THE CHAPMAN–ENSKOG SOLUTION OF THE BOLTZMANN EQUATION: A REFORMULATION IN TERMS OF IRREDUCIBLE TENSORS AND MATRICES

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CONTENTS

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Summary

The Chapman–Enskog method of solving the Boltzmann equation is presented in a simpler and more efficient form. For this purpose all the operations involving the usual polynomials are carried out in spherical polar coordinates, and the Racah–Wigner methods of dealing with irreducible tensors are used throughout. The expressions for the collision integral and the associated bracket expressions of kinetic theory are derived in terms of Talmi coefficients, which have been extensively studied in the harmonic oscillator shell model of nuclear physics. These expressions are most convenient for exhibiting some formal properties and can be programmed for machine computations in the most general form. The Chapman–Enskog linearization leads to linear matrix equations whose solutions can be expressed compactly in matrix notation. Calculations of transport coefficients up to any order can be performed almost automatically, without requiring the elaborate manipulations in each new order as is the case with the usual theory. The simple gas is treated in detail, and the treatment for the case of a gas mixture is outlined.

All calculations have been carried out without reference to the theory of integral equations. The role of summational invariants and of the subsidiary conditions becomes more transparent in this presentation. The relationship of the formulation to the usual one is also discussed.

I. Introduction

The best-known method for solving the Boltzmann equation is the Chapman–Enskog method (Enskog 1911*a*, 1911*b*; Chapman 1912; Burnett 1935*a*, 1935*b*; Chapman and Cowling 1939, 1952)—for calculating the transport coefficients of a gas from the knowledge of intermolecular interactions it is practically the only method available. The solution that is obtained by this method is a special, the so-called "normal", solution. It has often been thought that this solution is of an asymptotic nature and that it is related to the existence of relaxations of different characteristic times in the system. The latter circumstance is related to the contraction in the description of the temporal development of the system. Among others, Uhlenbeck (Uhlenbeck and Ford 1963, p. 111) has maintained that it is a "very fundamental insight" and has stressed the similarity between the Chapman–Enskog assumptions and those used by Bogoliubov (1962) (see Uhlenbeck and Ford 1963, p. 123) in solving the Liouville equation. However, there is no universal agree-

ment on these points, and doubts have often been expressed regarding the meaning and interpretation of this method. On the basis of an extensive investigation of the fundamental questions with the aid of certain models and alternative procedures Ikenberry and Truesdell (1956; see p. 8) concluded that " . . . the results of this [their own] research divest the classical approach to the kinetic theory, as presented for example in Boltzmann's lectures and in the treatise of Chapman and Cowling . . . , of most of the relevance it was once fancied to have". While stating that "little toward solving problems of flow can be expected from the Chapman–Enskog method", they nonetheless concede that the "formula which the Chapman–Enskog method produces for viscosity in terms of the molecular models may be correct" (Ikenberry and Truesdell 1956, p. 118). A part of this uncertainty about the mathematical status of the method is no doubt due to its great algebraic complexity, because of which calculations even in the third approximation, which is in fact only the second significant approximation, become quite difficult and have never been fully carried out (see, however, Burnett 1935*a*, 1935*b* and Chapman and Cowling 1952).

It is evident that any reduction in the algebraic work associated with the Chapman–Enskog procedure would not only benefit those interested in the computation of transport coefficients but may also bring some further lucidity in the discussion of the fundamental questions.

The first step towards simplifying the calculations was taken by Burnett (1935*a*, 1935*b*), who noted the peculiar appropriateness of the use of the Sonine polynomials in the treatment of transport problems. Since then these polynomials have been widely used in kinetic theory (Chapman and Cowling 1939, 1952; Wang-Chang and Uhlenbeck 1952; Hirschfelder, Curtiss, and Bird 1954; Waldmann 1960; Uhlenbeck and Ford 1963; Louck and De Vault 1964). In connection with the present work the use of spherical polar coordinates by Wang-Chang and Uhlenbeck (1952) (see also Louck and De Vault 1964) and, in a different context, the use of the considerations of irreducibility and of the vector coupling coefficients by Waldmann (1960, 1963) is noteworthy.

Further algebraic simplifications arise out of the following three main observations.

The first is that the most effective way of exploiting the irreducibility properties is to use spherical polar coordinates along with the algebra of irreducible tensors as developed in connection with the quantum mechanical problems of angular momentum coupling, which is encountered in atomic and nuclear physics and is mainly associated with the names of Wigner and Racah. Our standard reference on this topic will be the book by Fano and Racah (1959). In checking the details of calculations it may also be useful to consult some books on the theory of angular momentum (Condon and Shortley 1953; Edmonds 1957; Rose 1957; Brink and Satchler 1962).

The second observation arises out of the work on the harmonic oscillator shell model of nuclear physics. It was noted by Talmi (1952) that a product of the harmonic oscillator wavefunctions of two particles, $\psi(\mathbf{r}_1)\,\psi(\mathbf{r}_2)$, can be expressed as a sum of the products of similar wavefunctions, $\psi(\mathbf{R})\,\psi(\mathbf{r})$, based on the centre-of-mass and relative coordinates of the two particles. The constant coefficients of this trans-

formation are called Talmi coefficients and have been extensively studied and tabulated (for references, see Appendix I). Now, in the Burnett expansion (Burnett 1935*a*, 1935*b*; Chapman and Cowling 1939, 1952) of the distribution function the Sonine polynomials always occur multiplied with a spherical harmonic; this product is the same as the coefficient of the exponential term in harmonic oscillator wavefunctions and, therefore, undergoes the Talmi transformation with the same coefficients. The structure of the collision integral can therefore be simplified by a Talmi transformation to separate the dependence on the centre-of-mass and relative velocities in the integrand.

Both in nuclear physics and in kinetic theory it is the practice to separate the three-dimensional polynomials into their one-dimensional parts using either polar or Cartesian coordinates. The only exception is a work like that of Grad (1949*a*, 1949*b*, 1960), in which three-dimensional polynomials have been used. However, no special advantage is gained in his work, because he still uses the Cartesian system and the polynomials do not form a minimal set. As a supplement to the second observation one may note that many calculations, both in nuclear physics and in kinetic theory, can be performed without separating the radial and angular parts of the polynomials. This point has been demonstrated in connection with the calculation of Talmi coefficients (Kumar 1966*b*), and further examples occur in Appendix III of the present paper.

A part of the difficulty of the Chapman–Enskog procedure lies in the fact that the initial discussions are always made in terms of the usual integro-differential form of the Boltzmann equation. The streaming and collision terms are analysed to make a successive approximation scheme possible, and the complexities introduced by the existence of the collision invariants are handled by invoking the theorems on integral equations. The final results are, however, expressed in terms of infinite determinants, which shows that in fact, as far as the velocity dependence is concerned, a matrix inversion is involved. Therefore, the third observation is that the presentation of the method would be simplified if from the very beginning the problem were discussed in a matrix form. Contrary to the impression created by the usual presentations, it is not necessary to put the problem in the form of an integral equation in order to deal with the successive approximation procedure or the subsidiary conditions associated with the occurrence of the collision invariants. With appropriate choice of notation and proper use of the irreducibility considerations all these things can be handled in a fairly compact matrix form. The elements of these matrices are, of course, functions of operators depending on the position and time variables, and the development of the system in the configuration space is governed by the matrix-differential equations that are thus obtained.

In the present paper we wish to recast the Chapman–Enskog procedure in the light of the above observations. The appearance of our formulae and manipulations becomes quite different from what is familiar in kinetic theory. In fact, it may appear more familiar to those who have worked on the harmonic oscillator shell model of nuclear physics. Therefore, it seems appropriate to present this work in such a way that both types of specialists may be able to understand the basic problems—

for nuclear physicists it is the problem of applying familiar methods to a new and perhaps more complex physical situation, and to the kinetic theorist it is the problem of seeing the old method put into a new form. The scope of the paper is therefore limited to an exposition of the Chapman–Enskog method and the related scheme for the calculation of the transport coefficients, which suffices to exhibit various features of the new formulation and at the same time has some practical utility. We do not here enter into any discussion of the type of questions that were mentioned in the beginning of this introduction.

The main body of the paper is divided into three parts, which constitute Sections II, III, and IV. In Section II we introduce the polynomial expansion of the distribution function and the mathematical forms for physical quantities of interest. Here we also discuss the main integrals associated with the collision term of the Boltzmann equation.

Section III contains a discussion of the simple gas and the general scheme of the Chapman–Enskog method in terms of matrices. The reduction of the Boltzmann integro-differential equation to a nonlinear algebraic-differential equation for the coefficients in the expansion of the distribution function, with which that section begins, is, of course, independent of any approximations and may be used as a starting point in methods other than those of Chapman and Enskog.

Section IV is intended only to indicate the modifications necessary for handling the problem of gas mixtures.

Further indications about the presentation may be seen in the table of contents.

