A STABILITY CRITERION FOR A RADIALLY CONSTRICTED GAS DISCHARGE BETWEEN ELECTRODES

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Summary

The stability of a centrally constricted, highly ionized deuterium discharge between electrodes is considered. Initially, the normal mode and energy principles for the examination of plasma stability are reviewed and the results contrasted. Teller's powerful stability criterion is next obtained in a useful integral form by means of a thermodynamic analysis of interchange instability. This general geometrical result is then applied to the centrally constricted plasma between electrodes, and a sufficient stability criterion derived. For this type of system, initially stabilized by a strong, external guiding magnetic field, it is found that the onset of instability occurs for a discharge current $I_0^*(w_b) \sim 1 \cdot 6\varphi_E/z_e$, where φ_E from the external solenoid is the total flux through any discharge equipotential surface, z_e is the discharge semi-length, and practical simplifying assumptions of high radial compression ratio ν and maximum discharge radius $\rho_1 \gtrsim \frac{1}{5} z_e$ have been made.

It is suggested that experimental confirmation of the above result would justify a more detailed stability analysis.

I. Introduction

By taking an appropriate moment of the Boltzmann transport equation for ionized gas particles, an equation of momentum transfer is obtained. Specialization of this result for ions and electrons gives a pair of equations often termed the two-fluid equations of a plasma. If the pressure tensor can be approximated by a scalar pressure, addition of these equations of motion for the ion and electron fluids leads to the linearized single-fluid equation of motion (Spitzer 1956; Linhart 1960) for a plasma in a magnetic field,

$$\rho \partial \mathbf{v}/\partial t = \mathbf{j} \times \mathbf{H} - \nabla p, \tag{1.1}$$

where ρ is the mean mass density,
$\mathbf{v}$ is the mean mass velocity,
$\mathbf{j}$ is the mean current density,
$\mathbf{H}$ is the magnetic field,
p is the scalar kinetic pressure,
and we have neglected gravitational effects.

As Artsimovich (1958) has indicated, equation (1.1) includes two extreme cases. If the gas pressure gradient is small, the body force $\mathbf{j} \times \mathbf{H}$ is balanced by the inertial force term $\rho \partial \mathbf{v}/\partial t$, and the plasma acquires a directed velocity. This magnetohydrodynamic phase is characterized by its short-duration transient nature; it serves to permit the conversion of kinetic energy of motion into heat by a suitably selected process.

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If, however, the plasma has negligible acceleration, the inertial term may be dropped from equation (1.1), and the body force is then balanced by the gradient of the scalar pressure,

$$\nabla p = \mathbf{j} \times \mathbf{H}, \tag{1.2}$$

independent of the time.

Early experimental work in the plasma physics field unfortunately showed that the stable plasma configurations that might profitably be associated with the steady-state equation (1.2) could not, in practice, be realized. Instabilities of various types were found to develop before significant plasma containment time had been achieved, and the desirable stable plasma configurations were destroyed. A major theoretical task thus became that of predicting the stability requirements of chosen plasma geometries. Mainly, cylindrical gaseous conductors received consideration, and further reference to this theoretical work will be made in Section II.

In earlier papers (Seymour 1961*a*, 1961*b*) methods were provided for estimating the distribution of temperature with axial distance along a steady-state centrally constricted, highly ionized deuterium discharge between electrodes, first neglecting thermoelectric effects, and then including them. The boundary surface of this discharge was conveniently approximated mathematically by a hyperboloid of one sheet. In the present paper we take up the problem of the stability of this particular plasma configuration against small-amplitude perturbations, and develop in Section IV a simple, sufficient stability criterion.

II. Stability Analyses of Cylindrical Gaseous Conductors: Teller's Stability Criterion

At present two main methods exist for the examination of stability. In the normal mode method an attempt is made to solve the linearized equations of motion for small-amplitude perturbations about an equilibrium state. If any perturbation grows in time the system is unstable; stability corresponds to bounded perturbation oscillation about an equilibrium state. The normal mode method encounters mathematical difficulties unless the plasma geometry is reasonably simple, and so, where specific knowledge of perturbation growth rates and frequencies is not required, an energy principle, based on the work of Lundquist (1951, 1952), has been developed by Bernstein *et al.* (1958) for determination of the stability of more complicated systems. This method depends on the fact that, if any realizable perturbation of a steady-state plasma equilibrium tends to reduce the potential energy of the system, the corresponding increase in kinetic energy can feed the perturbation so that its amplitude continues to grow with time. Mathematically, stability exists if the second-order variation of system potential energy due to perturbation of the plasma is positive definite.

