tau_e \gg 1$. For flow having $q$ and $j$ parallel to the magnetic field $H$ at every point within the discharge (strictly longitudinal flow), it was seen that the results obtained for $\omega_e \tau_e \ll 1$ applied for all $\omega_e \tau_e$.

In this paper we examine the same cases as above, under the same approximations, except that we include thermoelectric effects by generalizing the previous analysis, which destroys the symmetry of the temperature and voltage distributions about the median plane. Since strictly longitudinal flow is considered, it is convenient and sufficient to employ a vector method only.

Initial consideration of the problem suggested that for simplicity thermoelectric effects should be excluded in the first attempt at solution, and included later if possible.

This solution procedure proved satisfactory, and showed that it was not desirable to combine the separate results obtained, hence the presentation of results in separate papers.

II. ISOTROPIC FORMS OF $j$ AND $q$ (SEEBECK AND PELTIER EFFECTS ONLY INCLUDED)

When the magnetic field $H$ is negligible or parallel to the electric field $E$ and the temperature gradient $\nabla T$, the general equations for $j$ and $q$ include only the Seebeck and Peltier effects respectively, as in (3.8) and (3.9) of S. For $\omega_e \tau_e \ll 1$ these equations reduce to the isotropic forms

$$j = \sigma E + \alpha \nabla T,$$

and

$$q = -K \nabla T - \beta E,$$

where $\sigma$ and $K$ are the scalars obtained in Section IV (a) of S, and we have replaced Marshall's (1957) $\varphi$ and $\xi$ by Spitzer and Härn's (1953) $\alpha$ and $-\beta$ respectively, and used Table 1 (constructed from pp. 67 and 69 of Marshall's report) to show that $\alpha^I = \alpha^{II} = \alpha^I \approx 0$; $\beta^I = \beta^{II} = \beta^I \approx 0$, when $\omega_e \tau_e \ll 1$.

<table>
<thead>
<tr>
<th>Components of $[\alpha]$</th>
<th>Components of $[\beta]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha^I = 1.554 \frac{kn_e e \tau_e}{m_e}$</td>
<td>$\beta^I = 6.38 \frac{kn_e e \tau_e T}{m_e}$</td>
</tr>
<tr>
<td>$\alpha^{II} = -1.5 \frac{kn_e e \tau_e}{m_e}$</td>
<td>$\beta^{II} = 2 \frac{kn_e e \tau_e T}{m_e}$</td>
</tr>
<tr>
<td>$\alpha^{III} = -4.3 \frac{kn_e e \tau_e}{m_e}$</td>
<td>$\beta^{III} = 2 \frac{kn_e e \tau_e T}{m_e}$</td>
</tr>
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TABLE 1
COMPONENTS OF THE THERMOELECTRIC TENSORS

$$\beta^{III} = 2 \frac{kn_e e \tau_e T}{m_e} \left[ 1.25 \frac{2 \omega_e e \tau_e}{m_e} + 7.627 \right]$$
Comparing $\alpha$ and $\beta$ with their values in a Lorentz gas, we find from Spitzer and Härm that when the atomic number $Z=1$,

$$\alpha = 3 \left( \frac{2}{\pi} \right)^{3/2} \frac{k_{5/2} T^{3/2}}{m_e^{1/2} e^2 \ln \lambda} \gamma_r, \tag{2.3}$$

and

$$\beta = 8 \left( \frac{2}{\pi} \right)^{3/2} \frac{k_{5/2} T^{5/2}}{m_e^{1/2} e^2 \ln \lambda} \delta_E, \tag{2.4}$$

where the electron charge $e$ has been taken in e.s.u. Here $\alpha$ and $\beta$ for an actual gas have been expressed in terms of their values in a Lorentz gas by means of the transport coefficients $\gamma_r$ and $\delta_E$, given by Spitzer and Härm as $\gamma_r = 0.2727$, $\delta_E = 0.4652$. Spitzer and Härm's $\alpha$ and $\beta$ agree closely with those of Marshall, who, however, does not specifically relate actual and Lorentz gas values.

