MASSIVE STARS WITH UNIFORM COMPOSITION

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Summary

It is the purpose of the present paper to derive a set of tables which greatly facilitate the construction of models of massive stars with uniform composition, to show how these tables can be used and finally to apply the method to derive models of stars composed of pure hydrogen in which a small amount of $^{13}$C is present.

I. INTRODUCTION

Three circumstances facilitate the construction of models of massive stars (Schwarzschild and Härm 1958). (1) Convective envelopes must not be considered since their surface temperatures are so high. (2) Their convective cores are very extensive and contain almost all the nuclear energy production. (3) Bound-free and free-free transitions do not contribute appreciably to the opacity, and electron scattering constitutes the main source of opacity.

On the other hand, the effects of radiation pressure must be taken fully into account.

Instead of solving the differential equations, giving the four basic equilibrium conditions, directly in terms of the physical quantities, it is usual to transform them.

First, since for massive stars we can assume that almost all the energy generation takes place in the convective core, it is possible to detach the luminosity equation and to evaluate it as a separate quadrature. In this way we have to solve a system of three differential equations in both core and envelope.

Secondly, it is convenient to eliminate the unknown radius by using as variable $RT$ instead of $T$.

A third useful simplification is possible when the composition is uniform and the opacity is due to electron scattering. In this case it is possible to eliminate explicit reference to $\mu$ in the equations, resulting in one set of solutions which cover all models of massive stars with uniform composition.

The transformations used here are:

\[
\begin{align*}
m &= \mu^2 M_r, \\
l &= \kappa \mu^2 L, \\
x &= r/R, \\
t &= \mu RT,
\end{align*}
\]

where the opacity $\kappa$ is given by

\[
\kappa = 0.2004(1+X),
\]

$X$ being the abundance by weight of hydrogen.

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These, together with the equation of state
\[ \rho = \frac{a\mu}{3R} \cdot T^3 \cdot \frac{\beta}{1-\beta'} \]
lead to the following differential equations.

(1) **In the core**
\[
\frac{dm}{dx} = \frac{4\pi a}{3R} x^2 t^3 \frac{\beta}{1-\beta'} \quad (I_c)
\]
\[
\frac{dt}{dx} = -\frac{2(4-3\beta)}{32-3\beta^2-24\beta} \cdot \beta G \cdot x^2 \cdot \mu, \quad (II_c)
\]
\[
\frac{d\beta}{dx} = \frac{G\beta(1-\beta)}{Rx^2} \cdot \frac{m}{t} \cdot \frac{3\beta^2}{1-3\beta^2-24\beta+32} \quad (III_c)
\]

(2) **In the envelope**
\[
\frac{dm}{dx} = \frac{4\pi a}{3R} x^2 t^3 \frac{\beta}{1-\beta'} \quad (I_e)
\]
\[
\frac{dt}{dx} = -\frac{\beta}{16\pi cR(1-\beta)} \cdot \frac{l}{x^2} \quad (II_e)
\]
\[
\frac{d\beta}{dx} = \frac{1}{R} \cdot \frac{\beta(1-\beta)}{t} \cdot \frac{1}{x^2} \left\{ \frac{l}{4\pi c(1-\beta)} \cdot mG \right\}. \quad (III_e)
\]

\(\beta\) is used as a variable, in preference to the total pressure, on account of its smaller variation near the surface of the star.

At the centre the boundary condition \( m = 0 \) was used, values close to the centre being obtained by the expansions
\[
m = \frac{4\pi a}{9R} \cdot \frac{\beta_c}{1-\beta_c} \cdot t^3 \cdot x^3,
\]
\[
\beta = \beta_c + \frac{2\pi aG}{3R^2} \cdot \frac{\beta_c^4}{32-24\beta_c-3\beta_c^2} \cdot x^2,
\]
\[
t = t_c - \frac{2\pi aG}{9R^2} \cdot \frac{\beta_c^3}{1-\beta_c} \cdot \frac{8-6\beta_c}{32-24\beta_c-3\beta_c^2} \cdot t^2 \cdot x^2,
\]
to \(x = 0.01\).