(*a*) *Examples of Normal Mode Analysis*

As a first example of the method of small oscillations, we refer to the paper by Kruskal and Schwarzschild (1954). Mainly, these authors considered the stability of a plasma supported against gravity by the pressure of a horizontal magnetic field; and the stability of a cylindrical gaseous conductor radially pinched under the action of its self-magnetic field (see equation (1.2)). In each of these cases the electrical conductivity was assumed infinite, and suitable surface distributions of current and charge were chosen. The analyses showed both equilibria to be unstable, the first in close analogy to the Rayleigh-Taylor instability of a heavy fluid supported in a uniform gravitational field by a weightless fluid (Lamb 1945), and the second against lateral distortions.

Using the latter case to illustrate the method, the solution of the linearized equations for the perturbed plasma is effected by assuming that the disturbance has been analysed into its normal modes, and that the dependence on the cylindrical coordinates θ, z, and the time t of the perturbed quantities is given by $\exp\{i(m\theta+kz)+\omega t\}$, where m is an integer, k a wave number, and $e^{\omega t}$ represents the rate of growth of the instability. The solution gives ω as a function of m and k. If a choice of m and k results in ω being real and positive, perturbations of growing amplitude are obtained, and the system is unstable. Conversely, for purely imaginary ω, bounded oscillations of the perturbations about their means occur, and the system is stable. The phenomenon of overstability (oscillations of growing amplitude), corresponding to complex ω with positive real part, will not occur for static, conservative plasma equilibria (Bernstein *et al.* 1958, p. 22).

For this second case of self-magnetic field confinement, Kruskal and Schwarzschild limited their analysis to the condition $m=1$. In general this gives a corkscrew type of perturbation, for which the axis of the pinched gaseous conductor assumes helical form. To obtain lateral or kink type distortion of the plasma boundary surface, these authors superpose perturbation quantities having helices of opposite twist. This results in a lateral sinusoidal perturbation.

In correspondence to Kruskal and Schwarzschild's analysis of a plasma cylinder in a toroidal magnetic field, Tayler (1957*a*) has obtained results for a volume, rather than a surface, current, with m arbitrary. Particularly, Tayler shows that the infinite rate of growth for vanishingly small perturbation wavelength found by Kruskal and Schwarzschild is removed for a volume current distributed according to a power law.

Recalling that for infinite conductivity the magnetic lines of force appear to be frozen-in to the plasma (Spitzer 1956, pp. 36–37; Cowling 1957; Linhart 1960, p. 127), and that magnetic lines of force behave like elastic strings with a longitudinal tension $H^2/4\pi$ (Lundquist 1950; Cowling 1957, p. 9), physical reasoning suggests that the addition of an axial magnetic field to the interior of the type of plasma configuration discussed by Kruskal and Schwarzschild should produce stabilizing forces to compete with the destabilizing forces. An axial magnetic field external to the plasma would combine with the circular lines of force of the discharge self-magnetic field which exists in the vacuum region (the self-magnetic field vanishes within the plasma for an assumed surface current) to produce a helical resultant magnetic field. Again reasoning physically, for a helical perturbation of the same pitch as this resultant field, the lines of force can be moved into new positions with little bending, and hence with little effort. The magnetic field thus does little to suppress this interchange, and instability results. Interchange instability is considered in detail in Section III.

Using the linear theory, the instability of a pinched conducting fluid in the presence of a longitudinal magnetic field has been treated mathematically by a number of investigators, notably Kruskal and Tuck (1953, 1958), Rosenbluth (1957), Shafranov (1957), and Tayler (1957*b*). Broadly, the above physical considerations are supported by the analyses.

To summarize, when the longitudinal magnetic field is everywhere zero, the $m=0$ (sausage) instabilities, and the $m=1$ (corkscrew, kink) instabilities develop most readily. Particularly, theory predicts that the growth rate of the short-wavelength sausage mode, a purely radial perturbation, exceeds that of all others, and so increase in the pinch containment time should depend initially on suppression of this $m=0$ instability. Appropriate simple distribution of the axial magnetic field inside and outside the plasma theoretically results in suppression of the sausage instability.

