CHARGED PARTICLE MOTION IN A TIME-DEPENDENT AXIALLY SYMMETRIC MAGNETIC FIELD

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Summary

Following a short review of the drift theory of plasma radial compression, an exact solution for the motion of a charged particle in an axially symmetric time-dependent magnetic field is obtained. The method gives forms for the cylindrical coordinates r and θ of the charged particle that have a simple interpretation, the z-motion being of constant velocity. As examples, the exact results are discussed for a simple power law and an exponential time dependence of the magnetic field and, using the latter results, the drift theory of plasma radial compression is qualitatively verified.

I. Introduction

In an earlier paper (Seymour 1963), a review was included of the drift theory of radial compression of a tenuous plasma suitably contained within a long, straight solenoid through which is passing a time-dependent electric current, and this was followed by a trajectory approach to the same problem using the equations of motion for a charged particle moving in a time-dependent axially symmetric magnetic field, $\mathbf{B}(t)$. A simple, but not very adequate, approximation then led to a flux-conserving result similar in form to, but different in nature from, that obtained by the drift analysis. At that stage it was evident that an exact solution for the trajectory approach would be helpful, since it should permit determination of the various charged particle motions that may occur in collisionless plasma approximation, but mathematical difficulties were encountered.

In the present paper, the general form of the exact microscopic solution for the motion of a charged particle in a time-dependent axially symmetric magnetic field is obtained, and applied to particular cases of interest. This gives insight into the macroscopic motions of charged particles in a plasma of low number density, in which interparticle collisions are rare. For the convenience of the reader, the salient features of the particle drift and trajectory approaches described by Seymour (1963, pp. 436–43) appear in the next section, prior to the derivation of the exact solution.

II. Charged Particle Guiding-Centre and Trajectory Approaches to Plasma Radial Compression

As is well known, drift analysis is primarily concerned with the motion of the guiding centre of a spiralling charged particle, whereas the trajectory approach seeks to determine the classical path traversed by the charged particle. For the magnetic field specified in Section I, we consider the above approaches separately.

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(a) Guiding-Centre Approach

The geometry of interest is that of a long, straight solenoid through which is passing a current that varies with time. Within the solenoid, clear of the ends, the resulting time-dependent magnetic field possesses axial symmetry, and is simply $\mathbf{B}(t) = \mathbf{k}B_z(t)$, in the usual notation. Accordingly, it is plausible to assume that the induced electric field, $\mathbf{E}$, has circular field lines, concentric with the longitudinal axis of symmetry of the solenoid. To calculate E_θ, we evaluate the line integral of $\mathbf{E}$ around a circular path P of radius r, bounding a surface S lying in a plane normal to the z axis of the solenoid. Employing electromagnetic units, we have from Maxwell's equation for the curl of $\mathbf{E}$, and Stokes's theorem,

$$\oint_P \mathbf{E}.\mathrm{d}\mathbf{l} = -\int_S \dot{\mathbf{B}}.\mathbf{k}\,\mathrm{d}S, \tag{2.1}$$

where $\mathrm{d}\mathbf{l}$ is taken along the boundary of P. Since $\dot{\mathbf{B}}$ has the same value at all points on the surface S, and also has the z direction of the unit vector $\mathbf{k}$, (2.1) yields

$$E_\theta = -\tfrac{1}{2}r\dot{B}, \tag{2.2}$$

where for convenience we omit the subscript z from B_z.

The effect of this induced electric field in the presence of the magnetic field is to produce a drift of a charged particle's guiding centre at a velocity

$$\mathbf{v}_\mathrm{d} = \frac{\mathbf{E}\times\mathbf{B}}{B^2}. \tag{2.3}$$

Since a tenuous plasma is under consideration, we assume, for a reasonably strong magnetic field, a collisionless approximation in which the charged particles spiral about the magnetic field lines and do not transfer from one magnetic field line to another. In other words, diffusion transverse to the magnetic field is absent in this approximation, and typically, at some instant of time, guiding centres of certain plasma particles would be found lying on the path P of radius r, used in the calculation of E_θ. In the dynamic situation encountered here the time rate of change of r must therefore be equated to the radial drift velocity E_θ/B, to obtain the equation

$$\frac{2}{r}\dot{r}+\frac{1}{B}\dot{B} = 0, \tag{2.4}$$

with the solution

$$Br^2 = \text{constant}. \tag{2.5}$$

For the case of interest, $\dot{B} > 0$, this result states that the total flux, $\phi = \pi r^2 B$ enclosed by a circle of radius r, concentric with the solenoid axis, is conserved as r shrinks, and guiding centres lying on this circle move inward in the axially symmetric time-dependent magnetic field. Graphically, the particle guiding centres stick to the surface of some collapsing flux tube of the time-dependent magnetic field.