At the surface the boundary condition was taken as \( t = 0 \) together with the expansions
\[
\beta = \beta_e,
\]
\[
m = \mu^2 M,
\]
\[
t = \frac{\beta_e G \cdot m_e}{4R} \left( \frac{1}{x} - 1 \right).
\]
The condition \( \beta = \beta_e \) leads to the mass-luminosity relation
\[
l = 4\pi G \cdot c \cdot m_e (1-\beta_e).
\]

This boundary condition is the same as the one used by Hoyle and Fowler (1963) in their study of more massive stars, and is equivalent to our assumption
that $\beta$ is constant in the outer layers of the star, as is at once evident from equation (III$_e$).

Dividing equation (III$_e$) by (II$_e$) we obtain the differential equation

$$
\frac{d\beta}{dt} = \frac{4}{l} \left( 1 - \beta \right) \left\{ \frac{4\pi Gm_e}{l} (1 - \beta) - 1 \right\}
$$

for the outermost layers of the star, where $m$ is approximately constant and equal to its value $m_e$ at the surface. Integration of this equation gives

$$
\frac{1 - \beta}{1 - \beta - l/4\pi Gm_e} = KT^4,
$$

where $K$ is a constant of integration, depending on the actual surface conditions.

As $T$ increases from the outside this shows that $1 - \beta$ converges rapidly to the value $l/4\pi Gm_e$, which is the approximate boundary condition used for the present integrations.

**Table 1**

Principal Characteristics of the Models

<table>
<thead>
<tr>
<th>$\beta_e$</th>
<th>$\beta_f$</th>
<th>$\beta_e$</th>
<th>$x_e$</th>
<th>$q_f$</th>
<th>$l \times 10^{-38}$</th>
<th>$t_c \times 10^{-19}$</th>
<th>$m_e \times 10^{-34}$</th>
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<td>0.9718</td>
<td>0.9844</td>
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<td>0.75</td>
<td>0.8301</td>
<td>0.8718</td>
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<td>0.625</td>
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**II. Construction and Use of Tables**

The above system of differential equations was solved on the IBM 1620 computer of the Research School of Physical Sciences, Australian National University, and the results of the computations are given in the following tables.

Table 1 gives:

- the values of $\beta$ at the centre, fitting point, and boundary of the star,
- the position of the fitting point $x_f$,
- the fraction of the total mass contained in the core $q_f$,
- the value of $l = \kappa \mu^2 L$,
- the value of $t$ at the centre $t_c = \mu RT_c$,
- the value of $m$ at the boundary $m_e = \mu^2 M$. 

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### Table 2

**Variation of $\beta$ throughout the core**

<table>
<thead>
<tr>
<th>$10^3x$</th>
<th>$\beta \times 10^4$</th>
</tr>
</thead>
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<tr>
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</tr>
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</tr>
<tr>
<td>8</td>
<td>9517 9026 8530 8031 7531 7029 6527 6025 5523 5020 4517 4015 3512 3010 2507 2005</td>
</tr>
<tr>
<td>10</td>
<td>9527 9040 8546 8048 7548 7046 6543 6040 5535 5031 4527 4023 3519 3015 2511 2008</td>
</tr>
<tr>
<td>12</td>
<td>9538 9058 8567 8069 7569 7066 6561 6065 5551 5045 4539 4033 3527 3021 2516 2011</td>
</tr>
<tr>
<td>14</td>
<td>9552 9078 8590 8094 7593 7089 6583 6086 5580 5076 4572 4065 3560 3052 2547 2041</td>
</tr>
<tr>
<td>16</td>
<td>9567 9102 8618 8123 7622 7116 6609 6099 5595 5091 4585 4078 3568 3058 2548 2040</td>
</tr>
<tr>
<td>18</td>
<td>9584 9128 8648 8155 7654 7147 6637 6126 5623 5119 4618 4109 3600 3079 2566 2065</td>
</tr>
<tr>
<td>20</td>
<td>9602 9156 8682 8191 7689 7181 6669 6155 5653 5151 4646 4138 3627 3098 2583 2090</td>
</tr>
<tr>
<td>22</td>
<td>9621 9187 8720 8231 7729 7219 6704 6187 5668 5158 4648 4138 3628 3098 2583 2088</td>
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<td>24</td>
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<td>26</td>
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<tr>
<td>32</td>
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<tr>
<td>34</td>
<td>9416 9003 8539 8040 7519 6985 6443 5896 5348 4798 4250 3703 3159 2617 2080</td>
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<td>8667 8171 7647 7105 6552 5994 5432 4870 4310 3751 3195 2644 2098</td>
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<tr>
<td>42</td>
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<td>7868 7312 6741 6161 5578 4992 4410 3830 3256 2687 2126</td>
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<td>46</td>
<td>7389 6811 6223 5631 5037 4447 3859 3277 2703 2136</td>
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<tr>
<td>48</td>
<td>6885 6289 5687 5084 4485 3890 3300 2719 2147</td>
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<tr>
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<td>6964 6358 5747 5134 4525 3921 3324 2735 2158</td>
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<tr>
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</tr>
<tr>
<td></td>
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</tr>
<tr>
<td></td>
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</tr>
<tr>
<td></td>
<td>4144 3489 2850 2231</td>
</tr>
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<td></td>
<td>4189 3521 2872 2245</td>
</tr>
<tr>
<td></td>
<td>3556 2896 2260</td>
</tr>
<tr>
<td></td>
<td>2921 2275</td>
</tr>
<tr>
<td></td>
<td>2292</td>
</tr>
<tr>
<td></td>
<td>2309</td>
</tr>
</tbody>
</table>
Table 2 gives the variation of $\beta$ throughout the core, with a step length equal to one-fiftieth of the radius, each column listing the values of $\beta$ for a separate model.

