SIMPLE S AND D DEUTERON GROUND STATE WAVEFUNCTIONS ASSUMING CENTRAL AND r^{-2} TENSOR POTENTIALS

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Summary

The coupled wave equations for the S and D parts of the deuteron ground state in the presence of central and tensor forces are shown to reduce to one equation if the tensor potential has an r^{-2} radial dependence. One wavefunction is then a multiple of the other. It is shown that a simple Hulthen wavefunction may be adjusted to give the observed electric quadrupole moment together with a D-state probability having any desired value between $0 \cdot 1$ and $5 \cdot 6\%$.

I. Introduction

There have been many discussions of the proton–neutron interaction in terms of two-body forces, in particular of central, tensor, and spin-orbit kinds, in which meson dynamics are ignored and non-relativistic quantum mechanics is applied. The observed magnetic and electric quadrupole moments of the deuteron lead (see e.g. Sachs 1953) to a description of the bound state in terms of central and tensor forces; while analyses of high-energy scattering data have led to well-fitting descriptions of the interaction in terms of central, tensor, and spin-orbit forces, each assumed to have a Yukawa-like radial dependence (Gammel and Thaler 1957; Signell and Marshak 1958). One of the obstacles to discussion is the fact that the introduction of the tensor potential into the Schroedinger equation yields two coupled radial equations which are relatively intractable: an analytic solution cannot be given even for a square-well potential. All solutions obtained so far involve numerical procedures. In these it is usually assumed that the tensor potential has the same radial dependence as the central potential.

In the present paper it is shown that if the deuteron ground state may be specified by central and tensor potentials only and if the tensor potential has an r^{-2} radial dependence, then the coupled S- and D-state equations reduce to one equation and the two wavefunctions are the same but for a constant. A simple form for this wavefunction is found to be adjustable to give the observed value of the electric quadrupole moment together with a suitable amount of D-state in the ground state.

II. r^{-2} Tensor Potential

We suppose a neutron and proton to interact through central and tensor two-body potentials of the form

$$V_c(r) = -\frac{\hbar^2}{M} W_c(r),$$

$$V_t(r) = -\frac{\hbar^2}{M} W_t(r). \tag{1}$$

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It is well known that the wavefunction of the deuteron ground state then takes the form

$$\psi_1^m=\left[\frac{1}{r}u(r)+\frac{1}{r}w(r)\mathbf{S}_{np}\right]\chi_1^m, \tag{2}$$

which is a mixture of S- and D-state functions $\frac{1}{r}u(r)$ and $\frac{1}{r}w(r)$. Here $\mathbf{S}_{np}$ is the tensor operator

$$\mathbf{S}_{np}=\frac{3(\boldsymbol{\sigma}_n\cdot\mathbf{r})(\boldsymbol{\sigma}_p\cdot\mathbf{r})}{r^2}-(\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_p).$$

The normalization condition is

$$4\pi\int_0^\infty u^2(r)\mathrm{d}r+32\pi\int_0^\infty w^2(r)\mathrm{d}r=1. \tag{3}$$

The Schroedinger equation then reduces to the following coupled equations for the functions u and w:

$$u''+(W_c-\alpha^2)u+8W_tw=0, \tag{4}$$

$$w''+\left(W_c-2W_t-\frac{6}{r^2}-\alpha^2\right)w+W_tu=0. \tag{5}$$

Now suppose that

$$w=\beta u, \tag{6}$$

β a constant. Then (4) and (5) reduce to

$$u''+(W_c-\alpha^2+8W_t\beta)u=0,$$

$$u''+\left(W_c-2W_t-\frac{6}{r^2}-\alpha^2+\frac{W_t}{\beta}\right)u=0.$$

If these equations are to hold for any $r>0$ they must be identical. Therefore (6) is true if and only if

