DAMPED WAVES ASSOCIATED WITH THERMAL CONVECTION

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Summary

Damped temperature-time oscillations associated with thermal convection are analysed and the energy of thermal inductance is identified with the free energy of entropy flow from the convection chimney.

I. Mathematical Analysis

In an earlier paper by two of the present authors (Bosworth and Groden 1960) the possible solutions for thermal transients associated with natural convection were enumerated on the basis of a postulated equivalent electrical circuit. One of the possible solutions included trigonometrical functions in which multiple steady states were in principle expected. In 1960 Wecksler realized that such conditions could be obtained (lecture to The Institution of Radio Engineers Australia, Radio and

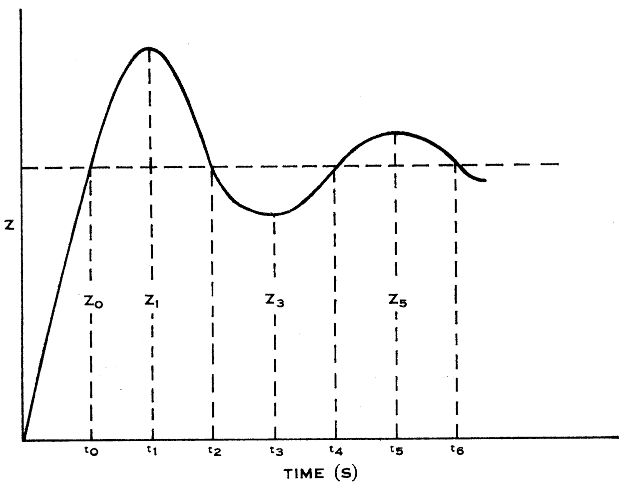

Fig. 1.—Oscillatory transients, Z = temperature.

Electronic Engineering Convention, Sydney, 1961). The thermal transient for a heated wire took the form (temperature versus time) of a damped wave in which the actual temperature of the wire reached the same value as the final asymptotic value after the lapse of t_0, t_2, t_4, t_6 seconds from the addition of the heating current while the temperature passed through maxima in t_1, t_5, etc. seconds and through minima in t_3, t_7, etc. seconds (Fig. 1).

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The amplitudes associated with the various critical times are denoted by:

Z_0 at $t = t_0 = t_2 = t_4, \ldots,$
Z_1 at $t = t_1$, first maximum,
Z_3 at $t = t_3$, first minimum,
Z_5 at $t = t_5$, second maximum, etc.

Using the notation of the previous paper we have

$$X = (L-RCr)/2r\surd(LC) \gtrless 0, \tag{1}$$

$$Y = (1/R)\surd(L/C) > 0, \tag{2}$$

$$\tau = t/\surd(LC), \tag{3}$$

$$A = \tau_1/\tau_2 < 1,$$

or

$$C = 1/A = \tau_2/\tau_1 > 1, \tag{4}$$

and for $|X| \leqslant 1$,

$$Z = \frac{rR}{r+R}\left[1-\exp\left\{-\left(X+\frac{1}{Y}\right)\tau\right\}\left\{\cos(1-X^2)^{\frac{1}{2}}\tau-\frac{X+Y}{(1-X^2)^{\frac{1}{2}}}\sin(1-X^2)^{\frac{1}{2}}\tau\right\}\right]. \tag{5}$$

When $\tau \to \infty$,

$$Z \to Z_0 = rR/(r+R). \tag{6}$$

This value also is attained when

$$\cos(1-X^2)^{\frac{1}{2}}\tau-\frac{X+Y}{(1-X^2)^{\frac{1}{2}}}\sin(1-X^2)^{\frac{1}{2}}\tau = 0, \tag{7}$$

or when

$$(1-X^2)^{\frac{1}{2}}\tau = \tan^{-1}\left\{\frac{(1-X^2)^{\frac{1}{2}}}{X+Y}\right\}+n\pi, \qquad n = 0, 1, 2, \ldots, \tag{8}$$

where $\tan^{-1}\{(1-X^2)^{\frac{1}{2}}/(X+Y)\}$ is the principal value of the inverse tangent function. Then

$$\tau_2 = \frac{1}{(1-X^2)^{\frac{1}{2}}}\left[\pi+\tan^{-1}\left\{\frac{(1-X^2)^{\frac{1}{2}}}{X+Y}\right\}\right]. \tag{9}$$