In closing the Introduction it should be said that since the subject is complicated there will be some complicated derivations and formulae even in this paper. They may also appear difficult to those not familiar with the algebra. In deciding upon the utility of the method the psychological feelings generated by lack of familiarity must not be allowed to intrude. There is an objective simplicity behind the appearances, which is related to conceptual simplifications in enumerating the tensorial quantities. The number of indices may be large, but the pattern of their occurrences is determined by simple rules. The preparation takes perhaps a bit longer than for Cartesian tensor analysis, but the final formulae are of greater generality; they include more detail and can be surveyed in a way in which the older formulae cannot.

II. Construction of Irreducible Tensors and Collision Integrals

(*a*) *Expansion of the Distribution Function*

(i) *Irreducible Tensors*

In kinetic theory we wish to analyse the distribution function $f(\mathbf{c}, \mathbf{r}, t)$, which is a scalar function of velocity and position vectors $\mathbf{c}$ and $\mathbf{r}$. Such a function can be formed from sums of scalar products of tensors formed from $\mathbf{c}$ and $\mathbf{r}$ separately. It is evident that to achieve greatest economy one must use a set of tensors based on $\mathbf{c}$ and on $\mathbf{r}$ that has the least possible number of members and is complete. This is the origin of the consideration of irreducibility in our context. The irreducible tensors

are defined with respect to rotations of a three-dimensional coordinate system. They can be standard or contrastandard (Fano and Racah 1959). A contrastandard, irreducible tensor of rank l is a set of $(2l+1)$ objects that transform like a spherical harmonic of rank l under rotations and are denoted by a superscript l in square brackets indicating the rank and a subscript m indicating a particular member of the set. For a given l we have $m = l, l-1, \ldots, -l+1, -l$, while l takes on the values 0, 1, 2, 3, ... According to the phase convention of Fano and Racah (1959) a spherical harmonic is defined as

$$\mathcal{Y}^{[l]}{}_m(\theta,\varphi) = \mathrm{i}^l(-)^{\frac{1}{2}(m+|m|)}\left\{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right\}^{\frac{1}{2}} P_l^{|m|}(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\varphi}, \tag{1a}$$

with

$$P_l^{|m|}(\cos\theta) = \frac{(-)^l}{2^l\,l!}(\sin\theta)^{|m|}\frac{\mathrm{d}^{l+|m|}(1-\cos^2\theta)^l}{(\mathrm{d}\cos\theta)^{l+|m|}}. \tag{1b}$$

The corresponding standard tensors are denoted by a superscript in round brackets and in the present convention are related to the contrastandard sets by

$$\mathcal{Y}^{(l)}{}_m = \mathcal{Y}^{[l]*}{}_m = (-)^{l-m}\,\mathcal{Y}^{[l]}{}_{-m}. \tag{2}$$

We have the normalization

$$\int \mathcal{Y}^{(l)}{}_m(\theta,\varphi)\,\mathcal{Y}^{[l]}{}_m(\theta,\varphi)\,\mathrm{d}(\cos\theta)\,\mathrm{d}\varphi = \delta_{ll'}\,\delta_{mm'}. \tag{3}$$

It is convenient to have another normalization of the spherical harmonics with a different notation

$$\mathfrak{C}^{[l]}{}_m(\theta,\varphi) = \left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}\mathcal{Y}^{[l]}{}_m(\theta,\varphi). \tag{4}$$

To distinguish between tensors of the same rank and standardization a further index may be included with the superscript. Thus, for example, we deal with irreducible tensor polynomials of the velocity $\phi^{[\nu l]}{}_m(\mathbf{C})$,

$$\phi^{(\nu l)}{}_m(\mathbf{C}) = \phi^{[\nu l]*}{}_m(\mathbf{C}) = (-)^{l-m}\,\phi^{[\nu l]}{}_{-m}(\mathbf{C}). \tag{5}$$

In this notation the components of a Cartesian vector $\mathbf{C}$ are written as

$$\left.\begin{aligned} C^{[1]}{}_{\pm1} \equiv C^{(1)*}{}_{\pm1} &= \frac{\mp\mathrm{i}C_1+C_2}{\sqrt{2}} = \mp\frac{\mathrm{i}}{\sqrt{2}}C\sin\theta\,\mathrm{e}^{\pm\mathrm{i}\varphi}, \\ C^{[1]}{}_0 = C^{(1)*}{}_0 &= \mathrm{i}C\cos\theta. \end{aligned}\right\} \tag{6}$$

The angles of the vector $\mathbf{C}$ are abbreviated as $\hat{\mathbf{C}}$ and an element of surface by $\mathrm{d}\hat{\mathbf{C}}$, so that the volume element is

$$\mathrm{d}\mathbf{C} = C^2\,\mathrm{d}C\,\mathrm{d}(\cos\theta)\,\mathrm{d}\varphi = C^2\,\mathrm{d}C\,\mathrm{d}\hat{\mathbf{C}}. \tag{7}$$

The corresponding components of the gradient operator associated with $\mathbf{C}$ will be denoted as follows:

$$\left.\begin{aligned} \partial_C{}^{[1]}{}_{\pm 1} &= \frac{1}{\sqrt{2}}\left(\mp \mathrm{i}\frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_2}\right), \\ \partial_C{}^{[1]}{}_0 &= \mathrm{i}\frac{\partial}{\partial C_3}. \end{aligned}\right\} \tag{8}$$

(1) *Coupling rule.*—Any two irreducible tensors $\mathfrak{a}^{(j_1)}{}_{m_1}$ and $\mathfrak{b}^{(j_2)}{}_{m_2}$ may be combined to form another irreducible tensor of rank j according to the formula

$$\left[\mathfrak{a}^{(j_1)} \times \mathfrak{b}^{(j_2)}\right]^{(j)}{}_m = \sum_{m_1,m_2} (j_1 j_2\, jm \mid j_1 m_1\, j_2 m_2)\, \mathfrak{a}^{(j_1)}{}_{m_1}\, \mathfrak{b}^{(j_2)}{}_{m_2}. \tag{9}$$

The coefficients on the right-hand side, which are known as Clebsch–Gordan or Wigner coefficients, are real numbers. The symbol is written in many forms and variously abbreviated

$$\begin{aligned} (j_1 j_2\, jm \mid j_1 m_1\, j_2 m_2) \equiv (j_1 m_1\, j_2 m_2 \mid j_1 j_2\, jm) &= (j_1 m_1\, j_2 m_2 \mid jm) \\ &= (m_1 m_2 \mid jm) \\ &= (jm \mid m_1 m_2). \end{aligned} \tag{10}$$

These coefficients vanish unless the indices j_1, j_2, j are related by the triangular conditions $j_1 + j_2 \geqslant j \geqslant |j_1 - j_2|$ and similar inequalities obtained by permuting the indices. The indices m satisfy the relation $m = m_1 + m_2$. For other properties we refer to the textbooks (Edmonds 1957; Rose 1957; Fano and Racah 1959; Brink and Satchler 1962).

(2) *Wigner–Eckart theorem.*—This theorem plays an important role in quantum mechanical calculations. In kinetic theory calculations, also, it is found to be of great importance as the guiding principle in determining the form of the integrals and in their classification. We state the theorem in a form adapted to our needs and refer to the book by Fano and Racah (1959) for a discussion in the context of quantum mechanics. The theorem is as follows:

> If the tensors $\phi^{(\nu l)}{}_m$ and $\phi^{[\nu' l']}{}_{m'}$ are irreducible polynomials in $\mathbf{C}$, and if the tensor $\mathscr{T}^{[\lambda]}{}_\mu$ depends only on $\mathbf{C}$ and the differential operators $\mathfrak{d}_C$, and if $w(C)$ is any scalar function, then the following formula holds:

$$\int w(C)\, \phi^{(\nu l)}{}_m(\mathbf{C})\, \mathscr{T}^{[\lambda]}{}_\mu(\mathbf{C}, \mathfrak{d}_C)\, \phi^{[\nu' l']}{}_{m'}(\mathbf{C})\, \mathrm{d}\mathbf{C} = (lm \mid l'm'\, \lambda\mu)\, (\nu l \parallel \mathscr{T}^{[\lambda]} \parallel \nu' l'). \tag{11}$$

This formula shows that for given νl, $\nu' l'$, and λ the ratios of integrals for different values of m, m', and μ are independent of ν, ν', and the nature of the operator $\mathscr{T}^{[\lambda]}{}_\mu$. At the same time it provides a definition of the second symbol on the right-hand side. Following the practice in nuclear physics we call this quantity the reduced integral. We have not tried to develop the analogy with quantum mechanics to the extent of expressing the integrals exactly in the form of a transition amplitude and to interpret them as such, although it is quite possible to do so. We have preferred to stay close to the usages of kinetic theory and simplify only the mathematical organization.