Bearing in mind the assumptions made above, one now wonders whether it is possible to demonstrate radial compression of the plasma under consideration by a non-drift analysis.

(b) Trajectory Approach

Using cylindrical coordinates, the velocity of a particle at r, θ, z may be written as

$$\mathbf{v} = \mathbf{r}_0 v_r + \boldsymbol{\theta}_0 v_\theta + \mathbf{k} v_z, \tag{2.6}$$

where $\mathbf{r}_0$, $\boldsymbol{\theta}_0$, $\mathbf{k}$ are unit vectors, and

$$v_r = \dot{r}, \qquad v_\theta = r\dot{\theta}, \qquad v_z = \dot{z}. \tag{2.7}$$

With similar resolution of $\mathbf{E}$ and $\mathbf{B}$, and recalling that $\mathrm{d}\mathbf{r}_0/\mathrm{d}\theta = \boldsymbol{\theta}_0$, $\mathrm{d}\boldsymbol{\theta}_0/\mathrm{d}\theta = -\mathbf{r}_0$, the non-relativistic equation of motion for a particle of charge q, mass m,

$$m\dot{\mathbf{v}} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \qquad \text{(e.m.u.)}, \tag{2.8}$$

in which radiation damping has been permissibly neglected, yields the component equations

$$\ddot{r}-r\dot{\theta}^2 = (q/m)[E_r+(r\dot{\theta}B_z-\dot{z}B_\theta)], \tag{2.9}$$

$$2\dot{r}\dot{\theta}+r\ddot{\theta} = (q/m)[E_\theta+(\dot{z}B_r-\dot{r}B_z)], \tag{2.10}$$

$$\ddot{z} = (q/m)[E_z+(\dot{r}B_\theta-r\dot{\theta}B_r)]. \tag{2.11}$$

To facilitate analysis, the current in the solenoid may be reasonably approximated as $\mathbf{j} = \boldsymbol{\theta}_0 j_\theta$, whereupon symmetry considerations (see, for example, Smythe 1950) suggest that the magnetic vector potential is simply

$$\mathbf{A} = \boldsymbol{\theta}_0 A_\theta. \tag{2.12}$$

Then, neglecting contributions to $\mathbf{E}$ from the scalar potential ϕ in

$$\mathbf{E} = -\dot{\mathbf{A}}-\nabla\phi, \tag{2.13}$$

we obtain (cf. Seymour 1963, pp. 437–9)

$$A_\theta = \tfrac{1}{2}rB_z, \tag{2.14}$$

and

$$E_\theta = -\tfrac{1}{2}r\dot{B}_z \qquad \text{as in (2.2)}, \tag{2.15}$$

by means of a proof which shows formally that the induced electric field lines are circles concentric with the z axis of the solenoid. The result (2.14) depends on the fact that B_z has no spatial dependence here, as can be deduced from the Maxwell field equations

$$\operatorname{div}\mathbf{B} = 0, \tag{2.16}$$

$$\operatorname{curl}\mathbf{B} = 0, \tag{2.17}$$

the latter equation applying inside the solenoid, where it is assumed that the displacement current may be neglected, and that the charged particle motions do not contribute to the magnetic field, so that $\mathbf{B}(t) = \mathbf{k}B_z(t)$ has $j_\theta(t)$ alone as its source.