The extreme values of Z occur at the points when $\mathrm{d}Z/\mathrm{d}\tau = 0$. This condition yields

$$\tan(1-X^2)^{\frac{1}{2}}\tau = (1-X^2)^{\frac{1}{2}}/X, \tag{10}$$

or

$$\left.\begin{aligned}(1-X^2)^{\frac{1}{2}}\tau &= \tan^{-1}\{(1-X^2)^{\frac{1}{2}}/X\}+n\pi, \qquad n = 0, 1, 2, \ldots\\ &= \cos^{-1}X+n\pi.\end{aligned}\right\} \tag{11}$$

Hence, for the first maximum ($n = 0$)

$$\tau_1 = \frac{\cos^{-1}X}{(1-X^2)^{\frac{1}{2}}} = \frac{T}{\sin T}, \tag{12}$$

if

$$\cos T = X. \tag{13}$$

Similarly, for the first minimum ($n = 1$)

$$\tau_3 = (T+\pi)/\sin T. \tag{14}$$

These two values yield, in turn,

$$Z_1 = \frac{rR}{r+R}\left[1+Y\exp\left\{-\left(X+\frac{1}{Y}\right)\frac{T}{\sin T}\right\}\right] \quad \text{(a maximum)}, \tag{15}$$

and

$$Z_3 = \frac{rR}{r+R}\left[1-Y\exp\left\{-\left(X+\frac{1}{Y}\right)\frac{T+\pi}{\sin T}\right\}\right] \quad \text{(a minimum)}. \tag{16}$$

Now

$$\left.\begin{aligned} C = \frac{\tau_2}{\tau_1} &= \frac{\pi+\tan^{-1}\{(1-X^2)^{\frac{1}{2}}/(X+Y)\}}{\cos^{-1}X} \\ &= \frac{\pi+\tan^{-1}\{\sin T/(\cos T+Y)\}}{T}, \end{aligned}\right\} \tag{17}$$

or, solving for Y, we get

$$Y = -\frac{\sin(C-1)T}{\sin CT} > 0. \tag{18}$$

Also let

$$D = \frac{Z_1}{Z_0} = 1+Y\exp\left\{-\left(X+\frac{1}{Y}\right)\frac{T}{\sin T}\right\} = 1+B, \tag{19}$$

where

$$B = Y\exp\left\{-\left(X+\frac{1}{Y}\right)\frac{T}{\sin T}\right\} > 0. \tag{20}$$

Eliminating Y we then get

$$B = -\frac{\sin(C-1)T}{\sin CT}\cdot e^{T\cot(C-1)T} > 0. \tag{21}$$

Since B must be positive, $\sin(C-1)T/\sin CT$ must be negative. This condition leads to the following restrictions for T:

$$\left.\begin{aligned} &(a)\ \text{when}\ C \geqslant 2, && \text{then}\ \frac{\pi}{C} < T < \frac{\pi}{C-1}, \\ &(b)\ \text{when}\ 1 < C \leqslant 2, && \text{then}\ \frac{\pi}{C} < T < \frac{2\pi}{C}, \end{aligned}\right\} \tag{22}$$

and it can be shown that in both ranges of T, B is always finite (note that (21) becomes infinite when $(C-1)T = n\pi$).

Using the experimental values t_1, t_2, Z_0, and Z_1, and obtaining B from (19) and C from (4), T is then taken from Table 1 (which is based on (21)), and then X from (13), and Y from (18); t_0 and t_3 can be obtained from (8), (12), and (14), and