(ii) *Polynomials*

The polynomials used here for expanding the distribution function are basically the same as those used by Burnett (1935*a*, 1935*b*), Chapman and Cowling (1939, 1952), Wang-Chang and Uhlenbeck (1952), Waldmann (1960), and others. However, it is economical to introduce them by means of a generating function

$$G(\mathbf{a}, \mathbf{C}/\sqrt{2}) \equiv \exp\{-\mathbf{a}^2+2\mathbf{a}\cdot(\mathbf{C}/\sqrt{2})\} = \sum_{\nu=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} N_{\nu l}\, \chi^{(\nu l)}{}_m(\mathbf{a})\, \phi^{[\nu l]}{}_m(\mathbf{C}), \tag{12}$$

where

$$N_{\nu l}^2 = \frac{2\pi^{3/2}\Gamma(\nu+1)}{\Gamma(\nu+l+\frac{3}{2})}, \tag{13}$$

$$\chi^{(\nu l)}{}_m(\mathbf{a}) = \frac{(-)^\nu}{\nu!} a^{2\nu+l}\, \mathfrak{Y}^{(l)}{}_m(\hat{\mathbf{a}}). \tag{14}$$

Equations (12)–(14) define the polynomials $\phi^{[\nu l]}{}_m(\mathbf{C})$, whose explicit form may be obtained by expanding the exponential in terms of Bessel functions and using the corresponding generating function for Sonine or generalized Laguerre polynomials (e.g. Erdelyi *et al.* 1953, p. 189, equation (18); see also Kumar 1966*b*)

$$\begin{aligned}\phi^{[\nu l]}{}_m(\mathbf{C}) &= N_{\nu l}\,(C/\sqrt{2})^l\, S^{(\nu)}{}_{l+\frac{1}{2}}(\tfrac{1}{2}C^2)\, \mathfrak{Y}^{[l]}{}_m(\theta,\varphi)\\ &\equiv R_{\nu l}(C)\, \mathfrak{Y}^{[l]}{}_m(\hat{\mathbf{C}})\\ &\equiv \phi^{[\nu]}(\mathbf{C}).\end{aligned} \tag{15}$$

The last two equations respectively define the radial part $R_{\nu l}$ of the polynomial and an abbreviated notation. The polynomials satisfy the orthogonality relation

$$\int \bar{w}(\alpha, C)\, \phi^{(\nu l)}{}_m(\alpha\mathbf{C})\, \phi^{[\nu' l']}{}_{m'}(\alpha\mathbf{C})\, \mathrm{d}\mathbf{C} = \delta_{\nu\nu'}\, \delta_{ll'}\, \delta_{mm'} \equiv \delta_{\nu\nu'}, \tag{16}$$

where the weight function $\bar{w}$ is given by

$$\left.\begin{aligned}\bar{w}(\alpha, C) &= (\alpha^2/2\pi)^{3/2} \exp(-\tfrac{1}{2}\alpha^2 C^2),\\ \int \bar{w}(\alpha, C)\, \mathrm{d}C &= 1.\end{aligned}\right\} \tag{17}$$

This polynomial system depends on five parameters. Four of these are associated with the weight function and serve to fix the system in its essentials. Of these four, one is α and the other three are hidden in the choice of the origin of the coordinate system in the velocity space. The fifth parameter is related to the normalization and can be variously fixed. In our case it is done through (16) and (17).

The first few polynomials are

$$\phi^{[00]}{}_0(\alpha\mathbf{C}) = 1, \tag{18}$$

$$\phi^{[01]}{}_m(\alpha\mathbf{C}) = \alpha\, C^{[1]}{}_m, \tag{19}$$

$$\phi^{[02]}{}_m(\alpha\mathbf{C}) = (1/\sqrt{3})\alpha^2C^2\,\mathfrak{C}^{[2]}{}_m\,, \tag{20}$$

$$\phi^{[10]}{}_0(\alpha\mathbf{C}) = (1/\sqrt{6})(3-\alpha^2C^2), \tag{21}$$

$$\phi^{[11]}{}_m(\alpha\mathbf{C}) = (1/\sqrt{10})(5-\alpha^2C^2)\,\alpha\,C^{[1]}{}_m\,. \tag{22}$$

(1) *Transformation to centre-of-mass and relative velocities.*—We take

$$\alpha_1^2 = m_1/kT, \qquad \alpha_2^2 = m_2/kT\,, \tag{23}$$

$$\Gamma^2 = \alpha_1^2+\alpha_2^2 = (m_1+m_2)/kT\,, \tag{24}$$

$$\gamma^{-2} = \alpha_1^{-2}+\alpha_2^{-2}, \text{ or } \gamma^2 = (\alpha_1\alpha_2/\Gamma)^2 = m_1m_2/(m_1+m_2)kT\,, \tag{25}$$

where m_1 and m_2 are the masses, T is the temperature, and k is Boltzmann's constant. Then the transformation in question is $(\mathbf{C}_1, \mathbf{C}_2) \to (\mathbf{G}, \mathbf{g})$:

$$\Gamma^2\mathbf{G} = \alpha_1^2\,\mathbf{C}_1 + \alpha_2^2\,\mathbf{C}_2, \qquad \mathbf{g} = \mathbf{C}_1-\mathbf{C}_2\,, \tag{26}$$

$$\mathbf{C}_1 = \mathbf{G} + (\alpha_2/\Gamma)^2\,\mathbf{g}, \qquad \mathbf{C}_2 = \mathbf{G} - (\alpha_1/\Gamma)^2\,\mathbf{g}\,. \tag{27}$$

In a collision the velocities change: $(\mathbf{C}_1, \mathbf{C}_2) \to (\mathbf{C}_1', \mathbf{C}_2')$. There are equations corresponding to (26) and (27) which relate $(\mathbf{C}_1', \mathbf{C}_2')$ to $(\mathbf{G}', \mathbf{g}')$.

Conservation of energy and momentum give respectively

$$\mathbf{G} = \mathbf{G}', \qquad g^2 = g'^2\,. \tag{28}$$

Under the transformation of coordinates to the centre-of-mass and relative velocities a product of ϕ functions undergoes a Talmi transformation

$$\phi^{[\nu_1 l_1]}{}_{m_1}(\alpha_1\,\mathbf{C}_1)\,\phi^{[\nu_2 l_2]}{}_{m_2}(\alpha_2\,\mathbf{C}_2)$$
$$= \sum_{NLM,\nu lm} T\left(\begin{matrix}(\Gamma)\ NLM \\ (\gamma)\ \ \nu lm\end{matrix}\,\middle|\,\begin{matrix}(\alpha_1)\ \nu_1 l_1 m_1 \\ (\alpha_2)\ \nu_2 l_2 m_2\end{matrix}\right)\phi^{[NL]}{}_M(\Gamma\mathbf{G})\,\phi^{[\nu l]}{}_m(\gamma\mathbf{g})\,. \tag{29}$$

The chief merit of this transformation is that for given $\nu_1 l_1 m_1$, $\nu_2 l_2 m_2$ the constant coefficients T have nonvanishing values only for a limited set of values of NLM and νlm. This set is easily determined. Some properties of these coefficients and references to the literature and to tables of these coefficients will be found in Appendix I.

In abbreviated notation (29) is written as

$$\phi^{[\nu_1]}(\alpha_1\,\mathbf{C}_1)\,\phi^{[\nu_2]}(\alpha_2\,\mathbf{C}_2) = \sum_{\mathbf{N},\nu} T(\mathbf{N}, \boldsymbol{\nu} \mid (\alpha_1)\,\boldsymbol{\nu}_1, (\alpha_2)\,\boldsymbol{\nu}_2)\,\phi^{[\mathbf{N}]}(\Gamma\mathbf{G})\,\phi^{[\nu]}(\gamma\mathbf{g})\,. \tag{30}$$

The special case of one function plays an important role in our work:

$$\phi^{[\nu_1]}(\alpha_1\,\mathbf{C}_1) = \sum_{\mathbf{N},\nu} T(\mathbf{N}, \boldsymbol{\nu} \mid (\alpha_1)\,\boldsymbol{\nu}_1\,, (\alpha_2)\,\mathbf{0})\,\phi^{[\mathbf{N}]}(\Gamma\mathbf{G})\,\phi^{[\nu]}(\gamma\mathbf{g})\,. \tag{31}$$

When no confusion is likely one may also drop the scale parameters α_1 and α_2 in T.

In view of (28) it follows that if in (30) and (31) $\mathbf{C}_1$ and $\mathbf{C}_2$ are replaced respectively by $\mathbf{C}_1'$ and $\mathbf{C}_2'$ then the only change in the formulae would be to replace $\mathfrak{g}$ on the right-hand side by $\mathfrak{g}'$.