Tables 1 and 2 can now be used to derive the various physical characteristics of massive stars.

The luminosity of the star is given by the condition of thermal equilibrium, which in our notation can be written

$$\frac{l}{\kappa} = \frac{4\pi t^3}{\mu T^3} \int_0^{x_f} x^2 \rho \, dx.$$  

(1)

Dividing equation (II, c) by (III, c) we obtain the following differential equation

$$\frac{dt}{d\beta} = \frac{-2(4 - 3\beta)t}{3\beta^3(1 - \beta)},$$

(2)

which can be integrated to give the temperature $T$ as a function of $\beta$,

$$T = T_e \left( \frac{\beta e^{-4/\beta_c}}{(1 - \beta_c) \times \frac{1 - \beta}{\beta} e^{-4/\beta}} \right)^{2/3}.$$  

(3)

Using the equation of state we get

$$\rho = \frac{a\mu}{3k} \cdot T_e^3 \left( \frac{\beta e^{-4/\beta_c}}{(1 - \beta_c)} \right)^{2} \frac{1 - \beta}{\beta} e^{8/\beta}.$$  

(4)

### Table 3

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Accurate Integration</th>
<th>Interpolation Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log L$</td>
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<td>39.0781</td>
</tr>
<tr>
<td>$\log T_e$</td>
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<tr>
<td>$\log T_c$</td>
<td>8.3014</td>
<td>8.3021</td>
</tr>
</tbody>
</table>

By this means $T$ and $\rho$ are obtained as functions of $\beta$ and $T_c$ only. Since the rate of energy generation per unit mass $\epsilon$ is a function of $T$ and $\rho$, we see that the integrand of the luminosity integral (1) can be expressed as a function of $\beta$ and $T_c$. In order to evaluate this integral numerically it is therefore necessary to have tables, such as Table 2, giving the variation of $\beta$ with $x$.

For a given value of $\beta_c$ the values of $l$ and $t_c$ are known from Table 1.

Since in Table 2, $\beta$ is given as a function of $x$ in the range 0 to $x_f$ it is possible, using equations (3) and (4) and the appropriate value of $\epsilon$ to find a value of $T_c$ which satisfies equation (1).

Once $T_c$ has been determined the radius of the star will be given by the formula

$$R = t_c/\mu T_c,$$

and it is then possible to derive the effective temperature.

This gives then the position of the star in the Herzsprung-Russell diagram.
If a model is required for an intermediate value of $\beta_c$ it is possible, by interpolation in Tables 1 and 2, to obtain a stellar model with reasonable accuracy.