Table 1

Values of B for given C and T

$$B = -\frac{\sin(C-1)T}{\sin CT} \,.\, e^{T\cot(C-1)T}$$

	C \ T	1·2	1·4	1·6	1·8	2·0	2·2	2·4	2·6	3·0	3·5	4·0
$\pi/C < T < 2\pi/C$	1·2									102·4	47·14	35·04
	1·3								41·24	12·31	6·545	4·996
	1·4							20·29	8·300	3·433	1·834	1·409
	1·5						17·31	5·355	2·883	1·262	0·608	0·521
	1·6					34·47	4·607	2·114	1·208	0·485	0·177	
	1·7					5·445	1·899	0·947	0·522	0·161	0·028	
	1·8				12·787	2·131	0·883	0·424	0·205	0·033		
	1·9				3·326	0·998	0·410	0·169	0·060	0·001		
	2·0			16·341	1·446	0·481	0·171	0·049	0·008			
$\pi/C < T < \pi/(C-1)$	2·05			6·046	1·005	0·328	0·102	0·021	0·001			
	2·10			3·337	0·705	0·216	0·054	0·006				
	2·15			2·109	0·494	0·136	0·025	0·001				
	2·20			1·420	0·342	0·080	0·009					
	2·25		90·827	0·988	0·231	0·042	0·002					
	2·30		8·664	0·697	0·150	0·019						
	2·35		4·043	0·492	0·091	0·006						
	2·40		2·405	0·344	0·051	0·001						
	2·45		1·577	0·235	0·025							
	2·50		1·085	0·156	0·010							
	2·55		0·764	0·098	0·003							
	2·60		0·542	0·057								
	2·65	14·199	0·383	0·029								
	2·70	4·941	0·267	0·012								
	2·75	2·712	0·181	0·004								
	2·80	1·721	0·117									
	2·85	1·166	0·072									
	2·90	0·816	0·040									
	2·95	0·580	0·019									
	3·00	0·412	0·007									

Z_3 from (16). The calculated values of t_0, t_3, and Z_3 may be compared with the experimental values (Table 2). From Bosworth and Groden (1960), the equations (2b), (14), and (17) and Table 2 enable the parameters r, R, L, and C to be calculated.

II. Experimental Apparatus

The experimental system consisted of a horizontal platinum wire, electrically heated, immersed in a liquid. The temperature rise of the wire was measured by making it part of the Wheatstone bridge network shown in Figure 2. AB was the platinum wire (approximately 0·5 Ω) and BC a manganin resistor adjusted to be equal in resistance to the unheated platinum wire. AD and DC were exactly equal resistors of about 60 Ω. A constant voltage was maintained across AC by a voltage stabilizer, and the bridge output across BD, which gave the temperature rise, was measured by an electronic recorder. If a constant voltage is maintained across two resistances in series, one fixed and one variable, the power in the variable resistor remains substantially constant against considerable changes in resistance (Rosengren 1961).

Table 2

MEASUREMENTS AND CALCULATED PARAMETERS FROM TRANSIENTS

Heating Current (A)	t_1 (s)	t_2 (s)	t_3 (s)	Z_0 (degC)	Z_1 (degC)	Z_3 (degC)	Calculated t_3 (s)	Calculated Z_3 (degC)
0·681	2·42	5·2	7·5	4·94	5·49	4·84	6·1	4·93
0·766	1·95	4·4	6·6	6·06	6·74	5·97	5·1	6·03

Heating Current (A)	I_s (J (degC)$^{-1}$ s^{-1} cm^{-1})	r_s ((degC)^2 cm s J^{-1})	R_s ((degC)^2 cm s J^{-1})	L_s ((degC)^2 s^2 cm J^{-1})	C_s (J (degC)$^{-2}$ cm^{-1})
0·681	$2\cdot49\times10^{-4}$	$5\cdot3\times10^{4}$	$3\cdot2\times10^{4}$	$2\cdot4\times10^{4}$	$4\cdot5\times10^{-5}$
0·766	$3\cdot15\times10^{-4}$	$3\cdot9\times10^{4}$	$3\cdot8\times10^{4}$	$2\cdot5\times10^{4}$	$3\cdot5\times10^{-5}$

Thus, for an increase in resistance of 3% in the platinum wire AB, which at room temperature is given by a temperature rise of about 8 degC, the power decreases by only 0·02%. The circuit used therefore gives a constant heating rate in the platinum wire for the temperature rises produced.

The platinum wires used had diameters 0·005, 0·01, 0·015, and 0·02 cm. The liquids studied were water, ethanol, *n*-butanol, glycerol, toluene, and aqueous solutions of methyl cellulose. With toluene, a series of damped temperature-time oscillations, with a period of about 10 s, and 5–10 discernible maxima, were obtained. With the other liquids only one temperature maximum and one minimum were obtained, only the distances t_0, t_1, t_2, and t_3 (Fig. 1) being measurable. The temperature rises of the wires were up to about 20 degC.