If we now specialize the equations of motion (2.9), (2.10), and (2.11) to accord with the fields

$$\mathbf{E} = (0, -\tfrac{1}{2}r\dot{B}_z, 0), \tag{2.18}$$

$$\mathbf{B} = (0, 0, B_z), \tag{2.19}$$

we obtain, for the solenoid problem outlined,

$$\ddot{r} - r\dot{\theta}^2 = (q/m)B_z r\dot{\theta}, \tag{2.20}$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = -(q/m)(\tfrac{1}{2}r\dot{B}_z + \dot{r}B_z), \tag{2.21}$$

$$\ddot{z} = 0. \tag{2.22}$$

Equation (2.21) immediately yields a first integral

$$\dot{\theta} = \omega_{\mathrm{L}}(t) + Cr^{-2}, \tag{2.23}$$

where $\omega_{\mathrm{L}}(t)$ is the Larmor angular frequency

$$\omega_{\mathrm{L}}(t) = -(q/2m)\, B_z(t), \tag{2.24}$$

and C is a real constant. Elimination of $\dot{\theta}$ from equation (2.20) by means of (2.23) gives the differential equation governing the behaviour of r,

$$\ddot{r} + \omega_{\mathrm{L}}^2 r - C^2 r^{-3} = 0. \tag{2.25}$$

From (2.22) it is seen that the z-direction motion is trivial in this case, v_z being constant. The significant results representing charged particle motions in this particular time-dependent magnetic field are therefore the integrated form of (2.23),

$$\theta = \int_0^t \omega_{\mathrm{L}}\, \mathrm{d}t' + C\int_0^t \frac{\mathrm{d}t'}{r^2} + \theta_0, \tag{2.26}$$

where θ_0 is the value of θ at $t = 0$, and the solution of equation (2.25) for r.

III. Solution of the r Equation and Modification of the θ Equation

To obtain the general solution of equation (2.25), we commence with the *ansatz*

$$\dot{r} = M(t)\, r + N(t)\, r^{-1}, \tag{3.1}$$

where M and N are functions of time to be determined, so that

$$\ddot{r} = (M^2 + \dot{M})r + \dot{N}r^{-1} - N^2 r^{-3} \tag{3.2}$$

contains terms in r of the type appearing in (2.25). In fact, for (3.2) to assume precisely the form of (2.25), we see at once that the relationships

$$M^2 + \dot{M} = -\omega_{\mathrm{L}}^2, \tag{3.3}$$

$$\dot{N} = 0, \tag{3.4}$$

$$N^2 = -C^2, \tag{3.5}$$

must hold. The last two equations are consistent, and give

$$N = \pm \mathrm{i}C, \tag{3.6}$$

where, from equation (2.23), C is a real constant.

Equation (3.3) is a particular form of Riccati's differential equation. The introduction of a new variable $s(t)$, such that

$$(\ln s)^{\cdot} = M, \tag{3.7}$$

transforms (3.3) to the form

$$\ddot{s} + \omega_{\mathrm{L}}^2 s = 0. \tag{3.8}$$

To express r in terms of s, where s is determined by equation (3.8) for chosen forms of $\omega_{\mathrm{L}}(t)$, we integrate (3.1), modified by insertion of $N = +\mathrm{i}C$, say, and M from equations (3.6) and (3.7) respectively, by use of the integrating factor s^{-2}, to obtain

$$r = s\left(A + 2\mathrm{i}C \int_0^t \frac{\mathrm{d}t'}{s^2}\right)^{\frac{1}{2}}, \tag{3.9}$$

from which it is evident that, in general, $s(t)$ and the integration constant A are complex quantities. Clearly, the form of this equation can be simplified by the introduction of a further complex variable

$$u(t) = A^{\frac{1}{2}} s(t). \tag{3.10}$$

Then (3.9) becomes

$$r = u\left(1 + 2\mathrm{i}C \int_0^t \frac{\mathrm{d}t'}{u^2}\right)^{\frac{1}{2}}, \tag{3.11}$$

and (3.8) transforms to

$$\ddot{u} + \omega_{\mathrm{L}}^2 u = 0. \tag{3.12}$$

Equation (2.25) is now represented by the pair of equations (3.11) and (3.12). The integral appearing in equation (3.11) can be performed in some cases of interest (an example will be given in Section VI), but in fact it can be avoided by developing an alternative form of solution for r. The equation

$$\dot{r} = \frac{\dot{s}}{s} r + \frac{\mathrm{i}C}{r}, \tag{3.13}$$

which led, through the use of the integrating factor s^{-2}, to (3.9), can also be rearranged to the form

$$\frac{\dot{r}}{r} - \frac{\dot{u}}{u} = \frac{\mathrm{i}C}{r^2}, \tag{3.14}$$

where (3.10) has been used to eliminate $s(t)$.

Equation (3.14) immediately yields by integration the inverted form of (3.9)

$$u = r \exp\left(-\mathrm{i}C \int_0^t \frac{\mathrm{d}t'}{r^2}\right), \tag{3.15}$$

where the multiplicative constant of integration can be seen to be unity by comparing (3.15) with (3.11) at $t = 0$.