Writing (2.3) as

$$\alpha = \alpha_0 T^{3/2} / \ln \lambda, \tag{2.5}$$

and (2.4) as

$$\beta = \beta_0 T^{5/2} / \ln \lambda, \tag{2.6}$$

we obtain by inserting numerical values

$$\alpha_0 = 9.285 \times 10^{-9} \text{ A cm}^{-1} \text{ deg}^{-5/2}, \tag{2.7}$$

and

$$\beta_0 = 4.223 \times 10^{-8} \text{ J V}^{-1} \text{ cm}^{-1} \text{ s}^{-1} \text{ deg}^{-5/2}. \tag{2.8}$$

To complete these equations we recall from S, Section V, that

$$\sigma = \sigma_0 T^{3/2} / \ln \lambda, \tag{2.9}$$

and

$$K = K_0 T^{5/2} / \ln \lambda, \tag{2.10}$$

where

$$\sigma_0 = 1.53 \times 10^{-4} \text{ \Omega}^{-1} \text{ cm}^{-1} \text{ deg}^{-3/2}, \tag{2.11}$$

and

$$K_0 = 4.396 \times 10^{-12} \text{ J s}^{-1} \text{ cm}^{-1} \text{ deg}^{-7/2}. \tag{2.12}$$

### III. Solution of the Plasma Energy Equation for Strictly Longitudinal Flow

Again, the analysis of a general, constricted discharge having geometric symmetry about a chosen plane is conveniently handled by introduction of an orthogonal curvilinear coordinate system, which can be specialized later to deal with linear discharges and those having hyperbolic streamlines. Before developing these solutions, however, the form of the differential equation for strictly longitudinal flow to be finally integrated can readily be obtained by a simple continuity argument.

(a) Curved Stream Tube of a Constricted Discharge

Figure 1 shows a curved current stream tube possessing geometric symmetry about a chosen plane. Suppose also that there exists an equipotential surface,
normal to the curved axis of the tube, and nearer its lower end in Figure 1, on which \( q = 0 \) and on which it is convenient to take the constant electric potential as zero. Then, on that surface, with the notation of (5.15) of S, we have

\[ M = q + Vj = 0, \]  

(3.1.1)

and since from (5.14) \( M \) is solenoidal, it follows that \( M \) vanishes identically at every point within the discharge, and the flow is longitudinal. Inserting (2.1) and (2.2) in (3.1.1), and using \( E = -\nabla V \), together with (2.5), (2.6), (2.9), and (2.10) above, we obtain

\[ \nabla T / \nabla V = \left( \beta_0 T - \sigma_0 V \right) / \left( K_0 T - \alpha_0 V \right). \]  

(3.1.2)

From the mode of derivation we see that this equation is valid for all values of \( \omega_c \) if \( q \) and \( j \) are at every point parallel to \( H \), i.e. if the flow is strictly longitudinal.

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Fig. 1.—Curved stream tube of a constricted discharge.

(b) Solution of a Simplified Form of the Plasma Energy Equation using Complex Variables

For the detailed mathematical analyses of longitudinal flow the plasma energy equation, \( \text{div} \ q = j \cdot E \), is again simplified initially by introduction of the orthogonal curvilinear coordinates, \( \psi \) (the current stream function), \( \varphi \), and \( V \); and by consideration of axi-symmetric flow, as in S, Section VI (b). Integration leads to

\[ \partial T / \partial V = \left( \beta_0 T - \sigma_0 V \right) / \left( K_0 T - \alpha_0 V \right), \]  

(3.2.1)

which is a convenient form of (3.1.2).
A method of integrating (3.2.1) runs as follows: first introduce a parameter $p$, such that

\[
\begin{align*}
\partial T/\partial p &= \beta_0 T - \sigma_0 V, \\
\partial V/\partial p &= K_0 T - \sigma_0 V.
\end{align*}
\] (3.2.2)

(3.2.3)

Assume now that the physical temperature and electric potential are the imaginary parts of complex quantities $\overline{T}$ and $\overline{V}$ respectively, so that, if Im is the imaginary part operator,

\[
T = \text{Im} \overline{T},
\] (3.2.4)

\[
V = \text{Im} \overline{V}.
\] (3.2.5)

Further, assume that

\[
\overline{T} = A e^{mp},
\] (3.2.6)

and

\[
\overline{V} = B e^{mp},
\] (3.2.7)

where $m$ is a complex quantity. Then

\[
\begin{align*}
mA &= \beta_0 A - \sigma_0 B, \\
mB &= K_0 A - \sigma_0 B.
\end{align*}
\] (3.2.8)