(iii) *Expansion of the Distribution Function*

The purpose of this expansion is to separate the velocity and space dependence of the distribution function. The parameters that determine the expansion can be functions of $\mathbf{r}$ and t, that is to say, the polynomial system used for the expansion is allowed to change from point to point and instant to instant through a variation in its parameters. These parameters will be called $n \equiv n(\mathbf{r}, t)$, $\mathbf{c}_0 \equiv \mathbf{c}_0(\mathbf{r}, t)$, $\alpha \equiv \alpha(\mathbf{r}, t)$. We take the origin of the coordinate system of velocities at $\mathbf{c}_0$ and thus deal with the *peculiar velocity* $\mathbf{C} = \mathbf{c} - \mathbf{c}_0$. The distribution function, which is actually $f(\mathbf{r}, \mathbf{c}, t) \equiv f(\mathbf{r}, \mathbf{C}+\mathbf{c}_0, t)$, will be written in the following equations as $f(\mathbf{r}, \mathbf{C}, t)$ according to the usual convention of kinetic theory (Chapman and Cowling 1939, 1952, p. 27). We then have

$$f(\mathbf{r}, \mathbf{C}, t) = n\bar{w}(\alpha, C) \sum_{\nu=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mathfrak{F}^{(\nu l)}{}_m(\alpha, \mathbf{r}, t)\, \phi^{[\nu l]}{}_m(\alpha \mathbf{C}), \tag{32}$$

$$\mathfrak{F}^{(\nu l)}{}_m(\alpha, \mathbf{r}, t) = \frac{1}{n} \int \phi^{(\nu l)}{}_m(\alpha \mathbf{C})\, f(r, \mathbf{C}, t)\, d\mathbf{C}. \tag{33}$$

If we choose

$$n \equiv n(\mathbf{r}, t) = \int f(\mathbf{r}, \mathbf{C}, t)\, d\mathbf{C}, \tag{34}$$

we have the simple result that the coefficients of the expansion in (32) are the average values of the polynomial tensors

$$\mathfrak{F}^{(\nu l)}{}_m(\alpha, \mathbf{r}, t) = \langle \phi^{(\nu l)}{}_m(\alpha \mathbf{C}) \rangle. \tag{35}$$

The left-hand side here is a linear combination of moments of maximum degree $2\nu+l$. In obtaining the Chapman–Enskog hierarchy of equations it is convenient to introduce a small parameter ϵ in terms of which the distribution function is expanded, thus, $f = \Sigma\, \epsilon^r f^r$. Although no mathematical significance may be attached to the orders defined by the powers of ϵ, it nonetheless remains a most convenient way of stating the procedure to be followed. Corresponding to this we have

$$\mathfrak{F}^{(\nu l)}{}_m(\alpha, \mathbf{r}, t) = \sum_{r=0} \epsilon^r\, {}^r\mathfrak{F}^{(\nu l)}{}_m(\alpha, \mathbf{r}, t). \tag{36}$$

When molecules of different species are involved we distinguish the quantities that depend on the nature of the molecules by using a subscript. Thus, we have m_s, α_s, $\mathbf{C}_s$, $\mathbf{c}_s$, f_s, and n_s. Accordingly,

$$\begin{aligned} {}^r\mathfrak{F}^{(\nu l)}{}_m(\alpha_s, \mathbf{r}, t) &= {}^r\langle \phi^{(\nu l)}{}_m(\alpha_s \mathbf{C}_s) \rangle_s \\ &\equiv {}^r\mathfrak{F}^{(\nu)}(s). \end{aligned} \tag{37}$$

In addition to the one above, other abbreviations will be made in particular instances by dropping inessential indices.

(b) *Local Equilibrium*

(i) *Parameters that Define Local Equilibrium*

In a gas mixture the distribution function $f_s(\mathbf{r}, \mathbf{C}_s, t)$ for each species is different and requires five parameters for the specification of its expansion. As we have seen, if $n_s(\mathbf{r}, t)$ is the number density for the species s, then, from (32)–(34),

$$\mathfrak{F}^{(00)}{}_0(s) = 1 . \tag{38}$$

The total number density is

$$n = \sum_s n_s . \tag{39}$$

The temperature is defined by the mean kinetic energy density:

$$(1/n) \sum_s n_s m_s \langle C_s^2 \rangle_s = 3kT ; \qquad T \equiv T(\mathbf{r}, t) . \tag{40}$$

If the parameters α_s are required to satisfy $\alpha_s^2 = m_s/kT$, then, by (21) and (35), since

$$\mathfrak{F}^{(10)}{}_0(s) \equiv \sqrt{\tfrac{1}{6}} \langle (3 - \alpha_s^2 C_s^2) \rangle_s ,$$

we have

$$\sum_s n_s \mathfrak{F}^{(10)}{}_0(s) = 0 . \tag{41}$$

From (19) and (35),

$$\mathfrak{F}^{(01)}{}_m(s) = \alpha_s \langle C_s{}^{(1)}{}_m \rangle_s . \tag{42}$$

Since $\mathbf{C}_s = \mathbf{c}_s - \mathbf{c}_0$, and $\mathbf{c}_0$ is to be chosen equal to the mass average velocity defined by

$$\rho \mathbf{c}_0 = \sum \rho_s \langle \mathbf{c}_s \rangle_s , \tag{43}$$

where the mass densities ρ_s for the species s and the total mass density ρ are given by

$$\rho_s = n_s m_s ; \qquad \rho = \sum \rho_s ; \tag{44}$$

then,

$$\mathbf{c}_0 = \frac{\sum_s n_s \alpha_s^2 \langle \mathbf{c}_s \rangle_s}{\sum_s n_s \alpha_s^2} . \tag{45}$$

It follows that

$$\sum_s n_s \alpha_s \mathfrak{F}^{(01)}{}_m(s) = 0 . \tag{46}$$

These three equations may be regarded as a definition of $\mathbf{c}_0$.

For a simple gas (41) and (46) reduce to

$$\mathfrak{F}^{(10)}{}_0 = \mathfrak{F}^{(01)}{}_m = 0 . \tag{47}$$

(ii) *Pressure Tensor*

Consider first a simple gas. If the Cartesian axes are denoted by subscripts i, j (= 1, 2, or 3), then the pressure tensor has the form

$$P_{ij} = \rho \langle C_i C_j \rangle, \qquad p \equiv \tfrac{1}{3} \sum_{i=1}^{3} P_{ii} = nkT, \tag{48}$$

where p is the kinetic or hydrostatic pressure. Equation (48) may be put in the irreducible tensor notation by using the relation

$$C_i C_j = \tfrac{1}{3} C^2 \delta_{ij} + \tfrac{2}{3} \sum_{m=-2}^{2} (ij \mid 2m)^* C^2 \mathfrak{C}^{[2]}{}_m . \tag{49}$$

The coefficients $(ij \mid 2m)$ are easily calculated and are (see Kumar 1966*a*)

$$\left.\begin{aligned} (ij \mid 20) &= \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad (ij \mid 2\pm 1) = \sqrt{\frac{3}{8}}\begin{bmatrix} 0 & 0 & \pm 1 \\ 0 & 0 & \mathrm{i} \\ \pm 1 & \mathrm{i} & 0 \end{bmatrix}, \\ (ij \mid 2\pm 2) &= \sqrt{\frac{3}{8}}\begin{bmatrix} -1 & \mp \mathrm{i} & 0 \\ \mp \mathrm{i} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}\right\} \tag{50}$$

Hence, from (20),

$$P_{ij}/p = \delta_{ij} + \sqrt{\tfrac{4}{3}} \sum_{m=-2}^{2} (ij \mid 2m)^* \mathfrak{F}^{[02]}{}_m . \tag{51}$$

Alternatively, since

$$C^2 \mathfrak{C}^{[2]}{}_m = \sum_{i,j=1}^{3} (ij \mid 2m)\, C_i C_j , \tag{52}$$

from (20) and (48)

$$\mathfrak{F}^{[02]}{}_m = \sqrt{\tfrac{1}{3}} \sum_{i,j=1}^{3} (ij \mid 2m)\, P_{ij}/p . \tag{53}$$

The isotropic part p of the pressure is always equal to nkT. The tensors $\mathfrak{F}^{(02)}{}_m$ represent that part of the second-order tensor P_{ij} which is involved in the phenomenon of viscosity.

In the case of a gas mixture the partial pressures are denoted by $p_s = n_s kT$ and $\mathfrak{F}^{[02]}{}_m(s)$ defined in terms of averages of $\mathbf{C}_s$ in the manner of (53). The total pressure in that case is

$$\left.\begin{aligned} P_{ij} &= \sum_s P_{ij}(s), \qquad p = \sum_s p_s , \\ P_{ij} &= p\, \delta_{ij} + \sqrt{\tfrac{4}{3}} \sum_{m=-2}^{2} (ij \mid 2m)^* \Big(\sum_s p_s\, \mathfrak{F}^{[02]}{}_m(s) \Big). \end{aligned}\right\} \tag{54}$$

(iii) *Thermal Flux Vector*

The molecular energy $E = \frac{1}{2}mC^2$ at any point $\mathbf{r}$ determines the corresponding thermal energy density

$$Q = \bar{E} = \tfrac{1}{2}nm\langle C^2\rangle = \tfrac{3}{2}nkT = \tfrac{3}{2}p\,. \tag{55}$$

With proper interpretation this equation holds for the simple gas as well as for the gas mixtures. It is related to the definition of α_s and the equation (41). The thermal flux vector in the simple gas case is defined as

$$\mathbf{q} = \overline{E\mathbf{C}} = \tfrac{1}{2}nm\langle C^2\mathbf{C}\rangle\,. \tag{56}$$

Writing the components as in (6) and using (22) we have

$$\mathfrak{F}^{(11)}{}_m = -\sqrt{\frac{2}{5}}\frac{\alpha^3}{mn}q^{(1)}{}_m = -\sqrt{\frac{2}{5}}\frac{\alpha}{p}q^{(1)}{}_m\,, \tag{57}$$

since $\mathfrak{F}^{(01)}{}_m = 0$.