Since r, C, and t are real quantities, we see at once that

$$r = (uu^*)^{\frac{1}{2}}, \qquad r \geqq 0, \tag{3.16}$$

where the equation

$$u^* = r \exp\left(\mathrm{i}C \int_0^t \frac{\mathrm{d}t'}{r^2}\right) \tag{3.17}$$

corresponds to the choice $N = -\mathrm{i}C$ in equation (3.6).

Writing

$$u(t) = \alpha(t) + \mathrm{i}\beta(t), \tag{3.18}$$

where α and β are real quantities, we have from (3.15)

$$C \int_0^t \frac{\mathrm{d}t'}{r^2} = -\tan^{-1}(\beta/\alpha). \tag{3.19}$$

Hence the expression for θ given by (2.26) becomes

$$\theta = \int_0^t \omega_{\mathrm{L}}\, \mathrm{d}t' - \tan^{-1}(\beta/\alpha) + \theta_0. \tag{3.20}$$

A useful identification of the constant C is obtained as follows. Insertion of (3.18) into (3.14), use of (3.16), and equation of imaginary parts shows that

$$W(\alpha, \beta) = -C, \tag{3.21}$$

where $W(\alpha,\beta)$ is the Wronskian determinant $\begin{vmatrix} \alpha & \beta \\ \dot{\alpha} & \dot{\beta} \end{vmatrix}$.

Let now $I_1(t)$ and $I_2(t)$ be linearly independent, real solutions of equation (3.12), such that

$$\alpha = a_{11} I_1 + a_{12} I_2, \tag{3.22}$$

$$\beta = a_{21} I_1 + a_{22} I_2, \tag{3.23}$$

the a_{ij} being arbitrary real constants. Then (3.21) gives

$$W(I_1, I_2) = \begin{vmatrix} I_1 & I_2 \\ \dot{I}_1 & \dot{I}_2 \end{vmatrix} = -\frac{C}{|a_{ij}|}. \tag{3.24}$$

From (3.18) and the form of α and β,

$$u = \mu I_1 + \rho I_2, \tag{3.25}$$

where

$$\mu = a_{11} + \mathrm{i}a_{21} = |\mu| \exp \mathrm{i}\epsilon_1, \qquad \rho = a_{12} + \mathrm{i}a_{22} = |\rho| \exp \mathrm{i}\epsilon_2,$$

and so (3.16) gives, as the general solution of (2.25),

$$r = \{|\mu|^2 I_1^2 + |\rho|^2 I_2^2 + 2|\mu||\rho|\, I_1 I_2 \cos(\epsilon_1 - \epsilon_2)\}^{\frac{1}{2}}, \tag{3.26}$$

where, from (3.24),

$$\sin(\epsilon_1 - \epsilon_2) = \frac{C}{|\mu||\rho|\, W(I_1, I_2)}. \tag{3.27}$$

Further, in view of (3.22) and (3.23), θ of (3.20) can be written as

$$\theta = \int_0^t \omega_{\mathrm{L}}\, \mathrm{d}t' - \tan^{-1}\left\{\frac{|\mu|\, I_1 \sin\epsilon_1 + |\rho|\, I_2 \sin\epsilon_2}{|\mu|\, I_1 \cos\epsilon_1 + |\rho|\, I_2 \cos\epsilon_2}\right\} + \theta_0. \tag{3.28}$$

Although these results for r and θ are of complicated appearance, they can, in fact, be interpreted in a simple manner. From (3.26) we see that at any instant of time r can be regarded as the resultant of two vectors, of lengths $|\mu|\, I_1(t)$ and $|\rho|\, I_2(t)$, having a constant angle $\epsilon_1 - \epsilon_2$ between them. If we now set $|\mu|\, I_1$ at angle ϵ_1 ($\epsilon_1 > \epsilon_2$, say) to an axis moving in the $r\theta$-plane with angular velocity $\omega_{\mathrm{L}}(t)$ about the z axis, and set $|\rho|\, I_2$ at angle ϵ_2 to this moving axis, then it is readily verified that the angle between the resultant r measured relative to the moving axis is the second term on the right-hand side of (3.28). The general solution for the charged particle motion in the $r\theta$-plane is thus pictorially represented by the vectors of lengths $|\mu|\, I_1$ and $|\rho|\, I_2$, with constant angle $\epsilon_1 - \epsilon_2$ between them, moving about the origin of coordinates at the angular velocity $\omega_{\mathrm{L}}(t)$.