(3.2.9)

Combination of (3.2.8) and (3.2.9) yields

\[
A/B = -\sigma_0/(m-\beta_0) = (m+\sigma_0)/K_0,
\] (3.2.10)

and hence, from the right-hand side equation,

\[
m = \frac{1}{2}(\beta_0 - \sigma_0) + \frac{i}{2}(4K_0 \sigma_0 - (\sigma_0 + \beta_0)^2)^{1/4},
\] (3.2.11)

where $4K_0 \sigma_0 - (\sigma_0 + \beta_0)^2 > 0$, and the positive square root has been chosen for convenience of solution.

Writing

\[
a = (\beta_0 - \sigma_0)/(4K_0 \sigma_0 - (\sigma_0 + \beta_0)^2)^{1/4},
\] (3.2.12)

(3.2.11) becomes

\[
m = \frac{1}{2}(4K_0 \sigma_0 - (\sigma_0 + \beta_0)^2)^{1/4}(a + i).
\] (3.2.13)

We now define a variable $\theta$ in terms of $p$ as

\[
\theta = \frac{1}{2}(4K_0 \sigma_0 - (\sigma_0 + \beta_0)^2)^{1/4}p,
\] (3.2.14)

and then, on the permissible assumption that $B$ is a real constant, write (3.2.7) as

\[
V = Be^{\sigma_0 e^{\theta i}},
\] (3.2.15)

and, with the aid of (3.2.10), write (3.2.6) as

\[
T = (m+\sigma_0)/K_0 Be^{\sigma_0 e^{\theta i}}.
\] (3.2.16)
Using (3.2.11), it is a simple proposition to show that

\[ m + x_0 = (K_0 \sigma_0) e^{i \theta}, \tag{3.2.17} \]

where

\[ \varepsilon = \tan^{-1} \left[ \frac{4K_0 \sigma_0 - (x_0 + \beta_0)^2}{x_0 + \beta_0} \right]. \tag{3.2.18} \]

Hence, using (3.2.17), we may write (3.2.16) in the form

\[ T = (\sigma_0/K_0)^{1/4} Be^{\theta_0} \sin (\theta + \varepsilon). \tag{3.2.19} \]

Application of the operator $\text{Im}$ to the equations (3.2.15) and (3.2.19) yields the physical electric potential and temperature expressions

\[ V = Be^{\theta_0} \sin \theta, \tag{3.2.20} \]

and

\[ T = (\sigma_0/K_0)^{1/4} Be^{\theta_0} \sin (\theta + \varepsilon). \tag{3.2.21} \]

To determine the real constant $B$, we observe that when $\partial T/\partial \theta = 0$, $T = T_m$, the maximum temperature, and $\theta = \theta_m$. From (3.2.21), $\partial T/\partial \theta = 0$ gives

\[ \tan (\theta_m + \varepsilon) = -1/a. \tag{3.2.22} \]

Since $\varepsilon$ and $a$ can be calculated from the available data, $\theta_m$ is determined by (3.2.22). $B$ is therefore obtained from (3.2.21) as

\[ B = \frac{(K_0/\sigma_0)^{1/4} T_m}{(\exp a \theta_m) \sin (\theta_m + \varepsilon)}. \tag{3.2.23} \]

Proceeding numerically, use of equations (2.7), (2.8), (2.11), and (2.12) in (3.2.12) and (3.2.18) gives

\[
\begin{align*}
a &= 5.446, \\
\varepsilon &= 0.1169 \text{ radians (6.7°).}
\end{align*}
\tag{3.2.24}
\]

Insertion of these results into (3.2.22) gives

\[ \theta_m = 2.8431 \text{ radians (162.9°)}, \tag{3.2.25} \]

and so, using (2.11), (2.12), (3.2.24), and (3.2.25), (3.2.23) gives

\[ B = 1.7713 \times 10^{-10} T_m \text{ deg}^{-1} V, \tag{3.2.26} \]

and the factor $(\sigma_0/K_0)^{1/4} B$ becomes

\[ (\sigma_0/K_0)^{1/4} B = 1.045 \times 10^{-6} T_m. \tag{3.2.27} \]