For a gas mixture the heat flux due to each component is to be added up:

$$q^{(1)}{}_m = \sum_s q^{(1)}{}_m(s) = \tfrac{1}{2}\sum_s n_s m_s \langle C_s^2\, C_s^{(1)}{}_m\rangle_s\,. \tag{58}$$

From (6) and (22) this becomes

$$q^{(1)}{}_m = \sqrt{\tfrac{5}{2}}\sum_s (p_s/\alpha_s)\big(\sqrt{\tfrac{5}{2}}\,\mathfrak{F}^{(01)}{}_m(s) - \mathfrak{F}^{(11)}{}_m(s)\big)\,. \tag{59}$$

(c) *Distortion of Local Equilibrium*

The parameters that define the expansion of the distribution function are obtained by performing velocity averages at any given point $(\mathbf{r}, t)$ of space and time. Because of the free movement and of the collisions of molecules these parameters undergo changes from point to point. The rates of change of the parameters can be classified according to their tensor character.

(i) *Time Derivatives*

The time derivatives always occur in certain combinations with other factors. It is convenient to define related quantities, which all have the dimensions of $(\text{time})^{-1}$:

$$\text{Change in number density} \quad : \quad N = \frac{1}{n}\frac{\partial n}{\partial t}\,, \tag{60}$$

$$\text{Change in temperature} \quad : \quad A = \frac{1}{\alpha}\frac{\partial \alpha}{\partial t}\,, \tag{61}$$

$$\text{Change in local mass-average velocity} \quad : \quad \mathbf{S} = \alpha\frac{\partial}{\partial t}\mathbf{c}_0\,. \tag{62}$$

Components of $\mathbf{S}$ according to (6) are $S^{[1]}{}_\mu$. From (62) we have, since $\mathbf{C} = \mathbf{c} - \mathbf{c}_0$,

$$\mathbf{S} = -\alpha\frac{\partial}{\partial t}\mathbf{C}\,. \tag{63}$$

(ii) *Space Derivatives*

The gradient operator with respect to space coordinates will be denoted by $\mathfrak{d}$ and its components according to (6) by $\mathfrak{d}^{[1]}{}_\mu$ etc. We then have the related quantities

$$\text{Density gradient} \quad : \quad \mathscr{N} = \frac{1}{\alpha n}\mathfrak{d}n, \tag{64}$$

$$\text{Temperature gradient} : \quad \mathscr{A} = \frac{1}{\alpha^2}\mathfrak{d}\alpha. \tag{65}$$

Since $\alpha^2 = m/kT$,

$$\mathscr{A} = -\tfrac{1}{2}(kT/m)^{\frac{1}{2}}\,\mathfrak{d}(\log T). \tag{66}$$

The gradient operator acting on the vector $\mathbf{c}_0$ produces three irreducible tensors

$$\begin{aligned}\mathscr{S}^{[l]}{}_m &= \sum_\mu (1\ \mu\ 1\ m-\mu \mid lm)\ \partial^{[1]}{}_\mu\ \mathbf{c}_0{}^{[1]}{}_{m-\mu} \\ &= -\sum_\mu (1\ \mu\ 1\ m-\mu \mid lm)\ \partial^{[1]}{}_\mu\ C^{[1]}{}_{m-\mu},\end{aligned} \tag{67}$$

corresponding to the three allowed values of $l = 0,\ 1,\ 2$. In addition to these a vector

$$\mathscr{C} = \alpha(\mathbf{c}_0\cdot\mathfrak{d})\mathbf{c}_0 \tag{68}$$

is needed. All these script symbols involve differentiation with respect to $\mathbf{r}$ and have dimensions of (time)$^{-1}$.

For $l = 0$, equation (67) gives the scalar

$$\begin{aligned}\mathscr{S}^{[0]} &= \sum_\mu (1\ \mu\ 1-\mu \mid 00)\ \partial^{[1]}{}_\mu\ c_0{}^{[1]}{}_{-\mu} \\ &= \sqrt{\tfrac{1}{3}}\ \mathrm{div}\,\mathbf{c}_0.\end{aligned} \tag{69}$$

(1) *Rate-of-shear tensor.*—In Cartesian coordinates the rate-of-shear tensor is of second order and traceless:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial c_{0i}}{\partial x_j} + \frac{\partial c_{0j}}{\partial x_i}\right) - \frac{1}{3}\left(\sum_{k=1}^{3}\frac{\partial c_{0k}}{\partial x_k}\right)\delta_{ij}. \tag{70}$$

The corresponding quantity in the present notation is the irreducible tensor $\mathscr{S}^{[2]}{}_m$. That it is symmetric with respect to the three axes is seen from the fact that $(1\ \mu\ 1\ \nu \mid 2\ \mu{+}\nu) = (1\ \nu\ 1\ \mu \mid 2\ \mu{+}\nu)$. By substituting specific values in (67) it can be shown that

$$\begin{aligned}\mathscr{S}^{[2]}{}_m &= \sqrt{\tfrac{2}{3}}\sum_{i,j=1}^{3}(ij \mid 2m)\ \tfrac{1}{2}(\partial_i c_{0j} + \partial_j c_{0i}) \\ &= \sqrt{\tfrac{2}{3}}\sum_{i,j=1}^{3}(ij \mid 2m)\ S_{ij}.\end{aligned} \tag{71}$$

The inverse relation is

$$S_{ij} = \sqrt{\tfrac{2}{3}} \sum_{m=-2}^{2} (ij \mid 2m)^* \, \mathscr{S}^{[2]}{}_m . \tag{72}$$

With (48), (53), and (69) this gives

$$\sum_{i,j=1}^{3} P_{ij} \, \partial_i c_{0j} = p\left(\sqrt{2} \sum_{m=-2}^{2} \mathscr{S}^{(2)}{}_m \, \mathfrak{F}^{[02]}{}_m + \sqrt{3}\, \mathscr{S}^{(0)}\right). \tag{73}$$

(2) *Vortex vector.*—Because of the relation $(1\,\mu\,1\,\nu \mid 1\,\mu{+}\nu) = -(1\,\nu\,1\,\mu \mid 1\,\mu{+}\nu)$,

$$\mathscr{S}^{[1]}{}_m = \tfrac{1}{2} \sum_{\mu} (1\,\mu\,1\,m{-}\mu \mid 1m)\left(\partial^{[1]}{}_\mu \, c_0{}^{[1]}{}_{m-\mu} - \partial^{[1]}{}_{m-\mu} \, c_0{}^{[1]}{}_\mu\right). \tag{74}$$

This is equivalent to the vorticity vector, which in Cartesian coordinates is expressed by $\frac{1}{2}\operatorname{curl} \mathbf{c}_0$.

(*d*) *Some Integrals*

We consider now the integrals associated with the collision integral and bracket expressions of the kinetic theory (Chapman and Cowling 1939, 1952, Sections 3.54 and 4.4). These integrals are over eight variables, six of which are the velocity components of the colliding particles, before and after the collision, and the remaining two express the relation between the velocities before and after. There are various ways of choosing the last two variables. The most common choice is that given in the book by Chapman and Cowling (1939, 1952), in which the two variables are taken to be the directions of the apse line of the orbit. If the unit vector in the direction of the apse line is denoted by $\mathbf{k}$, then the eight-dimensional volume element is denoted by $\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{c}_1\,\mathrm{d}\mathbf{c}_2$ (see Chapter 3 of Chapman and Cowling). On the other hand Waldmann (1960) has used the unit vector in the direction of the final relative velocity $\mathbf{e}'$ in place of $\mathbf{k}$, so that his volume element is $\mathrm{d}\mathbf{e}'\,\mathrm{d}\mathbf{c}_1\,\mathrm{d}\mathbf{c}_2$. In either case, to give meaning to the integrals one has to make a change of variables to integrate over the inverse collisions. In our notation $\hat{\mathbf{g}} = \mathbf{e}$, $\hat{\mathbf{g}}' = \mathbf{e}'$, and we observe that, in view of the conservation relations $\mathbf{G} = \mathbf{G}'$ and $g = g'$, the volume elements of Waldmann can be written in the form

$$\begin{aligned} \mathrm{d}\hat{\mathbf{g}}\,\mathrm{d}\hat{\mathbf{g}}'\,\mathrm{d}\mathbf{G} &\equiv \mathrm{d}\hat{\mathbf{g}}'\,\mathrm{d}\mathbf{g}\,\mathrm{d}\mathbf{G} \\ &= g^2\,\mathrm{d}g\,\mathrm{d}\hat{\mathbf{g}}'\,\mathrm{d}\hat{\mathbf{g}}\,\mathrm{d}\mathbf{G} . \end{aligned} \tag{75}$$

In the last form no tacit understanding for changing the volume elements for direct and inverse collisions is necessary.