IV. Particular Solutions of the u Equation, and Corresponding Forms of r and θ

In this section we develop solutions for some simple time dependences of $\omega_{\mathrm{L}}(t)$.

Case (a) $\omega_L(\mathrm{t})$ *Obeys the Power Law* $\pm\gamma \mathrm{t}^{\mathrm{k}}$, $\mathrm{k} \neq -1$

If we adopt the functional transformation

$$v(t) = t^{-\frac{1}{2}}\, u(t), \tag{4.1}$$

equation (3.12) assumes the form

$$\frac{\ddot{v}}{\omega_{\mathrm{L}}^2} + \frac{\dot{v}}{t\,\omega_{\mathrm{L}}^2} + \left(1 - \frac{1}{4t^2\omega_{\mathrm{L}}^2}\right) v = 0, \tag{4.2}$$

which bears some resemblance to Bessel's differential equation

$$y'' + \frac{y'}{x} + \left(1 - \frac{p^2}{x^2}\right) y = 0, \tag{4.3}$$

where the prime denotes differentiation with respect to x. Comparison of (4.2) and (4.3) suggests a further transformation

$$y(x) = v(t), \tag{4.4}$$

where the new variable $x = x(t)$. Then (4.2) becomes

$$\frac{\dot{x}^2}{\omega_{\mathrm{L}}^2} y'' + \frac{1}{\omega_{\mathrm{L}}^2}\left(\ddot{x} + \frac{\dot{x}}{t}\right) y' + \left(1 - \frac{1}{4t^2\omega_{\mathrm{L}}^2}\right) y = 0, \tag{4.5}$$

and this assumes the Bessel form if

$$\dot{x} = \pm\omega_{\mathrm{L}}, \tag{4.6}$$

$$\ddot{x} + \frac{\dot{x}}{t} = \frac{\omega_{\mathrm{L}}^2}{x}, \tag{4.7}$$

$$\frac{p}{x} = \pm\frac{1}{2t\omega_{\mathrm{L}}}. \tag{4.8}$$

From (4.6) and (4.8) we find that

$$x = \frac{\gamma}{k+1} t^{k+1}, \tag{4.9}$$

where γ and k are constants, with $p = \frac{1}{2}(k+1)^{-1}$. From (4.6) it follows immediately that

$$\omega_{\mathrm{L}} = \pm\gamma t^{k}, \tag{4.10}$$

so that ω_{L} obeys a simple power law here, and $x = \pm\omega_{\mathrm{L}} t(1+k)^{-1}$. We note that (4.7) is consistent with (4.9) and (4.10).

In terms of the real cylindrical functions $\mathrm{J}_p(x)$ and $\mathrm{Y}_p(x)$, the Bessel functions of the first and second kind respectively, of order p and real argument x, the solution of (3.12) may be written, by use of (4.1) and (4.4), as

$$u(t) = t^{\frac{1}{2}}\{\lambda\,\mathrm{J}_p(x) + \sigma\,\mathrm{Y}_p(x)\}, \tag{4.11}$$

where, in general, $\lambda = |\lambda| \exp \mathrm{i}\psi_1$ and $\sigma = |\sigma| \exp \mathrm{i}\psi_2$ are complex constants. Since

$$\mathrm{Y}_p(x) = \frac{\mathrm{J}_p(x) \cos p\pi - \mathrm{J}_{-p}(x)}{\sin p\pi}, \tag{4.12}$$

we recall that, when p is not an integer, $\mathrm{Y}_p(x)$ in (4.11) can be replaced by J_{-p} to again give the general solution. However, when $p = n$, an integer, J_n and J_{-n} are no longer linearly independent, being related by

$$\mathrm{J}_{-n} = (-1)^n\,\mathrm{J}_n, \tag{4.13}$$

and so we retain the form (4.11), modified by introduction of J_n and Y_n, where

$$\mathrm{Y}_n(x) = \lim_{p\to n} \mathrm{Y}_p(x). \tag{4.14}$$

Considering, for example, non-integral p, (3.16) and the appropriate form of u described above give

$$r = [t\{|\lambda|^2\,\mathrm{J}_p^2 + |\sigma|^2\,\mathrm{J}_{-p}^2 + 2|\lambda|\,|\sigma|\,\mathrm{J}_p\,\mathrm{J}_{-p} \cos(\psi_1 - \psi_2)\}]^{\frac{1}{2}}. \tag{4.15}$$