Substitution of these results in (3.2.20) and (3.2.21) yields the numerical forms

\[ V = 1.7713 \times 10^{-10} T_m e^{0.4460} \sin \theta \text{ V deg}^{-1}, \tag{3.2.28} \]

and

\[ T = 1.045 \times 10^{-6} T_m e^{0.4460} \sin (\theta + 0.1169). \tag{3.2.29} \]
If the thermoelectric coefficients \( \alpha_0 \) and \( \beta_0 \) are excluded from (3.2.12) and (3.2.18),
\[
\begin{align*}
\alpha &= 0, \\
\beta &= 0
\end{align*}
\]  
and hence, from (3.2.22),
\[
\theta_m = 0.
\]  
With these results the expression (3.2.23) for \( B \) becomes simply,
\[
B = (K_0/\sigma_0)^{1/2} T_m,
\]
and accordingly (3.2.20) and (3.2.21) reduce respectively to
\[
V = (K_0/\sigma_0)^{1/2} T_m \sin \theta,
\]
and
\[
T = T_m \cos \theta,
\]
which can be combined to give
\[
T^2 + (\sigma_0/K_0) V^2 = T_m^2,
\]
as obtained in S, Section VI (a).

To complete the detailed mathematical analyses of non-constricted and constricted axi-symmetric discharges, we follow the procedure adopted in Section VI (d) of S, and introduce a more general orthogonal curvilinear coordinate system, \( w, v, u \), and assume henceforth that \( V \) and \( T \) are functions of \( u \) only. Again, the only non-zero component of \( j \) is \( j_u \), which, from (2.1), (2.5), (2.9), (3.2.14), (3.2.2), and (3.2.3), is given in curvilinear form as
\[
\begin{align*}
\dot{j}_u &= \frac{2(z_0 \beta_0 - K_0 \sigma_0)}{4K_0 \sigma_0 - (z_0 + \beta_0)^2} \frac{T^{5/2}}{(h_3 \ln \lambda)} \frac{\partial \theta}{\partial u'} \\
&= G(u)/h_3,
\end{align*}
\]  
where the notation adopted here is illustrated in Figure 2.

Integration of the steady-state form of the electrical equation of continuity, \( \text{div } j = 0 \), gives, for axi-symmetric flow,
\[
\dot{j}_u = F(w)/h_1 h_2.
\]  
Combination of (3.2.35) and (3.2.37) results in
\[
F(w) \int_0^u h_2 \frac{\dot{j}_u}{h_1 h_2} du' = \frac{2(z_0 \beta_0 - K_0 \sigma_0)}{4K_0 \sigma_0 - (z_0 + \beta_0)^2} \frac{T_{m}^{5/2}}{(h_3 \ln \lambda)} \int_0^\theta \left( \frac{T}{T_m} \right)^{5/2} d\theta',
\]  
where we assume that the electrodes are held at \( T = 0 \), so that from (3.2.21)
\[
\begin{align*}
\theta_1 \text{ (at cathode)} &= -0.1169 \text{ radians } (-6.7^\circ), \\
\theta_2 \text{ (at anode)} &= +3.0246 \text{ radians } (+173.3^\circ).
\end{align*}
\]
Fig. 2.—*w* and *u* as orthogonal curvilinear coordinates (axi-symmetric flow).

Fig. 3.—Discharge cross sections in the *p-z* plane. (a) Streamlines parallel to *oz*, (b) streamlines curved.
Equation (3.2.38) can be readily solved if the variables in \( h_3/(h_1h_2) \) are separable. We now consider its application to (1) the linear, (2) the constricted discharge.

(e) *Specialization of Results for a Linear Discharge*

The cross section in the \( \rho-z \) plane for this case is shown in Figure 3 (a).