The angle χ between the initial and final relative velocities is given by

$$\cos\chi = \hat{\mathbf{g}} \cdot \hat{\mathbf{g}}' . \tag{76}$$

When the intermolecular potential is spherically symmetric all integrals involve a factor $g\sigma(g, \chi)$, where the functional form of σ is determined by the nature of the

potential. This function may be expanded in terms of spherical harmonics to separate the dependence on the angles of $\hat{\mathbf{g}}$ and $\hat{\mathbf{g}}'$:

$$\sigma(g,\chi) = \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \sigma_\lambda(g)\, \mathcal{Y}^{[\lambda]}{}_\mu(\hat{\mathbf{g}})\, \mathcal{Y}^{(\lambda)}{}_\mu(\hat{\mathbf{g}}'), \tag{77}$$

$$\sigma_\lambda(g) = 2\pi \int \sigma(g,\chi) P_\lambda(\cos\chi)\, \mathrm{d}(\cos\chi). \tag{78}$$

The procedure for determining $\sigma_\lambda(g)$ from the intermolecular potential and its connection to the quantities used in the previous works is given in Appendix II. Here we note that the function σ always occurs in the following combination with the volume elements:

$$\mathrm{d}\mathfrak{V} = \sigma(g,\chi)\, g^3\, \mathrm{d}g\, \mathrm{d}\hat{\mathbf{g}}'\, \mathrm{d}\hat{\mathbf{g}}\, \mathrm{d}\mathbf{G}. \tag{79}$$

Consider the integral

$$\int \bar{w}(\Gamma, G)\, \bar{w}(\gamma, g)\, \phi^{(\nu_3)}(\alpha_1 \mathbf{C}_1)\, \phi^{[\nu_1]}(\alpha_1 \mathbf{C}_1')\, \phi^{[\nu_2]}(\alpha_2 \mathbf{C}_2')\, \mathrm{d}\mathfrak{V}. \tag{80}$$

From the formulae (30) and (31) this becomes

$$\sum_{\mathbf{N},\mathbf{N}',\nu,\nu'} T(\Gamma\mathbf{N}, \gamma\nu \mid \alpha_1\nu_3, \alpha_2\mathbf{0})\, T(\Gamma\mathbf{N}', \gamma\nu' \mid \alpha_1\nu_1, \alpha_2\nu_2)$$

$$\times \int \bar{w}(\Gamma, G)\, \bar{w}(\gamma, g)\, \phi^{(\mathbf{N})}(\Gamma\mathbf{G})\, \phi^{[\mathbf{N}']}(\Gamma\mathbf{G})\, \phi^{(\nu)}(\gamma\mathbf{g})\, \phi^{[\nu']}(\gamma\mathbf{g}')\, \mathrm{d}\mathfrak{V}.$$

The integration over the centre-of-mass velocity gives $\delta_{\mathbf{N}\mathbf{N}'}$ from (16), and the integral over relative velocities becomes

$$\int \bar{w}(\gamma, g)\, \phi^{(\nu)}(\gamma\mathbf{g})\, \phi^{(\nu')}(\gamma\mathbf{g}')\, \sigma(g,\chi)\, g^3\, \mathrm{d}g\, \mathrm{d}\hat{\mathbf{g}}'\, \mathrm{d}\hat{\mathbf{g}}.$$

Using (15), (77), and the orthogonality of the spherical harmonics this reduces to (Kumar 1966*a*)

$$\delta_{ll'}\, \delta_{mm'} \int \bar{w}(\gamma, g)\, R_{\nu l}(\gamma g)\, \sigma_l(g)\, R_{\nu' l}(\gamma g)\, g^3\, \mathrm{d}g \equiv \delta_{ll'}\, \delta_{mm'}\, (\nu l \,||\, g\sigma_l \,||\, \nu' l), \tag{81}$$

so that, finally, the integral (80) becomes

$$\sum_{NLM,\nu lm,\nu'} T\begin{pmatrix} (\Gamma)\ NLM & (\alpha_1)\ \nu_3 l_3 m_3 \\ (\gamma)\ \nu lm & (\alpha_2)\ 000 \end{pmatrix} T\begin{pmatrix} (\Gamma)\ NLM & (\alpha_1)\ \nu_1 l_1 m_1 \\ (\gamma)\ \nu' lm & (\alpha_2)\ \nu_2 l_2 m_2 \end{pmatrix} (\nu l \,||\, g\sigma_l \,||\, \nu' l). \tag{82}$$

Since formally the sum goes over all values of ν and ν' it follows that (82) is also equal to the integral

$$\int \bar{w}(\Gamma, G)\, \bar{w}(\gamma, g)\, \phi^{(\nu_3)}(\alpha_1 \mathbf{C}_1')\, \phi^{[\nu_1]}(\alpha_1 \mathbf{C}_1)\, \phi^{[\nu_2]}(\alpha_2 \mathbf{C}_2)\, \mathrm{d}\mathfrak{V}. \tag{83}$$

On the other hand

$$\int \bar{w}(\Gamma, G)\,\bar{w}(\gamma, g)\,\phi^{(\nu_3)}(\alpha_1\,\mathbf{C}_1)\,\phi^{[\nu_1]}(\alpha_1\,\mathbf{C}_1)\,\phi^{[\nu_2]}(\alpha_2\,\mathbf{C}_2)\,\mathrm{d}\mathfrak{V}$$

$$= \sum_{NLM,\nu lm,\nu'} T\begin{pmatrix}(\Gamma)\ NLM \\ (\gamma)\ \nu lm\end{pmatrix}\begin{matrix}(\alpha_1)\ \nu_3 l_3 m_3 \\ (\alpha_2)\ 000\end{matrix} T\begin{pmatrix}(\Gamma)\ NLM \\ (\gamma)\ \nu' lm\end{pmatrix}\begin{matrix}(\alpha_1)\ \nu_1 l_1 m_1 \\ (\alpha_2)\ \nu_2 l_2 m_2\end{matrix}$$

$$\times (\nu l \,||\, g\sigma_0 \,||\, \nu' l)\,. \tag{84}$$

From (82) and (84)

$$\int \phi^{(\nu_3)}(\alpha_1\,\mathbf{C}_1)\,(f_1 f_2 - f_1' f_2')\,\mathrm{d}\mathfrak{V}$$

$$= n_1 n_2 \sum_{\nu_1,\nu_2,\mathbf{N},\nu,\nu'} T\begin{pmatrix}(\Gamma)\ NLM \\ (\gamma)\ \nu lm\end{pmatrix}\begin{matrix}(\alpha_1)\ \nu_3 l_3 m_3 \\ (\alpha_2)\ 000\end{matrix} T\begin{pmatrix}(\Gamma)\ NLM \\ (\gamma)\ \nu' lm\end{pmatrix}\begin{matrix}(\alpha_1)\ \nu_1 l_1 m_1 \\ (\alpha_2)\ \nu_2 l_2 m_2\end{matrix}$$

$$\times V^l_{\nu\nu'}\,\mathfrak{F}^{(\nu_1)}(\alpha_1)\,\mathfrak{F}^{(\nu_2)}(\alpha_2)\,, \tag{85}$$

where

$$V^l_{\nu\nu'} = (\nu l \,||\, g(\sigma_0 - \sigma_l) \,||\, \nu' l)$$

$$\equiv \int \bar{w}(\gamma, g)\,R_{\nu l}(\gamma g)\,R_{\nu' l}(\gamma g)\,\{\sigma_0(g) - \sigma_l(g)\}\,g^3\,\mathrm{d}g\,, \tag{86}$$

and $f_1 = f(\mathbf{C}_1)$, $f_2' = f(\mathbf{C}_2')$, etc., and we have used the expansion (32) for the f's.

From the equality of (80) and (83) with (82) we can replace $\phi(f_1 f_2 - f_1' f_2')$ by $\phi'(f_1' f_2' - f_1 f_2)$, so that we have the usual result

$$\tfrac{1}{2}\int \{\phi^{(\nu_3)}(\alpha_1\,\mathbf{C}_1) - \phi^{(\nu_3)}(\alpha_1\,\mathbf{C}_1')\}(f_1 f_2 - f_1' f_2')\,\mathrm{d}\mathfrak{V}$$

$$= \int \phi^{(\nu_3)}(\alpha_1\,\mathbf{C}_1)\,(f_1 f_2 - f_1' f_2')\,\mathrm{d}\mathfrak{V}\,. \tag{87}$$

The replacement of $\mathbf{C}_1$ by $\mathbf{C}_2$ in $\phi^{(\nu_3)}$ of (85) leads only to an interchange of indices in the last column of the Talmi coefficient, hence we have

$$\tfrac{1}{2}\int [\phi^{(\nu_3)}(1) + \phi^{(\nu_3)}(2) - \phi^{(\nu_3)}(1') - \phi^{(\nu_3)}(2')]\,(f_1 f_2 - f_1' f_2')\,\mathrm{d}\mathfrak{V}$$

$$= n_1 n_2 \sum_{\nu_1,\nu_2,\mathbf{N},\nu,\nu'} \left[T\begin{pmatrix}(\Gamma)\ NLM \\ (\gamma)\ \nu lm\end{pmatrix}\begin{matrix}(\alpha_1)\ \nu_3 l_3 m_3 \\ (\alpha_2)\ 000\end{matrix} + T\begin{pmatrix}(\Gamma)\ NLM \\ (\gamma)\ \nu lm\end{pmatrix}\begin{matrix}(\alpha_1)\ 000 \\ (\alpha_2)\ \nu_3 l_3 m_3\end{matrix}\right]$$

$$\times\, T\begin{pmatrix}(\Gamma)\ NLM \\ (\gamma)\ \nu' lm\end{pmatrix}\begin{matrix}(\alpha_1)\ \nu_1 l_1 m_1 \\ (\alpha_2)\ \nu_2 l_2 m_2\end{matrix} V^l_{\nu\nu'}\,\mathfrak{F}^{(\nu_1)}(\alpha_1)\,\mathfrak{F}^{(\nu_2)}(\alpha_2)\,. \tag{88}$$