The associated angle, θ, is obtained from (3.20), (4.10), and the appropriate u as

$$\theta = \pm\frac{\gamma}{k+1} t^{k+1} - \tan^{-1}\left(\frac{|\lambda|\,\mathrm{J}_p \sin\psi_1 + |\sigma|\,\mathrm{J}_{-p} \sin\psi_2}{|\lambda|\,\mathrm{J}_p \cos\psi_1 + |\sigma|\,\mathrm{J}_{-p} \cos\psi_2}\right) + \theta_0. \tag{4.16}$$

Similar results are obtained for p an integer.

When $k = -1$, we see from $p = \frac{1}{2}(k+1)^{-1}$ that the Bessel functions are of infinite order. In this case we proceed as follows.

Case (b) ω_L(t) Varies Inversely with the Time

For $k = -1$, equation (4.10) gives

$$\omega_{\mathrm{L}}(t) = \pm\gamma/t, \tag{4.17}$$

and hence equation (3.12) becomes

$$t^2\ddot{u} + \gamma^2 u = 0, \tag{4.18}$$

a simplified form of the Euler–Cauchy equation. The usual procedure here is to let

$$u(t) = v(x), \tag{4.19}$$

where

$$x = \ln t, \tag{4.20}$$

whereupon (4.18) transforms to the second-order linear differential equation with constant coefficients

$$v'' - v' + \gamma^2 v = 0, \tag{4.21}$$

with characteristic equation

$$\xi^2 - \xi + \gamma^2 = 0. \tag{4.22}$$

The general solution of (4.21) is, of course,

$$v = K_1 \exp \xi_1 x + K_2 \exp \xi_2 x, \tag{4.23}$$

where

$$\xi_1 = \tfrac{1}{2} + \eta, \qquad \xi_2 = \tfrac{1}{2} - \eta, \tag{4.24}$$

with

$$\eta = (\tfrac{1}{4} - \gamma^2)^{\frac{1}{2}}. \tag{4.25}$$

Using (4.19), (4.20), and (4.24), equation (4.23) becomes

$$u = t^{\frac{1}{2}}(K_1 t^{\eta} + K_2 t^{-\eta}), \tag{4.26}$$

where K_1 and K_2 are, in general, complex constants.

Three particular forms of (4.26) must now be considered:

(i) $\frac{1}{4} > \gamma^2, \quad \eta > 0.$ (4.27)

Equation (4.26) applies as it stands.

(ii) $\frac{1}{4} = \gamma^2, \quad \eta = 0.$ (4.28)

Equation (4.26) assumes the form

$$u = t^{\frac{1}{2}}(K_3 + K_4 \ln t), \tag{4.29}$$

where K_3 and K_4 are complex constants.

(iii) $\frac{1}{4} < \gamma^2, \quad \eta = \mathrm{i}\eta_0$, where $\eta_0 = (\gamma^2 - \frac{1}{4})^{\frac{1}{2}} > 0.$ (4.30)

Equation (4.26) here becomes

$$u = t^{\frac{1}{2}}\{K_5 \cos(\eta_0 \ln t) + K_6 \sin(\eta_0 \ln t)\}, \tag{4.31}$$

where K_5 and K_6 are complex constants.

The forms of r and θ associated with (i), (ii), and (iii) above are readily obtained from (3.16) and (3.20) respectively. As can be seen, the charged particle motions in each instance are markedly different.

Case (c) ω_L(t) Varies Exponentially with the Time

In this case we first transform (3.12) by assuming

$$u(t) = v(x), \tag{4.32}$$

where, with D and G real constants,

$$x = D\,\mathrm{e}^{Gt}. \tag{4.33}$$

The result,

$$G^2(x^2v''+xv')+\omega_L^2 v = 0, \tag{4.34}$$

can then be converted to the form of Bessel's equation (4.3), by a particular choice of ω_L, namely,

$$\omega_L^2 = G^2(x^2-p^2). \tag{4.35}$$

Then, for $G \neq 0$, (4.34) yields the required form

$$x^2v''+xv'+(x^2-p^2)v = 0, \tag{4.36}$$

and so from (4.32) the general solution of (3.12) for ω_L varying exponentially as in (4.35) is