Using cylindrical coordinates, with \( \psi=\rho, \ u=z, \) we have \( h_1=h_3=1, \ h_2=\rho, \) and \( V=V(z), \ T=T(z), \ \psi=\psi(\rho). \) Thus, in Figure 3 (a) the streamlines are parallel to \( oz, \) and the equipotential lines are normal to \( oz. \) Using the above scale factors, (3.3.36) and (3.3.37) yield

\[
j_z=G(z)=F(\rho)/\rho=\text{const.},
\]

and since (3.2.21) and (3.2.23) combine to give

\[
\frac{T}{T_m} = \frac{(\exp a\theta) \sin (\theta+\varepsilon)}{(\exp a\theta_m) \sin (\theta_m+\varepsilon)},
\]

use of the scale factors and these results in (3.3.38) gives

\[
z = \frac{2(\beta_0 \sigma_0-K_0\sigma_0)}{4K_0\sigma_0-(\beta_0+\sigma_0)^2} \frac{T_m^{5/2}}{j_z \ln \lambda} \int_0^\theta \left( \frac{(\exp a\theta') \sin (\theta'+\varepsilon)}{(\exp a\theta_m) \sin (\theta_m+\varepsilon)} \right)^{5/2} d\theta', \quad (3.3.2)
\]

or, using the earlier results for \( \alpha_0, \beta_0, \sigma_0, K_0, \) and taking \( \ln \lambda=10, \)

\[
z = -0.9273 \frac{T_m^{5/2}}{j_z} \int_0^\theta \left( \frac{(\exp a\theta') \sin (\theta'+\varepsilon)}{(\exp a\theta_m) \sin (\theta_m+\varepsilon)} \right)^{5/2} d\theta' \quad \text{A cm}^{-1} \text{deg}^{-5/2}. \quad (3.3.3)
\]

The value of the integral in the right-hand side of (3.3.3) can be obtained for chosen values of \( \theta \) in the range \( \theta_1<\theta<\theta_2 \) by numerical integration, since \( a, \ v, \ \varepsilon, \) and \( \theta_m \) are known. In particular, the length of the discharge, \( z_t>0, \) corresponds to \( \theta=\theta_2, \) and (3.3.3) then leads to

\[
\frac{z}{z_t} = \frac{1}{0.293} \int_0^\theta \left( \frac{(\exp a\theta') \sin (\theta'+\varepsilon)}{(\exp a\theta_m) \sin (\theta_m+\varepsilon)} \right)^{5/2} d\theta'. \quad (3.3.4)
\]

Equation (3.3.4) gives the fraction \( z/z_t \) for each value of \( \theta \) chosen, and therefore, using (3.2.28) and (3.2.29), \( V/T_m \) and \( T/T_m \) can be plotted against \( z/z_t, \) as in Figure 4 (b). With reference to S, Section VI (d), we also include for comparison Figure 4 (a), which shows \( V/T_m \) and \( T/T_m \) plotted against \( z/z_0 \) when thermoelectric effects are excluded.

Examination of (3.3.35) for this case suggests the form \( j_z = -j, \) where \( j>0. \) Using this result, the maximum temperature is derived from (3.3.3) as

\[
T_m = 2228 \left( \frac{1}{2} I(\rho) \right)^{2/15} A^{-2/15} \text{cm}^{2/15} \text{deg}, \quad (3.3.5)
\]

where \( I(\rho) = \pi \rho^2 j, \) and the semi-length, \( \frac{1}{2} z_t, \) has been introduced to facilitate comparison of the above result with S, (6.4.14).

From (3.2.28) and (3.2.29), we have at the anode of the discharge,

\[
V_2 = +2 \ 434 \times 10^{-4} T_m \text{ V deg}^{-1}, \quad (3.3.6)
\]

and at the cathode,

\[
V_1 = -1.0933 \times 10^{-11} T_m \text{ V deg}^{-1}. \quad (3.3.7)
\]
The latter result shows that for values of $T_m$ likely to be attained in practice, $V_1 \sim 0$, and accordingly the zero of potential may be conveniently considered located at the cathode. Then $V_2$ is the voltage across the discharge, and so, varying the subscript to $\iota$,

$$T_m = 3397V \cdot V^{-1} \text{ deg.}$$  \hspace{1cm} (3.3.8)
Combination of (3.3.5) and (3.3.8) results in
\[ V_t = 0.656 \left( \frac{\frac{1}{2} I_1(z_1)}{\rho_1^2} \right)^{2/5} A^{-2/5} \text{cm}^{2/5} V. \]  
(3.3.9)