Assessment of the effect of an axial field on the $m=1$ mode involves examination of two cases. First, if the helical perturbation twists in the opposite direction to the helical resultant magnetic field, theory shows that the presence of axial magnetic field stabilizes the corkscrew perturbation ; secondly, if the perturbation and resultant magnetic field helices twist in the same direction, it is shown that some corkscrew perturbations of long wavelength continue to grow in amplitude however strong the axial magnetic field. Since analysis also shows that the proximity of a coaxial conducting shell external to the discharge should have a suppressing effect on long wavelength instabilities (Bostick *et al.* 1953), it would appear that a properly chosen combination of axial magnetic field and external conducting shell should lead to complete stability. Conventional hydromagnetic surface current analyses tend to support this hypothesis, with the proviso that axial magnetic field must be minimized outside the plasma : also Chandrasekhar, Kaufman, and Watson (1958) have obtained very similar stability criteria, using a method in which the Boltzmann equation is taken as a starting point, and collisions between the gas constituents are ignored. However, an analysis by Tayler (1957*b*, pp. 1057–61) for uniform current through the discharge, and experience to date give contrary results !

(*b*) *Examples of Energy Principle Analysis*

First we consider the paper by Suydam (1958). Using the Bernstein energy principle, Suydam considers the stability of a linear pinch having a trapped longitudinal magnetic field. The plasma undergoes a displacement $\boldsymbol{\xi}$, and the variational principle is used to obtain the resultant change in system potential energy. If any $\boldsymbol{\xi}$ can produce a reduction of potential energy, the system is unstable : if every displacement increases the potential energy, this will require external work, and the system is stable. The theorem obtained by Suydam states :

" A *necessary* condition that $m\neq 0$ modes of a linear pinch be stable is that

$$\frac{H_z^2}{8\pi}\,\frac{r}{4}\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\ln\frac{H_\theta}{rH_z}\right)\right]^2+\frac{\mathrm{d}p}{\mathrm{d}r}\geqslant 0 \tag{2.1}$$

at every point in the plasma."

Here r, θ, z are cylindrical coordinates, H is the magnetic field, and p the kinetic pressure.

For plasma material at rest, Cowling (p. 4) derives a diffusion equation,

$$\frac{\partial \mathbf{H}}{\partial t}=\frac{1}{4\pi\mu\sigma}\nabla^2\mathbf{H}, \tag{2.2}$$

which suggests that the magnetic field leaks through the plasma with a characteristic time of decay,

$$\tau_d=4\pi\mu\sigma L^2, \tag{2.3}$$

where μ is the permeability, σ the electrical conductivity, and L a length comparable with the dimensions of the system.

Although it may be possible to establish initially a discharge having well-separated axial and azimuthal fields, this decay concept suggests that after a time of order τ_d these fields will diffuse and mix to such an extent that helical resultant magnetic fields will exist both inside and outside the plasma. This is the situation considered by Suydam. The trapped axial field tends to stabilize the plasma by twisting the magnetic lines of force so that a displacement will bend some of them. After a time diffusion will result in a reduction of the torsion of the lines of force, and a particular perturbation may produce a displacement with a minimum of field line bending, so leading to instability. Suydam emphasizes particularly the dangers of these fluted interchanges, whilst Spitzer (1959) discusses the use of " shear " magnetic fields for stabilization purposes. Such fields possess a resultant which changes direction with increase of radial distance from the discharge axis of symmetry, so that interchange of lines of force requires significant bending, and plasma stability should be improved.

The necessary stability criterion (2.1) is satisfied if the term involving H_θ and H_z can be made large near the plasma boundary ; this can be achieved if the sign of H_z is changed in that region. Using normal mode analysis, Rosenbluth (1958) has shown that a reversed axial magnetic field, applied externally to a plasma, pinched and containing the trapped, original axial field, can satisfy necessary and sufficient, or marginal, conditions. However, this procedure poses a practical problem because application of the reversed field must be suitably timed, and this requires extremely rapid establishment of its maximum value.