$$8\beta W_t=-2W_t-\frac{6}{r^2}+\frac{W_t}{\beta},$$

that is,

$$W_t=-6/r^2(8\beta+2-1/\beta)=\gamma/r^2. \tag{7}$$

Accordingly, if we now assume that

$$W_t=\frac{\gamma}{r^2}, \tag{7a}$$

it follows that

$$w=\beta u, \tag{6a}$$

in the deuteron ground state and that

$$u''+\left(W_0-\alpha^2+\frac{\lambda}{r^2}\right)u=0, \tag{8}$$

where

$$\beta=\frac{1}{8}\left\{-\left(1+\frac{3}{\gamma}\right)\pm\sqrt{\left[\left(1+\frac{3}{\gamma}\right)^2+8\right]}\right\}, \tag{9}$$

and

$$\lambda=8\beta\gamma. \tag{10}$$

The deuteron electric quadrupole moment is now (Sachs 1953)

$$Q=\frac{8\pi}{5}e\int_0^\infty (uw-w^2)r^2\mathrm{d}r$$
$$=\beta(1-\beta)\frac{8\pi}{5}e\int_0^\infty u^2r^2\mathrm{d}r, \tag{11}$$

and this is positive experimentally, so we would require

$$0<\beta<1. \tag{12}$$

The normalization condition (3) becomes

$$4\pi(1+8\beta^2)\int_0^\infty u^2\mathrm{d}r=1. \tag{13}$$

From (7)

$$\gamma=-6\beta/(4\beta-1)(2\beta+1), \tag{14}$$

so the range $(0,\frac{1}{4})$ for β corresponds to

$$0<\gamma<\infty$$

(yielding $0<\lambda<\infty$), an attractive tensor potential of any depth, while the range $(\frac{1}{4},1)$ for β corresponds to

$$-\infty<\gamma<-\tfrac{2}{3},$$

a repulsive tensor potential of minimum height $-\gamma=\frac{2}{3}$. It seems likely that the tensor potential would be attractive for any plausible central potential, narrowing the range of interest of β from that of (12) to

$$0<\beta<\tfrac{1}{4}. \tag{15}$$

This is not a severe restriction as $\beta=\frac{1}{4}$ defines from (13) a ground state which is 67% *S*-state and 33% *D*-state, while estimates of *D*-state probability based on the observed magnetic moment of the deuteron are commonly about 4%, corresponding to $\beta\approx0\cdot07$. This value of β makes $\gamma=0\cdot51$ and the tensor potential would then be

$$V_t=-21\cdot2/r^2\ \mathrm{MeV}, \tag{16}$$

if r is measured in fermis.

III. Wavefunctions

The wave equation (8) is the same as that for a system of angular momentum l and central potential W_c if $\lambda=-l(l+1)$: however, we are not restricted here to a λ corresponding to integral or even positive l, indeed the range (15) for β makes λ positive.

If we take for

$$W_0=W_c+\lambda/r^2, \tag{17}$$

any of the usual central potentials used in models of the deuteron, then we have the standard equation

$$u''+(W_0-\alpha^2)u=0, \tag{18}$$

and any of the numerically adjusted standard potentials, such as Yukawa or square-well may be applied at once in the present model. Any standard central potential W_0 which fits the deuteron binding energy may be taken, have added to it a repulsive core $-\lambda/r^2$, and we then have a central potential

$$W_c = W_0 - \lambda/r^2 \tag{19}$$

which, together with a tensor potential $W_t = \gamma/r^2$, describes a deuteron ground-state model through the known wavefunctions u (satisfying (18)) and $w = \beta u$, where λ, γ, and β are related through (7), (9), and (10); there is one independent parameter. This parameter may then turn out to be sufficiently adjustable to yield the right value for the quadrupole moment (11): it might also turn out that if that is done then β is of the right size for the D-state probability to lie in the plausible region around 4%, in order to accord with the observed magnetic dipole moment.