(d) Specialization of Results for a Discharge having Hyperbolic Streamlines

The cross section in the \( \rho-z \) plane for this case is shown in Figure 3 (b). The short solenoid producing the constricting, or guiding, magnetic field has axis \( oz \), and is assumed centrally located between anode and cathode of the discharge. With \( V = V(u), \varphi = \varphi(w) \), we can again approximate the current streamlines by hyperbolae to represent the practical situation. Mathematically, we relate \( u \) and \( w \) to \( \rho \) and \( z \) by the conformal transformation
\[ \rho + i(z - \frac{1}{2}z_1) = k \, \cosh (u + iw), \quad k \text{ defined below}, \]  
(3.4.1)
which expands to give
\[ \rho = k \, \cosh u \, \cos w, \]  
(3.4.2)
and
\[ z - \frac{1}{2}z_1 = k \, \sinh u \, \sin w. \]  
(3.4.3)

Then
\[ \rho^2/k^2 \cosh^2 u + (z - \frac{1}{2}z_1)^2/k^2 \sinh^2 w = 1, \]  
(3.4.4)
and
\[ \rho^2/k^2 \cos^2 w - (z - \frac{1}{2}z_1)^2/k^2 \sin^2 w = 1. \]  
(3.4.5)

As in S, Section VI (d), Case 2, we see that in the \( \rho-z \) plane the curves \( u = \text{const}, \, w = \text{const} \) form a family of confocal ellipses and hyperbolae, here with common foci at \(( \pm k, \, \pm \frac{1}{2}z_1)\). Again, for the scale factors we have
\[ h_1 = h_3, \]  
(3.4.6)
and
\[ h_2 = k \, \cosh u \, \cos w, \]  
(3.4.7)
and accordingly, from (3.2.36), (3.2.37), and the above two equations
\[ G(u) \cosh u = F(w)/k \cos w = \text{const.} = A. \]  
(3.4.8)

Following the procedure of S, Section VI (d), we can establish
\[ -I^*(w_b) = 2\pi k A (1 - \sin w_b). \]  
(3.4.9)\textsuperscript{†}

From Figure 3 (b) and equations (3.4.2), (3.4.3)
\[ \begin{align*}
\rho_0 &= k \cos w_b, \\
\rho_1 &= k \cosh u_c \cos w_b, \\
\frac{1}{2}z_1 &= k \sinh u_c,
\end{align*} \]  
(3.4.10)
where \( u = \pm u_c \) gives the electrodes.

\textsuperscript{†} The star superscript is used when necessary to indicate a Section III (d) quantity.
By proper choice of the left-hand side integral limits in (3.2.38), and use of (3.2.21), (3.2.23), and (3.4.6) to (3.4.9), is obtained

\[
\tan^{-1}(\sinh u) + \tan^{-1}(\sinh u') = \tan^{-1}(\sinh u)
\]

\[
\frac{4\pi k K_0 \sigma_0 - \sigma_0 \beta_0}{\ln \frac{\lambda}{\alpha}} \left[ \frac{4 K_0 \sigma_0 - \sigma_0 \beta_0}{(\alpha + \beta_0)^2} \right] \int_{\theta_1}^{\theta_2} \left( \frac{(\exp a \theta') \sin (\theta' + \varepsilon)}{(\exp a \theta_m) \sin (\theta_m + \varepsilon)} \right)^{5/2} d\theta'.
\]
With \( u = \pm u_0 \) corresponding to \( \theta = \theta_0 \), application to (3.4.11) of the steps that led to (3.3.4) yields

\[
\frac{\tan^{-1} \sinh u + \tan^{-1} \sinh u_0}{2 \tan^{-1} \sinh u_0} = \frac{1}{0.293} \int_{0}^{\theta_0} \left( \frac{(\exp a\theta') \sin (\theta' + \varepsilon)}{(\exp a\theta_m) \sin (\theta_m + \varepsilon)} \right)^{5/2} \, d\theta' = R \text{ say.}
\]

(3.4.12)

When \( w = \frac{1}{2} \pi \) we see from (3.4.2) and (3.4.3) that, as \( u \) varies, the point defined by the coordinates \( \rho \) and \( z \) moves along the symmetry axis \( oz \) in Figure 3 (b). Then, rearranging (3.4.12) and using (3.4.3) and the last equation of (3.4.10), we obtain

\[
z/z_1 = \frac{1}{2} \left[ 1 + \left\{ 1/(\nu^2 - 1)^{1/2} \right\} \tan \left( (2R - 1) \tan^{-1} (\nu^2 - 1)^{1/2} \right) \right],
\]