A second application of the energy principle (Bernstein *et al.* 1958, p. 31) is an examination of the stability of a plasma, confined by an external magnetic field but with no internal magnetic field. The result obtained may be stated as follows :

> " Calling **R** the vector from the centre of curvature of a magnetic line of force to a point on the line, then if **R** everywhere points away from the plasma the system is unstable : if **R** everywhere points towards the plasma, the system is stable."

Hereafter we shall refer to this result as Teller's criterion, because it was originally proposed on intuitive grounds by E. Teller at a Sherwood conference at Princeton, U.S.A., in 1954 (Bishop 1960). Teller's criterion is illustrated in Figure 1.

To conclude this section we observe that the detailed normal mode analyses yield results on instability frequencies and growth rates for various m, and provide either marginal or sufficiency conditions for stability. These conditions appear as rather simple relationships between parameters such as m, k, ω. Use of the energy principle leads to a necessary condition for stability, and confirmation of Teller's powerful criterion.

In the following section we give a thermodynamic treatment which provides useful insight into the phenomenon of interchange instability in a plasma having internal magnetic field. The approach follows broadly those of Rosenbluth and Longmire (1957) and Chandrasekhar and Trehan (1960), but unlike the latter authors, who discuss a curving, axially symmetric confining magnetic field, we give Teller's criterion in a useful mathematical form applicable to more general magnetic field configurations, and apply it in Section IV to the centrally constricted discharge described at the end of Section I.

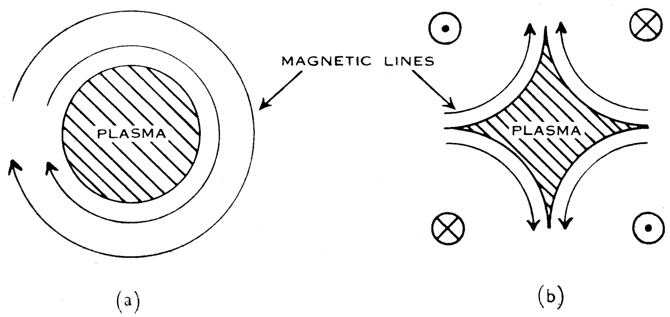

Fig. 1.—Illustration of Teller's criterion. (*a*) Unstable ; (*b*) stable.

III. Thermodynamic Derivation of Teller's Stability Criterion

The following derivation gives a useful integral form of Teller's criterion. We consider a general region in which static plasma and magnetic lines of force are embedded in each other, and choose a pair of associated tubes generated by the flux lines. Remembering that the potential energy of this system is the sum of the magnetic and internal energies of the plasma, we first obtain expressions for the change in magnetic energy, δW, and the change in gas energy, δU, resulting from an interchange of the magnetic flux and plasma in tube 1 of volume V_1, with that in tube 2 of volume V_2.

(*a*) *Calculation of* δW

If l is distance along a flux tube n, having cross sectional area $A(l)$ and volume element $\mathrm{d}V=A(l)\mathrm{d}l$, the total magnetic energy within the tube n is

$$W_n=\int_n \frac{H^2}{8\pi}A\,\mathrm{d}l, \tag{3.1}$$

where the subscript n signifies that the integral is for the complete tube n.

Since div $\mathbf{H}=0$, the total flux $\varphi=HA$ is constant along a flux tube. Accordingly, (3.1) becomes

$$W_n=\frac{\varphi_n^2}{8\pi}\int_n\frac{dl}{A}. \tag{3.2}$$

Using (3.2), we obtain

$$\delta W=\frac{1}{8\pi}\left[\left(\varphi_2^2\int_1\frac{dl}{A}+\varphi_1^2\int_2\frac{dl}{A}\right)-\left(\varphi_2^2\int_2\frac{dl}{A}+\varphi_1^2\int_1\frac{dl}{A}\right)\right]$$

or

$$\delta W=\frac{\varphi_2^2-\varphi_1^2}{8\pi}\left[\int_1\frac{dl}{A}-\int_2\frac{dl}{A}\right]. \tag{3.3}$$