If we have the functions

$$\left.\begin{aligned}\psi_1(\mathrm{C}_1) &= \sum_{\nu_3} \psi_1^{(\nu_3)}(\alpha_1)\, \phi^{[\nu_3]}(\alpha_1 \mathrm{C}_1)\,,\\ \psi_2(\mathrm{C}_2) &= \sum_{\nu_3} \psi_2^{(\nu_3)}(\alpha_2)\, \phi^{[\nu_3]}(\alpha_2 \mathrm{C}_2)\,,\end{aligned}\right\} \tag{89}$$

then in an integral similar to (88) the first Talmi coefficients would be multiplied by $\psi^{(\nu_3)}(\alpha_1)$, and using equation (A14) (Appendix I) we obtain

$$\begin{aligned}&\tfrac{1}{2}\int (\psi_1+\psi_2-\psi_1'-\psi_2')\,(f_1 f_2 - f_1' f_2')\,\mathrm{d}\mathfrak{V}\\ &\quad = \int (\psi_1+\psi_2-\psi_1'-\psi_2')\, f_1 f_2\,\mathrm{d}\mathfrak{V}\\ &\quad = n_1 n_2 \sum_{\nu_1,\nu_2,\nu_3} (\mathbf{v}_3 \mid J(\psi f) \mid \alpha_1\mathbf{v}_1, \alpha_2\mathbf{v}_2)\, \mathfrak{F}^{(\nu_1)}(\alpha_1)\, \mathfrak{F}^{(\nu_2)}(\alpha_2)\,,\end{aligned} \tag{90}$$

where

$$\begin{aligned}&(\mathbf{v}_3 \mid J(\psi f) \mid \alpha_1\mathbf{v}_1, \alpha_2\mathbf{v}_2)\\ &\quad = \sum_{\mathbf{N}_1\nu_1\nu'} [\psi_1^{[\nu_3]}(\alpha_1) + (-)^l(\alpha_2/\alpha_1)^{2N+L-2\nu-l}\, \psi_2^{[\nu_3]}(\alpha_2)]\, V^l_{\nu\nu'}\\ &\qquad \times T\begin{pmatrix}(\Gamma)\ NLM & (\alpha_1)\ \nu_3 l_3 m_3\\ (\gamma)\ \nu lm & (\alpha_2)\ 000\end{pmatrix} T\begin{pmatrix}(\Gamma)\ NLM & (\alpha_1)\ \nu_1 l_1 m_1\\ (\gamma)\ \nu' lm & (\alpha_2)\ \nu_2 l_2 m_2\end{pmatrix}.\end{aligned} \tag{91}$$

(i) *Summational Invariants*

This is not proposed as a simple method of obtaining the summational invariants but serves to introduce the way in which the calculations are to be carried out. We investigate the influence of the existence of these invariants on the structure of (91).

For a given set of values of $(\nu_3 l_3 m_3)$ the first T-coefficient in (91) can take nonvanishing values only for a certain set of values of the indices $(NLM, \nu lm)$ according to the rules given in equations (A4)–(A7) (Appendix I). We observe further that, from (86),

$$V^0_{\nu\nu'} \equiv 0\,. \tag{92}$$

The summational invariants then arise as follows.

(1) For $(\nu_3 l_3 m_3) = (000)$ the only allowed set of $(NLM, \nu lm)$ is $(000, 000)$; hence, from (92),

$$(000 \mid J(\psi f) \mid \alpha_1\mathbf{v}_1, \alpha_2\mathbf{v}_2) = 0\,, \tag{93}$$

and (91) vanishes for $\psi =$ constant.

(2) For $(\nu_3 l_3 m_3) = (01m)$ there are two allowed sets $(01m, 000)$ and $(000, 01m)$. The contribution of the first vanishes because of (92), and that of the second vanishes if

$$\psi_1^{[01]}{}_m(\alpha_1) - (\alpha_1/\alpha_2)\psi_2^{[01]}{}_m(\alpha_2) = 0\,.$$

If this were the only coefficient in ψ then, from (19), (90) would have vanished for $\psi_1 = \alpha_1^2 \mathbf{C}_1$, $\psi_2 = \alpha_2^2 \mathbf{C}_2$, or, symbolically, $\psi = m\mathbf{C}$. That is,

$$(01m \mid J(m\mathbf{C}, f) \mid \alpha_1 \mathbf{v}_1, \alpha_2 \mathbf{v}_2) = 0\,. \tag{94}$$

This is the expression for momentum conservation.

(3) For $(\nu_3 l_3 m_3) = (100)$ the possible sets are $(100, 000)$, $(000, 100)$, and $(01M, 01\, m_3 - M)$. The contribution of the first two vanishes from (92). That of the last one also vanishes if

$$\psi_1{}^{[10]}{}_0(\alpha_1) - \psi_2{}^{[10]}{}_0(\alpha_2) = 0\,,$$

that is, if

$$\psi_1 = \phi^{[10]}{}_0(\alpha_1 \mathbf{C}_1) = \psi_2 = \phi^{[10]}{}_0(\alpha_2 \mathbf{C}_2)\,.$$

From (21) the significant part here is seen to be $\alpha_1^2 C_1^2 \sim m_1 C_1^2$, which corresponds to the energy conservation

$$(100 \mid J(mC^2, f) \mid \alpha_1 \mathbf{v}_1, \alpha_2 \mathbf{v}_2) = 0\,. \tag{95}$$

It is convenient to use the symbol $\mathbf{v}^*$ for the five sets of $\mathbf{v}$ for which the above quantity vanishes:†

$$\mathbf{v}^* = 000 \text{ or } 01m \text{ or } 100\,. \tag{96}$$

(ii) *Bracket Expressions*

We refer here to the bracket expressions that occur in the later stages of a Chapman–Enskog calculation. The most general bracket is defined as (Chapman and Cowling 1939, 1952, Section 4.4, equation (9), p. 86)

$$[\psi_1 + \psi_2, \Psi_1 + \Psi_2]_{12} = \tfrac{1}{2} \int \bar{w}_1 \bar{w}_2 (\psi_1 + \psi_2 - \psi_1' - \psi_2')(\Psi_1 + \Psi_2 - \Psi_1' - \Psi_2')\, \mathrm{d}\mathfrak{V}\,. \tag{97}$$

This quantity arises if in (90) we substitute for f_1 and f_2 expressions of the form

$$f_1(\mathbf{C}_1) = n_1 \bar{w}_1 \{1 + \epsilon \psi_1(\mathbf{C}_1)\},$$

and collect the terms linear in ϵ.

Using an expansion of the form

$$\psi_1 = \sum_{\boldsymbol{\nu}} \psi_1{}^{(\nu)}\, \phi^{[\nu]}(\alpha_1 \mathbf{C}_1)\,,$$

we get from (90) and (91), with $\mathbf{v}' \equiv (\nu' l m)$,

$$\begin{aligned}[\psi_1 + \psi_2, \Psi_1 + \Psi_2]_{12} = 2 \sum_{\nu_1, \nu_2, \mathbf{N}, \nu, \nu'} & [\psi_1{}^{(\nu_1)} + (-)^l (\alpha_2/\alpha_1)^{2N+L-2\nu-l}\, \psi_2{}^{(\nu_1)}] \\ & \times [\Psi_1{}^{[\nu_2]} + (-)^l (\alpha_2/\alpha_1)^{2N+L-2\nu'-l}\, \Psi_2{}^{[\nu_2]}] \\ & \times V_{\nu\nu'}^{l}\, T(\Gamma\mathbf{N}, \gamma\mathbf{v} \mid \alpha_1 \mathbf{v}_1, \alpha_2 \mathbf{0})\, T(\Gamma\mathbf{N}, \gamma\mathbf{v}' \mid \alpha_1 \mathbf{v}_2, \alpha_2 \mathbf{0}).\end{aligned} \tag{98}$$

† Note the use of the five-pointed star in this context.

By specialization from this, all other brackets can be obtained. For instance, when the particles have the same mass, $\alpha = \alpha_1 = \alpha_2$, another bracket expression is defined by (Chapman and Cowling 1939, 1952, Section 4.4, equation (11))

$$[\psi, \Psi]_1 = [\psi_1, \Psi_1 + \Psi_2]_{12}. \tag{99}$$

According to (98), then,

$$[\psi, \Psi]_1 = \sum \psi^{(\nu_1)}\, \Psi^{[\nu_2]}\, \{1+(-)^l\}^2\, V^l_{\nu\nu'}\, T(\Gamma\mathbf{N}, \gamma\mathbf{v} \mid \alpha\mathbf{v}_1, \alpha\mathbf{0})\, T(\Gamma\mathbf{N}, \gamma\mathbf{v}' \mid \alpha\mathbf{v}_2, \alpha\mathbf{0}). \tag{100}$$

The importance of the relations (98) and (100) should be noted. They represent a complete working out of these bracket expressions, in the sense that using these a machine could evaluate the brackets. For comparison recall that in Chapman–Cowling techniques the bracket expressions have to be further reduced to the sums involving the integrals $\Omega^l(s)$. In order to determine which $\Omega^l(s)$ occur with what coefficients, Chapman and Cowling have to work out particular bracket expressions separately. Chapter 9 of their book is devoted to working out a certain number of bracket expressions. Here in (98) we have obtained the corresponding expression for the most general bracket. As shown in Appendix II, the $V^l_{\nu\nu'}$ are related to $\Omega^l(s)$, the quantities $\psi^{(\nu)}$ and $\Psi^{(\nu)}$ are expansion coefficients which would be needed in any case, and the T's are the Talmi coefficients for which formulae are given in Appendix I. The sum on the right-hand side can be formed mechanically.