III. Thermal Parameters

The solution for the parameters in the proposed equivalent electrical circuit (Bosworth and Groden 1960, Fig. 3) lies in the trigonometrical region. An attempt has been made to identify free energy characteristics of the thermal system with free energy values calculated from the thermal parameters.

If C_q is thermal capacitance, C_s entropy capacitance, T absolute temperature, and θ temperature rise, the entropy stored in a thermal capacitor is $C_q \ln\{(T+\theta)/T\}$, and if θ/T is small, this is $C_q\theta/T$ or $C_s\theta$. The free energy dissipated in the discharge of the entropy capacitor is $\frac{1}{2}C_s\theta^2$. The entropy flux inductance, L_s, of a system is defined by $L_s\, \mathrm{d}I_s/\mathrm{d}t = \theta$, where I_s is the entropy flux and t is time. The free energy dissipated in the discharge of an entropy inductance is $\frac{1}{2}L_s I_s^2$.

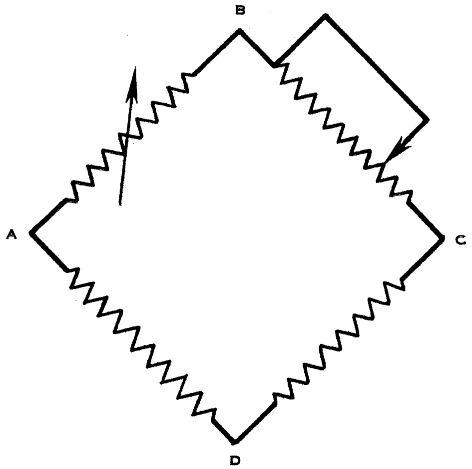

Fig. 2.—Bridge circuit.

Table 2 gives values from two temperature-time transients. The values are for entropy flux per centimetre of wire. The thermal systems were a horizontal platinum (impure; temperature coefficient of resistance 0·00315) wire (diameter 0·01 cm; length 4·00 cm; resistance 0·623 Ω at 30·0°C) in water (temperature 30·0°C; depth 12·0 cm), with heating currents of 0·681 and 0·766 A.

On comparing the calculated values of t_3 and Z_3 with the experimental values; the agreement is seen to be good with Z_3 and indifferent with t_3 (Table 2).

IV. The Free Energy Analogy

If we consider the 0·766 A system, the capacitive element is provided by a cylindrical mass of water 0·2 cm in diameter around the wire (Bosworth 1960). This water has a mean excess temperature of about 1 degC. The free energy associated with the flow of its excess entropy to the bulk of the liquid, which is approximately the heat capacity of the liquid multiplied by half the square of its mean excess temperature divided by the absolute temperature (309°K), is about 2×10^{-4} joules/cm. The calculated value of $\frac{1}{2}C_s\theta^2$ ($\theta = Z_0$) is $6\cdot4\times10^{-4}$ joules; the two free energy estimates are seen to be in rough agreement.

However, the value of $\frac{1}{2}L_sI_s^2$ (where I_s is the fraction of the total entropy flux which passes through L_s) is $3\cdot 2\times 10^{-4}$ joules, and this value is several hundred times the kinetic energy of motion of the convection chimney (Bosworth 1960). Another characteristic of the thermal system must therefore be sought for identification with the free energy of entropy-flow inductance. Taking the volume of heated liquid in the convection chimney to be 10 ml/cm of wire and to have a mean excess temperature of $0\cdot 2$ degC, the free energy dissipated in the flow of its excess entropy to the bulk of the liquid is about 3×10^{-3} joules, and this is seen to be of the order of the calculated value of $\frac{1}{2}L_sI_s^2$ ($3\cdot 2\times 10^{-4}$ joules).

Better knowledge of the convection behaviour, which can be obtained interferometrically, and fuller analysis of the temperature-time transients, may give exact agreement between the free energy quantities. It seems possible, therefore, that the analogy between thermal and electrical flow can be shown to be complete.

There was a large discrepancy between the experimental and calculated values of t_0. A possible explanation is that the single circuit proposed should be replaced by a number of similar networks connected in series, each corresponding to an isothermal shell around the wire.

V. References

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