(*b*) *Calculation of* δU

On the reasonable assumption that the interchange between the associated flux tubes 1 and 2 is locally adiabatic (i.e. heat gain by Joule heating and heat loss by conduction does not occur), the first law of thermodynamics becomes

$$0=dU+pdV. \tag{3.4}$$

Using the adiabatic equation of state

$$pV^\gamma=\text{const.}, \tag{3.5}$$

equation (3.4) integrates to the form

$$U=pV/(\gamma-1), \tag{3.6}$$

where U is the internal energy of the plasma per unit mass, p the kinetic pressure, V the specific volume, and γ the ratio of the specific heat at constant pressure to that at constant volume. Since the steady-state equation (1.2) implies that the pressure p is constant along a line of force, the change in gas energy resulting from the interchange between flux tubes 1 and 2 can now be written

$$\delta U=\frac{1}{\gamma-1}[(p_2'V_1+p_1'V_2)-(p_1V_1+p_2V_2)], \tag{3.7}$$

where p_2', the pressure in tube 1 after the interchange, and p_1', the pressure in tube 2 after the interchange are, from (3.5), given by

$$p_2'=(V_2/V_1)^\gamma p_2,\quad p_1'=(V_1/V_2)^\gamma p_1. \tag{3.8}$$

Elimination of p_2' and p_1' from (3.7) by means of (3.8) gives

$$\delta U=-\frac{1}{\gamma-1}[(p_2V_2^\gamma-p_1V_1^\gamma)(V_2^{1-\gamma}-V_1^{1-\gamma})]. \tag{3.9}$$

If the flux tubes are in close proximity,

$$p_2=p_1+\delta p,\qquad V_2=V_1+\delta V, \tag{3.10}$$

and it is then evident that (3.9) becomes

$$\delta U=-\frac{1}{\gamma-1}\delta(pV^\gamma)\delta(V^{1-\gamma}),$$

or

$$\delta U=p(\delta p/p+\gamma\delta V/V)\delta V. \tag{3.11}$$

(c) *The Flute Instability and Teller's Criterion*

Consider a magnetic field **H** in a region R, produced by given currents outside the region. Then in R, div $\mathbf{H}$=curl $\mathbf{H}=0$, and there exist magnetic vector and scalar potentials. Call the magnetic scalar potential Ψ, so that $\mathbf{H}=\text{grad}\,\Psi$. Further, consider within R a sub-region r, of volume V_r bounded by a surface S_r, and let a displacement of the matter in V_r vary **H** to $\mathbf{H}+\mathbf{h}$, where the perturbation **h** vanishes on and outside S_r. Then div $(\mathbf{H}+\mathbf{h})=0$, so that **h** is also solenoidal. The flux tubes generated by **h** are therefore closed, and confined within the volume V_r. The change in magnetic energy here is

$$\delta W'=\frac{1}{8\pi}\int_{V_r}(\mathbf{H}+\mathbf{h})\cdot(\mathbf{H}+\mathbf{h})\mathrm{d}V-\frac{1}{8\pi}\int_{V_r}H^2\mathrm{d}V$$

or

$$\delta W'=\frac{1}{4\pi}\int_{V_r}\text{grad}\,\Psi\cdot\mathbf{h}\mathrm{d}V+\frac{1}{8\pi}\int_{V_r}h^2\mathrm{d}V. \tag{3.12}$$

With div $\mathbf{h}=0$, application of Green's theorem to the first integral on the right-hand side of (3.12) yields

$$\Omega=\frac{1}{4\pi}\int_{S_r}(\Psi\mathbf{h})\cdot\mathrm{d}\mathbf{S}. \tag{3.13}$$

Since **h** vanishes on S_r, $\Omega=0$, and (3.12) reduces to

$$\delta W'=\frac{1}{8\pi}\int_{V_r}h^2\mathrm{d}V>0, \tag{3.14}$$

thus confirming that an irrotational magnetic field **H** is stable against a finite perturbation **h**. The proof given is similar to that by Lundquist (1952, p. 308).

We now apply the results (3.3), (3.11), and (3.14) to the diffuse boundary of a plasma in a magnetic field, considering particularly the type of interchange instability which flutes the plasma surface along the magnetic lines. More specifically, we take the magnetic field near the plasma boundary, where the gas is very tenuous and the kinetic pressure vanishingly small, to be closely irrotational. Then, if the interchange attempts to bend lines of force, (3.14) shows that the magnetic energy must be increased. Hence, near the plasma boundary the only type of interchange that may lead to instability is that which leaves the magnetic field unchanged. By locating flux tubes 1 and 2 close to the boundary, we can satisfy this condition by making $\varphi_1=\varphi_2$, whereupon (3.3) gives

$$\delta W=0. \tag{3.15}$$

The flute interchange is illustrated in Figure 2. The condition for stability now assumes the simple form

$$\delta U>0. \tag{3.16}$$

On the realistic assumption that p vanishes smoothly with radial approach to the boundary, we have the conditions

$$\left.\begin{aligned}p&>0,\\ \delta p/p&<0,\\ |\delta p/p|&>|\gamma\delta V/V|.\end{aligned}\right\} \tag{3.17}$$