IV. Hulthen Wavefunctions

The wavefunction

$$u = N(\mathrm{e}^{-\alpha r} - \mathrm{e}^{-\delta r}), \tag{20}$$

where $\delta > \alpha$, is a solution of (18) if the potential function has the form

$$W_0 = \frac{(\delta^2 - \alpha^2)\mathrm{e}^{-(\delta-\alpha)r}}{1 - \mathrm{e}^{-(\delta-\alpha)r}} \tag{21}$$

$$= A\frac{\mathrm{e}^{-\mu r}}{1 - \mathrm{e}^{-\mu r}},$$

so that the central potential W_c is like a Yukawa-attractive plus r^{-2}-repulsive potential:

$$\frac{A\mathrm{e}^{-\mu r}}{\mu r} - \frac{\lambda}{r^2},$$

for r near zero and an exponential-attractive plus r^{-2}-repulsive potential

$$A\mathrm{e}^{-\mu r} - \frac{\lambda}{r^2},$$

for r large. The parameter δ describes a family of wells all having a ground state (20) of energy α^2: as $\delta \to \infty$ the wells become deeper and of shorter range.

From the normalization condition (13),

$$N^2 = \alpha \frac{x(x+1)}{(x-1)^2 2\pi(1+8\beta^2)}, \tag{22}$$

where $x = \delta/\alpha > 1$; and the quadrupole moment (11) becomes

$$Q = \beta(1-\beta) \cdot \frac{8\pi}{5}e \cdot \frac{N^2}{4\alpha^3} \frac{(x-1)^2}{x^3(x+1)^3}[(x^2+1)^2 + 5x(x+1)^2]$$

$$= \frac{\beta(1-\beta)e}{(1+8\beta^2)\alpha^2}\left[\frac{(x^2+1)^2}{5x^2(x+1)^2} + \frac{1}{x}\right]. \tag{23}$$

For a given β, that is, a given percentage of D-state, Q decreases monotonically from

$$Q_{\max.} = \frac{6}{5} \frac{\beta(1-\beta)e}{(1+8\beta^2)\alpha^2},$$

for $x \to 1$, towards

$$Q_{\text{min.}} = \frac{1}{5}\,\frac{\beta(1-\beta)e}{(1+8\beta^2)\alpha^2},$$

for $x \to \infty$.

The observed moment is about $2 \cdot 77 \times 10^{-27}$e cm^2. Using the wavefunction (20) and taking $\beta = 0 \cdot 0722$, 4% of *D*-state, we obtain

$$Q_{\text{max.}} = 6 \times 2 \cdot 397 \times 10^{-27} \text{ e cm}^2,$$
$$Q_{\text{min.}} = 2 \cdot 397 \times 10^{-27} \text{ e cm}^2.$$

So to fit the observed value of Q with 4% of *D*-state requires a fairly large value of x, approximately 21. The limiting value of β which will fit the observed value of Q as $x \to \infty$ is about $0 \cdot 086$, corresponding to $5 \cdot 6$% of *D*-state. The other limiting value for β, which fits the observed Q as $x \to 1$, is about $0 \cdot 0125$, corresponding to $0 \cdot 12$% of *D*-state. So the observed Q may be obtained for any desired amount of *D*-state between $0 \cdot 1$% and $5 \cdot 6$%. The same (but for β) Hulthen wavefunction in both the *S*- and *D*-states, satisfying the coupled wave equations (4) and (5) when the central and tensor potentials are assumed to be given by (19), (21), and (7), related by (9) and (10), can yield the observed electric quadrupole moment of the deuteron and a *D*-state probability which seems to be of the right size.

V. References

Gammel, J. L., and Thaler, R. M. (1957).—*Phys. Rev.* **107** : 291.
Sachs, R. G. (1953).—" Nuclear Theory." (Addison-Wesley : Reading, Mass.)
Signell, P. S., and Marshak, R. E. (1958).—*Phys. Rev.* **109** : 1229.