(3.4.13)

where \( \nu = \rho_1/\rho_0 \) is the radial compression ratio. Clearly, \( \nu = 1 \) reduces this equation to (3.3.4) as required.
Using appropriate numerical techniques we can obtain from (3.4.13) the fraction $z/z_i$ for each value of $\theta$ chosen, with $v$ as a parameter, and hence, with the further aid of (3.2.28) and (3.2.29), $V/T_m^*$ and $T/T_m^*$ can be plotted versus $z/z_i$ for suitable values of $v$, as in Figures 5 (b) and 6 (b) respectively. Similarly, with reference to S, (6.4.33), and the associated expressions $T=T_m^* \cos \theta$, $V=(K_o/\sigma_0)^{1/2} T_m^* \sin \theta$, $-\frac{1}{2} \pi < \theta < +\frac{1}{2} \pi$, we can calculate for comparison the characteristics of Figures 5 (a) and 6 (a), which show respectively $V/T_m^*$ and $T/T_m^*$ versus $z/z_o$ when thermoelectric effects are excluded.

Using the corresponding values $u=+u_e$ and $\theta=\theta_2$, the maximum temperature is derived from (3.4.11) as

\[
T_m^* = \left( \frac{\ln \lambda I^*(w_b) \{4K_o \sigma_o - (\alpha_o + \beta_o)^2\}^{1/2} \tan^{-1} \left( \frac{\sinh u_e}{(0.293)2\pi k(1-\sin w_b)/(K_o \sigma_o - \alpha_o \beta_o)} \right)}{(0.293)2\pi k(1-\sin w_b)/(K_o \sigma_o - \alpha_o \beta_o)} \right)^{2/5}. \tag{3.4.14}
\]

The results (3.4.10) can be interpreted geometrically with the aid of Figure 3 (b) as in S, Section VI (d), and (3.4.14) accordingly becomes

\[
T_m^* = \left( \frac{\ln \lambda I^*(w_b) \{4K_o \sigma_o - (\alpha_o + \beta_o)^2\}^{1/2} \tan^{-1} \left( \frac{\sinh u_e}{(0.293)2\pi (\alpha_o \beta_o)/(K_o \sigma_o - \alpha_o \beta_o)[1-(\rho_1/\rho_2)^2v^{-2}(\alpha^2-1)]^{1/2}} \right)}{(0.293)2\pi (\alpha_o \beta_o)/(K_o \sigma_o - \alpha_o \beta_o)[1-(\rho_1/\rho_2)^2v^{-2}(\alpha^2-1)]^{1/2}} \right)^{2/5}. \tag{3.4.15}
\]

Again referring to (2.7), (2.8), (2.11), (2.12); taking $\ln \lambda=10$, and from physical considerations, $(\rho_1/\rho_2)^2<<1$, (3.4.15) reduces to the more attractive form

\[
T_m^* = 2228 \left( \frac{\frac{1}{2} \left( \frac{\rho_1}{\rho_2} \right)^2 I^*(w_b)}{\frac{1}{2} \left( \frac{\rho_1}{\rho_2} \right)^2} \right)^{2/5} \left( \frac{\frac{1}{2} \tan^{-1} \left( \frac{\sinh u_e}{(0.293)2\pi (\alpha_o \beta_o)/(K_o \sigma_o - \alpha_o \beta_o)[1-(\rho_1/\rho_2)^2v^{-2}(\alpha^2-1)]^{1/2}} \right)}{(\alpha^2-1)^{1/2}} \right)^{2/5} \text{A}^{-2/5} \text{cm}^{2/5} \text{deg}. \tag{3.4.16}
\]

Combining (3.4.16) with (3.3.8), which is also applicable here,

\[
V_e^* = 0.656 \left( \frac{\frac{1}{2} \left( \frac{\rho_1}{\rho_2} \right)^2 I^*(w_b)}{\frac{1}{2} \left( \frac{\rho_1}{\rho_2} \right)^2} \right)^{2/5} \left( \frac{\frac{1}{2} \tan^{-1} \left( \frac{\sinh u_e}{(0.293)2\pi (\alpha_o \beta_o)/(K_o \sigma_o - \alpha_o \beta_o)[1-(\rho_1/\rho_2)^2v^{-2}(\alpha^2-1)]^{1/2}} \right)}{(\alpha^2-1)^{1/2}} \right)^{2/5} \text{A}^{-2/5} \text{cm}^{2/5} \text{V}. \tag{3.4.17}
\]