$$u(t) = \mu\,\mathrm{J}_p + \rho\,\mathrm{Y}_p, \tag{4.37}$$

where μ and ρ are complex constants. For p not an integer the solution for r here is

$$r = \{|\lambda|^2\mathrm{J}_p^2 + |\sigma|^2\mathrm{J}_{-p}^2+2|\lambda||\sigma|\,\mathrm{J}_p\mathrm{J}_{-p}\cos(\psi_1-\psi_2)\}^{\frac{1}{2}}, \tag{4.38}$$

in the notation of Case (*a*) of this section. Correspondingly, the equation for θ becomes, from (3.20), (4.33), and (4.35),

$$\theta = \frac{\omega_L}{G} \mp p\sec^{-1}\left|\frac{x}{p}\right| - \tan^{-1}\left(\frac{|\lambda|\,\mathrm{J}_p\sin\psi_1 + |\sigma|\,\mathrm{J}_{-p}\sin\psi_2}{|\lambda|\,\mathrm{J}_p\cos\psi_1 + |\sigma|\,\mathrm{J}_{-p}\cos\psi_2}\right) + \theta_0, \tag{4.39}$$

after the performance of an elementary integration to obtain the first two terms on the right-hand side. Proceeding as in Case (*a*), we similarly obtain r and θ here for $p = n$, an integer.

V. Discussion of Results

When $p = \frac{1}{2}$, the Bessel functions J_p and J_{-p} become simply

$$\mathrm{J}_{\frac{1}{2}} = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin x, \tag{5.1}$$

and

$$\mathrm{J}_{-\frac{1}{2}} = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos x. \tag{5.2}$$

From the hierarchy of charged particle motions covered by the results derived above, it serves our purpose here to restrict the discussion to results for $p = \frac{1}{2}$. Thus, in Case (*a*) of the previous section, $k = 0$, and so from (4.10)

$$\omega_L = \pm\gamma, \quad \text{a constant}, \tag{5.3}$$

corresponding to a magnetic field B_z independent of the time (cf. equation (2.24)). Further, from (4.9),

$$x = \gamma t = \pm\omega_L t. \tag{5.4}$$

Thus, for charged particle motion in a constant magnetic field, expression (4.15) yields the required result

$$r = \{P+Q\cos(\omega_g t+\phi)\}^{\frac{1}{2}}, \tag{5.5}$$

where P, Q, and ϕ are constants, restricted by $\omega_g^2(P^2-Q^2) = 4C^2$, ω_g being the familiar gyrofrequency given by

$$\omega_g = 2\omega_L. \tag{5.6}$$

Further, from (4.16),

$$\theta = \pm\tfrac{1}{2}\omega_g t - \tan^{-1}\left(\frac{|\sigma|\sin\psi_2 + |\lambda|\sin\psi_1\tan(\frac{1}{2}\omega_g t)}{|\sigma|\cos\psi_2 + |\lambda|\cos\psi_1\tan(\frac{1}{2}\omega_g t)}\right) + \theta_0. \tag{5.7}$$

For $p = \frac{1}{2}$, Case (*c*) of Section IV is of particular interest, because the simple trigonometrical nature of the constant magnetic field solution is retained, but the time dependence of B_z is not lost. From (4.33) and (4.38),

$$r = e^{-\frac{1}{2}Gt}\{L + R\cos(2D\,e^{Gt} + \xi)\}^{\frac{1}{2}}, \tag{5.8}$$

where L, R, and ξ are constants, restricted by $D^2G^2(L^2 - R^2) = C^2$; and from (4.33), (4.39), and (5.6),

$$\theta = \frac{\omega_g}{2G} \mp \tfrac{1}{2}\sec^{-1}|2D\,e^{Gt}| - \tan^{-1}\left(\frac{|\sigma|\sin\psi_2 + |\lambda|\sin\psi_1\tan(D\,e^{Gt})}{|\sigma|\cos\psi_2 + |\lambda|\cos\psi_1\tan(D\,e^{Gt})}\right) + \theta_0, \tag{5.9}$$

where, from (4.33), (4.35), and (5.6),

$$\omega_g = \pm G(4D^2\,e^{2Gt} - 1)^{\frac{1}{2}}. \tag{5.10}$$

We see immediately from equation (5.8) that for $G > 0$, corresponding to a magnetic field B_z whose magnitude increases with the time, r decreases with time, thus supporting, for $p = \frac{1}{2}$, the drift theory hypothesis of radial compression of the plasma particles under such conditions. The approximate treatment given by Seymour (1963, p. 443) can be recovered from (5.8) by giving the constant G a sufficiently small value.