Hence (3.11) shows that the stability condition (3.16) becomes

$$\delta V < 0. \tag{3.18}$$

Since for a flux tube, $V = \varphi \int dl/H$, we have for the interchange considered the general result for stability

$$\delta \int dl/H < 0, \tag{3.19}$$

because $\varphi > 0$.

The inequality (3.19) can be usefully transformed as follows. From the flute illustration, Figure 2, we have for the non-coplanar magnetic lines 1 and 2, separated at distance l by perpendicular distance $a(l)$,

$$\int_2 H dl = \int_1 H dl, \tag{3.20}$$

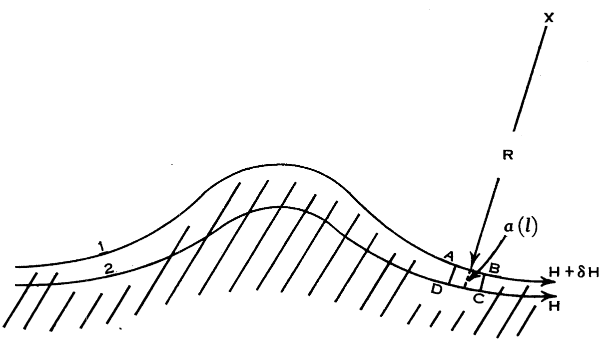

Fig. 2.—Flute instability at plasma boundary.

since the points connected by $a(l)$ are at the same magnetic scalar potential. The assumed condition curl $\mathbf{H} \sim 0$ in this region implies the vanishing of the line integral taken round the small rectangle $ABCD$ in Figure 2, which gives

$$\delta H/H = a/R. \tag{3.21}$$

Forming now

$$\delta \int \frac{dl}{H} = \int_2 \frac{dl}{H} - \int_1 \frac{dl}{H},$$

we obtain by means of (3.20)

$$\begin{aligned} \delta \int \frac{dl}{H} &= \int \left(\frac{1}{H_2^2} - \frac{1}{H_1^2} \right) H dl \\ &= \int \delta \left(\frac{1}{H^2} \right) H dl. \end{aligned} \tag{3.22}$$

Expanding under the integral sign, and eliminating $\delta H/H$ by means of (3.21),

$$\delta \int \frac{dl}{H} = -2 \int \frac{a dl}{RH}. \tag{3.23}$$

The stability condition (3.19) can now be written in the important general form

$$\int \frac{a\mathrm{d}l}{RH} > 0, \tag{3.24}$$

where $a>0$, $H>0$, and the derivation has been based on the magnetohydrodynamic approximation of a scalar pressure, with $p\sim 0$ at the plasma boundary.

With the usual convention that the concavity of the magnetic lines of force in Figure 2 is upwards for $R>0$ and downwards for $R<0$, equation (3.24) gives a compact integral form of Teller's criterion. Applied to the flute interchange of Figure 2, it predicts instability because, in the middle region where $R<0$, the magnetic field is least, and the net result is that the destabilizing forces exceed the stabilizing forces.

IV. Application of Teller's Criterion to a Radially Constricted Gas Discharge

Consistently with the findings of Section II, the result (3.24) supports the remark that the curvature of the magnetic lines of force is clearly in the unstable direction for any helical or azimuthal field (Berkowitz, Grad, and Rubin 1958).

In earlier papers by the present author (1961*a*, 1961*b*) the boundary surface of the discharge studied was represented by a hyperboloid of revolution of one sheet. Such a surface can be produced by rotation of a generating line about the axis of symmetry. The cross section of this discharge in the ρ-z plane is shown in Figure 3 (*a*), using the notation adopted earlier; a generator is also shown, and is to be imagined raised a vertical distance ρ_0 from the ρ-z plane. It is also helpful in the following discussions to imagine the discharge curved semi-surface above the ρ-z plane, produced by motion of this generator. Figure 3 (*b*) serves to define parameters associated with a generator.