IV. DISCUSSION OF RESULTS

To estimate the influence of thermoelectric effects, we first take from S the results (6.4.38) and (6.4.40), and combine them with (3.4.16) and (3.4.17) above to obtain for a given discharge the ratios

\[
\frac{T_m^*}{T_m^*} = 1.136, \quad \tag{4.1} \dagger
\]

and

\[
\frac{V_e^*}{2V_e^*} = 0.985. \quad \tag{4.2}
\]

Comparing Figures 4 (a) and 4 (b), we observe that, for the linear discharge, inclusion of thermoelectric effects results in a displacement of the zero of the heat flux vector and electric potential from midway between the electrodes virtually to the cathode. Also, in Figure 4 (b) it is evident that the slope of the electric

\dagger The curly bar is used when necessary to indicate a thermoelectric case quantity.
potential characteristic changes sign near the anode, and so, from (2.1), we see that in this region the constant current density \( j \) of the linear discharge is maintained by the term \( \varepsilon \mathbf{V} T \) acting against \( \sigma \mathbf{E} \). By forming the scalar product \( \mathbf{j} \cdot \mathbf{E} \), we see that the Joule heating per unit volume in this region is made up of a cooling term, \( \varepsilon \mathbf{E} \cdot \mathbf{V} T \), and a smaller heating term, \( \sigma E^2 \). Considering the nature of the Seebeck and Peltier effects included in the basic equations (2.1) and (2.2) for \( j \) and \( q \), this is perhaps not a surprising result.

Since, for the linear discharge, \( \varepsilon_j + V_j = 0 \), where the current density \( j_z = -j, j > 0 \), is a constant, it is of interest to note that the voltage characteristics of Figures 4 (a) and 4 (b) also represent the heat flux characteristics \( q_z / (j T_m) \) plotted against \( z/z_0 \) and \( z/z_i \) respectively.

Further comparison of Figures 4 (a) and 4 (b) shows that thermoelectric effects displace the temperature maximum from midway between the electrodes towards the anode, but here the displacement is not marked.

When the discharge is constricted at the plane of geometric symmetry and thermoelectric effects are excluded, the shape of the temperature versus distance characteristic varies with increase of radial compression ratio, as shown in Figure 6 (a). Since the central constriction produces an increase of plasma temperature at and near the plane of symmetry, the narrowing of the peaks of the characteristics in Figure 6 (a) with increase of radial compression ratio can be readily understood physically.

Figure 6 (b) gives corresponding characteristics when thermoelectric effects are included. Broadly, the above remark on Figure 6 (a) applies here, and, as would also be expected from physical reasoning, the displaced temperature maximum of the linear discharge tends to return towards the plane of geometric symmetry as the radial compression ratio assumes values greater than unity.

As can be verified by means of (3.2.36), (3.4.8), and S, (6.4.22), when the discharge is centrally constricted the current density is no longer constant, but becomes a function of \( w \) and \( u \). This remark also applies when thermoelectric effects are excluded, as can be seen from S, (6.4.5), (6.4.22), and (6.4.26). These and other facts complicate the possibility of obtaining a simple physical explanation of the trend of the voltage against distance characteristics of Figures 5 (a) and 5 (b) as radial compression increases.

The important conclusion that can be drawn from equation (4.1) is that the maximum temperature \( \tilde{T}_m^* \) is raised above \( T_m^* \) by some 14%, due to thermoelectric effects. However, by forming the temperature ratio, \( \tilde{T}_m^*/\tilde{R}_o \), from (3.4.16) and (3.5.5); and the resistance ratio, \( \tilde{R}_m^*/\tilde{R}_o \), from (3.4.17) and (3.3.9) for conditions outlined in S, Section VII, it is immediately evident that the characteristic of Figure 7 of that paper is also applicable when thermoelectric effects are included.

Equation (4.2) shows that the total voltage required to produce a given current in a particular geometry of discharge is about the same without and with thermoelectric effects.
V. ACKNOWLEDGMENT

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VI. REFERENCES