VI. Alternative Method of Calculating r

As mentioned in Section III, r can also be obtained from equation (3.11) if the integral $\int dt'/u^2$ can be evaluated. As a consistency check, this calculation was carried out for the exponential-law ω_L, Case (*c*) of Section IV. Briefly, we define a quantity

$$\Sigma = \frac{J_p}{u}, \tag{6.1}$$

where u is given by (4.37), and then calculate

$$\frac{d\Sigma}{dx} = -\frac{\rho\,W(J_p, Y_p)}{u^2}. \tag{6.2}$$

Since the Wronskian

$$W(J_p(x), Y_p(x)) = \begin{vmatrix} J_p & Y_p \\ J_p' & Y_p' \end{vmatrix} = \frac{2}{\pi x}, \tag{6.3}$$

where the prime denotes differentiation with respect to x, we obtain, with the help of (4.33),

$$\int \frac{dt'}{u^2} = \frac{1}{G}\int \frac{dx'}{x'u^2} = -\frac{\pi}{2G\rho}\frac{J_p}{u}. \tag{6.4}$$

Then, from equation (3.11),

$$r = \left\{(\mu\,\mathrm{J}_p + \rho\,\mathrm{Y}_p)^2 - \frac{\mathrm{i}C\pi}{G\rho}\,\mathrm{J}_p\,(\mu\,\mathrm{J}_p + \rho\,\mathrm{Y}_p)\right\}^{\frac{1}{2}}. \tag{6.5}$$

Again writing $\mu = |\mu|\exp\mathrm{i}\epsilon_1$ and $\rho = |\rho|\exp\mathrm{i}\epsilon_2$, we equate the imaginary part of (6.5) to zero to obtain real r, with the independent results

$$\left.\begin{aligned} |\mu|\,|\rho|\sin 2\epsilon_1 - \frac{\pi C}{G}\cos(\epsilon_1 - \epsilon_2) &= 0, \\ \sin 2\epsilon_2 &= 0. \end{aligned}\right\} \tag{6.6}$$

Use of these results in the real part of (6.5) yields real r, of the general form given by (4.38). Thus we have a satisfactory consistency check in the exponential-law case, and a similar procedure confirms consistency in the case of power-law ω_{L}.

VII. Wronskian Relationships

From the equations (3.18), (3.22), (3.23) and, say, (4.11) for the power-law ω_{L}, it follows that

$$I_1 = t^{\frac{1}{2}}(c_{11}\,\mathrm{J}_p + c_{12}\,\mathrm{Y}_p), \tag{7.1}$$

$$I_2 = t^{\frac{1}{2}}(c_{21}\,\mathrm{J}_p + c_{22}\,\mathrm{Y}_p), \tag{7.2}$$

where the c_{ij} are constants. Hence the Wronskian determinant in (3.24) can also be written as

$$W(I_1(t), I_2(t)) = |c_{ij}|\, W(\mathrm{J}_p(x), \mathrm{Y}_p(x))\,\dot{x}t. \tag{7.3}$$

With the help of (6.3), (3.24) and (7.3) can be combined to give

$$t(\ln x)^{\cdot} = \text{constant}. \tag{7.4}$$

Integration immediately yields the form of (4.9), a useful consistency check. Similarly, if (4.11) is replaced by (4.37) of the exponential ω_{L} case, we obtain

$$(\ln x)^{\cdot} = \text{constant}, \tag{7.5}$$

the integral of which is in agreement with (4.33).

VIII. Conclusions

In this analysis we have provided the general form of solution for the motion of a charged particle in the time-dependent magnetic field within a long solenoid, together with a simple pictorial interpretation. However, the particular examples we have discussed do not include the following important cases. First, approximation of the exact solution for quasi-static variation of the magnetic field, with the object of identifying the associated adiabatic invariants; secondly, examination of the case corresponding to harmonic variation of ω_{L} with time. These cases will be covered in a paper shortly to be published by one of the present authors (P.W.S.).

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X. References

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