Defining the azimuthal field due to $I^*(w_b)$ as H_φ, and the external guiding field due to the solenoid as H_E, we can at the discharge surface define an angle of inclination θ, of the resultant magnetic field to the direction of the axis of symmetry by the equation

$$\tan\theta \approx H_\varphi/H_E. \tag{4.1}$$

If we now regard the appropriately shaped H_E field as initially established at its full value, so that $\tan\theta$ can be increased by increasing H_φ (i.e. by increasing $I^*(w_b)$ from a threshold value corresponding to initiation of the discharge), the magnetic field resultant at a point on the generator in Figure 3 (*a*) can be made to swing from approximately the axial direction towards the direction of the generator. Further increase of H_φ leads to the magnetic field resultant crossing to the right of the generator. To permit a simple estimate of stability, it is assumed that the inclination θ_g of the generator to the axial direction is independent of $I^*(w_b)$ when central constriction is large, and, because of conservation of flux, that at the discharge surface H_E is approximately proportional to ρ^{-2}, where ρ is the radial distance from the symmetry axis; H_φ varies as ρ^{-1}. Then, when the magnetic field resultant is to the left of the generator, where $R>0$, equation (3.24) shows that stabilizing forces act. However, to the right of the

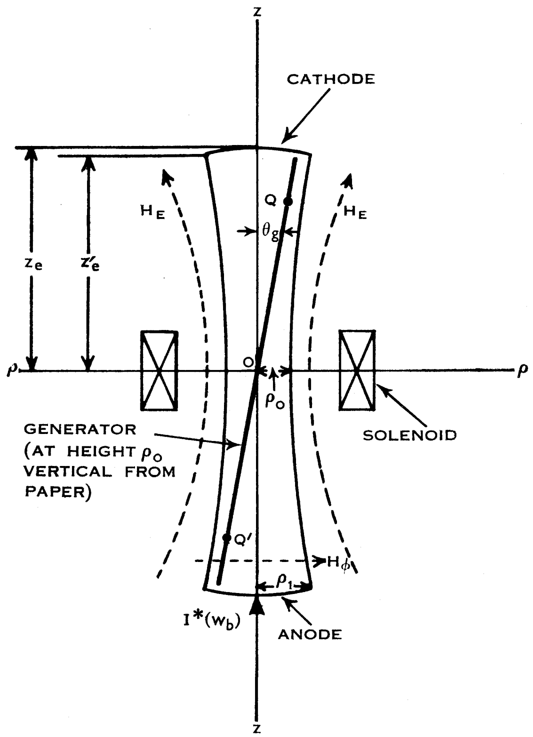

Fig. 3 (*a*).—Discharge cross section in the ρ-z plane.

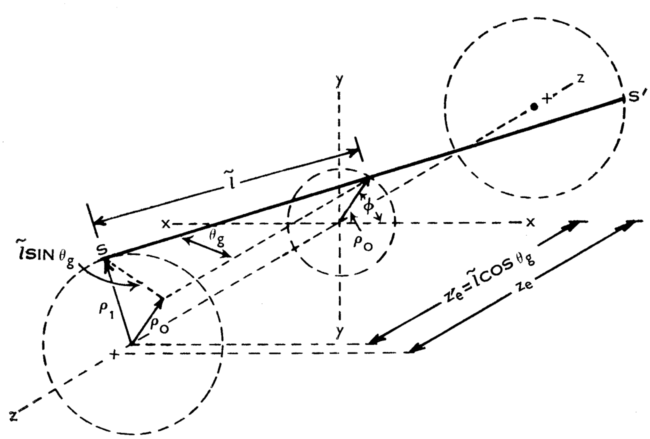

Fig. 3 (*b*).—Parameters associated with a generator *S*–*S'*.

generator, where $R<0$, the combined H_E and H_φ field forms a type of helix, and destabilizing forces contribute. Since $H_\varphi \propto \rho^{-1}$, $H_E \propto \rho^{-2}$, it follows from (4.1) that $\tan\theta \propto \rho$, and so from Figure 3 (*a*) we note that θ has a minimum at $z=0$ and equal maxima at the electrodes. Hence if θ of the field resultant matches θ_g at some point Q on a generator (Fig. 3 (*a*)), then these angles will again match at an image point Q', so that between Q and Q' stabilizing forces contribute, whilst between Q and the cathode and between Q' and the anode destabilizing forces act. Assessment of the net effect of these competing forces for chosen Q, Q' involves evaluation of the integral in (3.24). To overcome this difficulty and obtain a sufficient condition for stability, we set Q and Q' at the electrodes, so that stabilizing forces only contribute.