As an example we have, using linear interpolation in Table 2, computed a model of a star composed of pure helium, corresponding to a value of $\beta_c = 0.575$ and an energy generation given by

$$\epsilon_{3a} = 1.4 \times 10^{11} \frac{\rho^2 Y^3}{T^8} e^{-43.2/T} \text{ erg/s g.}$$

The values found from the interpolation method were then compared with those obtained from an accurate integration of the differential equations given in Section I of this paper, and it is seen from Table 3 that the agreement is quite good.

The value of log $L$ was found by linear interpolation between the values of log $l$ corresponding to $\beta_c = 0.55$ and $\beta_c = 0.6$ as given in Table 1.

<table>
<thead>
<tr>
<th>$\beta_c$</th>
<th>$X_C$</th>
<th>$T_e \times 10^{-8}$</th>
<th>log $T_e$</th>
<th>log $(L/L_\odot)$</th>
<th>$\beta_c$</th>
<th>$X_C$</th>
<th>$T_e \times 10^{-8}$</th>
<th>log $T_e$</th>
<th>log $(L/L_\odot)$</th>
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</table>

### IV. Stars composed of pure hydrogen

Models of stars composed of pure hydrogen are given by Boury (1960). As pointed out in this paper, it is to be expected that a certain amount of $^{12}C$ will have been formed during the period of contraction of these stars.

As shown by Ledoux and Boury (1959) the temperatures in the central regions of these massive stars will have reached $10^8 \text{ K}$ long before these stars have reached the main sequence.
In consequence, at these high temperatures, nuclear reactions of the proton-proton type will set in, followed by the triple alpha process, converting the helium formed in the previous process into carbon.

Only a small fraction of $^{12}\text{C}$ will be formed in this way. But this small amount of carbon will be sufficient, at these high temperatures, to influence the rate of energy production through the carbon cycle, and will influence the equilibrium configuration of the star in a marked manner.

![Fig. 1.—Positions of stars composed of pure hydrogen in the Herzsprung-Russell diagram, for various abundances of carbon. Curve A, $X_c = 0$; curve B, $X_c = 10^{-11}$; curve C, $X_c = 5 \times 10^{-11}$; D, $X_c = 10^{-10}$.

Using the method described above, we have computed models of massive hydrogen stars with small amounts of $^{12}\text{C}$ for the assumed abundances

$$X_c = 10^{-11}, 5 \times 10^{-11}, 10^{-10}, \text{ and } 0,$$

and for the following values of the rate of energy generation (Ledoux 1961)

$$\epsilon = \epsilon_{pp} + \epsilon_{CN},$$

where

\begin{align*}
\epsilon_{pp} &= 4 \cdot 19 \times 10^3 \alpha \rho X^2 \tau^2 e^{-\tau} \text{ erg/g s,} \\
\tau &= 33 \cdot 804 / T_6^4, \\
a &= 0.25, \text{ when } T_6 < 8, \\
a &= 0.5, \quad 8 < T_6 < 13, \\
a &= 0.96, \quad 13 < T_6 < 20, \\
a &= 0.71, \quad 20 < T_6.
\end{align*}

where

\begin{align*}
\epsilon_{CN} &= a \cdot 10^{23} \cdot \rho \cdot X \gamma \tau^2 e^{-\tau} \text{ erg/g s,} \\
(1) &
\end{align*}

when

(a) If $T_6 < 10 \quad a = 1.2, \quad \gamma = X_g, \quad \tau = 136 \cdot 7 / T_6^4,$

(b) If $10 < T_6 < 18 \quad a = 6.17, \quad \gamma = X_g + X_N, \quad \tau = 152 \cdot 3 / T_6^4,$

(c) If $18 < T_6 \quad a = 6.17, \quad \gamma = X_g + X_N + X_O, \quad \tau = 152 \cdot 3 / T_6^4.
The results are given in Table 4 and the position of these stars in the H–R diagram are given in Figure 1.

It is seen from this figure that a small amount of carbon formed during the contraction period has a marked effect on the position of the hydrogen main sequence. This is due to the fact that when a small amount of $^{12}$C is present, the energy generation is mainly due to the carbon cycle and no longer to the proton-proton cycle.

V. References