The current $I_0^*(w_b)$ which satisfies this sufficient stability condition can be estimated as follows. From Figure 3 (*b*), $z_e'=\tilde{l}\cos\theta_g$ and $(\rho_1^2-\rho_0^2)^{\frac{1}{2}}=\tilde{l}\sin\theta_g$, giving

$$\tan\theta_g=\frac{\rho_1}{z_e'\nu}(\nu^2-1)^{\frac{1}{2}}, \tag{4.2}$$

where $\nu=\rho_1/\rho_0$ is the radial compression ratio.

From equations (6.4.19), (6.4.28), and (6.4.31) (Seymour 1961*a*),

$$z_e'=z_e[1-(\rho_1/z_e)^2(1-1/\nu^2)]^{\frac{1}{2}},$$

where z_e is the discharge semi-length. Thus, eliminating z_e' from (4.2),

$$\tan\theta_g=\left(\frac{(\rho_1/z_e)^2(\nu^2-1)}{\nu^2-(\rho_1/z_e)^2(\nu^2-1)}\right)^{\frac{1}{2}}. \tag{4.3}$$

For the assumed ρ-dependences of H_φ and H_E, equation (4.1) gives at the electrodes

$$\tan\theta\approx\frac{\pi}{5}\,\frac{I^*(w_b)}{\varphi_E}\rho_1, \tag{4.4}$$

where the current unit is the ampere, and φ_E is the total flux from the solenoid through any equipotential surface terminating at the boundary surface of the discharge.

Setting $\theta=\theta_g$ in (4.4) and combining the result with (4.3),

$$I_0^*(w_b)\approx\frac{5\varphi_E}{\pi\rho_1}\left(\frac{(\rho_1/z_e)^2(\nu^2-1)}{\nu^2-(\rho_1/z_e)^2(\nu^2-1)}\right)^{\frac{1}{2}}. \tag{4.5}$$

V. Discussion of Results

For the type of discharge studied here, theory suggests that a stable plasma configuration can be obtained if the external field H_E, shaped to give the required centrally constricted boundary surface without too much assistance from the self-field H_φ, is initially strong enough.

Then, as $I^*(w_b)$ is increased, the destabilizing effect of the azimuthal H_φ becomes greater until, for a current of order $I_0^*(w_b)$ given by (4.5), the stable plasma configuration is destroyed. Since $(\rho_1/z_e)^2 \ll 1$ is a likely practical condition, and we desire $\nu^2 \gg 1$, equation (4.5) gives this current simply as

$$I_0^*(w_b) \sim 1 \cdot 6 \varphi_E/z_e. \tag{5.1}$$

In practice it may be more convenient to work in terms of $H_E(0)$ on the discharge median plane. Equation (5.1) then assumes the form

$$I_0^*(w_b) \sim 5 H_E(0) \rho_0^2/z_e. \tag{5.2}$$

It is important to observe that the above conclusions are based on an assumed infinite electrical conductivity, so that τ_d of equation (2.3) is also infinite. In practice we must contend with finite τ_d, which, from (2.3) and Seymour (1961*a*, equation (5.7)), may be written in terms of the temperature T as

$$\tau_d \sim 2 \times 10^{-13} L^2 T^{3/2} \text{ cm}^{-2} \text{ deg}^{-3/2} \text{ s}, \tag{5.3}$$

when $\ln \lambda = 10$, a result which is consistent with Spitzer's result (1956, p. 38). For practical values of L and T, τ_d is in the region of milliseconds, and so the " steady state " is of short duration !

However, it is thought that the procedure outlined in Section IV could form the basis of an interesting laboratory experiment for the observation of transition from stability to instability in a constricted gas discharge between electrodes. Then, if the practical results obtained from such an experiment were to support the theory reasonably, there would be a case for a more detailed theoretical stability study of this type of discharge.

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