III. The Simple Gas

(a) *The Boltzmann Equation and the Chapman–Enskog Hierarchy of Equations*

The simple gas is characterized by a single mass parameter so that in the Talmi coefficients $\alpha_1 = \alpha_2 = \alpha$, and they are then independent of α (Appendix I). The velocity variables inside a double integral are then represented by $\mathbf{C}$ and $\mathbf{C}_1$ rather than $\mathbf{C}_1$ and $\mathbf{C}_2$. The Boltzmann equation in this case is written as

$$\mathscr{D}f = -J(ff_1), \tag{101}$$

where

$$\mathscr{D}f = \frac{\partial}{\partial t}f + \mathbf{c}\cdot\frac{\partial}{\partial \mathbf{r}}f + \mathbf{F}\cdot\frac{\partial f}{\partial \mathbf{c}}, \tag{102}$$

$$J(ff_1) = \int (ff_1 - f'f'_1)\, g\, \sigma(g, \chi)\, \mathrm{d}\hat{\mathbf{g}}'\, \mathrm{d}\mathbf{c}_1. \tag{103}$$

These definitions are the same as in the book of Chapman and Cowling. The vector $\mathbf{F}$ is the force due to an external field; it depends on $\mathbf{r}$ and t but not on $\mathbf{c}$. Equation (101) may be converted to an equation for the coefficients $\mathfrak{F}$ in the expansion of the distribution function by using the expansion (32), multiplying from the left by $\phi^{(\nu_3)}(\alpha\mathbf{C})$, and integrating with respect to $\mathbf{C}$:

$$\sum_{\nu'_3} (\mathbf{v}_3 \mid \mathsf{D} \mid \mathbf{v}'_3)\, \mathfrak{F}^{(\nu'_3)} = -\sum_{\nu_1\nu_2} n\, (\mathbf{v}_3 \mid \mathsf{J} \mid \mathbf{v}_1, \mathbf{v}_2)\, \mathfrak{F}^{(\nu_1)}\, \mathfrak{F}^{(\nu_2)}, \tag{104}$$

where
$$(\mathbf{v}_3 \mid \mathsf{D} \mid \mathbf{v}_3') = \frac{1}{n}\int \phi^{(\nu_3)}\,(\mathscr{D} n\bar{w}(\alpha, C)\,\phi^{[\nu_3']})\,\mathrm{d}\mathbf{C}, \tag{105}$$

and the matrix J has been defined to be symmetric in $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\begin{aligned}(\mathbf{v}_3 \mid \mathsf{J} \mid \mathbf{v}_1, \mathbf{v}_2) = \tfrac{1}{2}\int \bar{w}(\alpha, C)\,\bar{w}(\alpha, C_1)\,\phi^{(\nu_3)}(\alpha\mathbf{C}) \\ \times[\phi^{[\nu_1]}(\alpha\mathbf{C})\,\phi^{[\nu_2]}(\alpha\mathbf{C}_1) - \phi^{[\nu_1]}(\alpha\mathbf{C}')\,\phi^{[\nu_2]}(\alpha\mathbf{C}_1') \\ +\phi^{[\nu_2]}(\alpha\mathbf{C})\,\phi^{[\nu_1]}(\alpha\mathbf{C}_1) - \phi^{[\nu_2]}(\alpha\mathbf{C}')\,\phi^{[\nu_1]}(\alpha\mathbf{C}_1')]\,\mathrm{d}\mathfrak{V}\,.\end{aligned} \tag{106}$$

In (105) the differentiations with respect to $\mathbf{C}$ are completed, but those with respect to $\mathbf{r}$ and t are left uncompleted. For instance, we write

$$\frac{\partial}{\partial t} n = \frac{\partial n}{\partial t} + n\frac{\partial}{\partial t},$$

so that the matrix element $(\mathbf{v}_3 \mid \mathsf{D} \mid \mathbf{v}_3')$ contains differential operators. Hence, in (104) it is important to keep it to the left of the coefficients $\mathfrak{F}^{(\nu)}(\mathbf{r}, t)$ on which it acts. Details of the structures of D and J will be discussed in the following sections.

To obtain the Chapman–Enskog hierarchy in terms of $\mathfrak{F}$, set, as in (29),

$$\mathfrak{F}^{(\nu)} = \sum_s \epsilon^s\, {}^s\mathfrak{F}^{(\nu)}. \tag{107}$$

As usual also replace J by $\epsilon^{-1}\mathsf{J}$, and separate the equations according to the powers of ϵ:

$$0 = \sum_{\nu_1,\nu_2} (\mathbf{v}_3 \mid \mathsf{J} \mid \mathbf{v}_2, \mathbf{v}_1)\, {}^0\mathfrak{F}^{(\nu_2)}\, {}^0\mathfrak{F}^{(\nu_1)}\,. \tag{108}$$

$$\sum_{\nu_3'} (\mathbf{v}_3 \mid \mathsf{D} \mid \mathbf{v}_3')\, {}^0\mathfrak{F}^{(\nu_3')} = -\sum_{\nu_1,\nu_2} 2n\,(\mathbf{v}_3 \mid \mathsf{J} \mid \mathbf{v}_2, \mathbf{v}_1)\, {}^0\mathfrak{F}^{(\nu_2)}\, {}^1\mathfrak{F}^{(\nu_1)}\,. \tag{109}$$

$$\begin{aligned}\sum_{\nu_3'} (\mathbf{v}_3 \mid \mathsf{D} \mid \mathbf{v}_3')\, {}^1\mathfrak{F}^{(\nu_3')} + \sum_{\nu_1,\nu_2} n\,(\mathbf{v}_3 \mid \mathsf{J} \mid \mathbf{v}_2, \mathbf{v}_1)\, {}^1\mathfrak{F}^{(\nu_2)}\, {}^1\mathfrak{F}^{(\nu_1)} \\ = -\sum_{\nu_1,\nu_2} 2n\,(\mathbf{v}_3 \mid \mathsf{J} \mid \mathbf{v}_2, \mathbf{v}_1)\, {}^0\mathfrak{F}^{(\nu_2)}\, {}^2\mathfrak{F}^{(\nu_1)}\,.\end{aligned} \tag{110}$$

. . . . . .

$$\begin{aligned}\sum_{\nu_3'} (\mathbf{v}_3 \mid \mathsf{D} \mid \mathbf{v}_3')\, {}^{(r-1)}\mathfrak{F}^{(\nu_3')} \\ + \sum_{\nu_1,\nu_2} n\,(\mathbf{v}_3 \mid \mathsf{J} \mid \mathbf{v}_2, \mathbf{v}_1)\,[{}^{(r-1)}\mathfrak{F}^{(\nu_2)}\, {}^1\mathfrak{F}^{(\nu_1)} + {}^{(r-2)}\mathfrak{F}^{(\nu_2)}\, {}^2\mathfrak{F}^{(\nu_1)} \ldots + {}^1\mathfrak{F}^{(\nu_2)}\, {}^{(r-1)}\mathfrak{F}^{(\nu_1)}] \\ = -\sum_{\nu_1,\nu_2} 2n\,(\mathbf{v}_3 \mid \mathsf{J} \mid \mathbf{v}_2, \mathbf{v}_1)\, {}^0\mathfrak{F}^{(\nu_2)}\, {}^r\mathfrak{F}^{(\nu_1)}\,.\end{aligned} \tag{111}$$

This is entirely analogous to the corresponding integro-differential equations and may be directly obtained from them. The first of these equations, (108), is nonlinear but has a simple solution, as will be seen in subsection (*c*) following. All others are linear algebraic equations for the successive coefficients ${}^r\mathfrak{F}$ and are to be solved one after another.

Three special features of the problem are to be noted.

(1) In each order of approximation the matrices are of infinite dimension, but we handle them as if they were finite matrices. This is justifiable to the same extent as are the usual manipulations of determinants in the last stages of calculation of transport coefficients.

(2) The coefficients $\mathfrak{F}$ are subject to five subsidiary conditions arising from the choice of local equilibrium in Section II(*b*), namely,

$$\mathfrak{F}^{(00)}{}_0 = 1, \qquad \mathfrak{F}^{(01)}{}_m = \mathfrak{F}^{(10)}{}_0 = 0. \tag{112}$$

Without loss of generality we may put ${}^0\mathfrak{F}^{(00)}{}_0 = 1$, then

$${}^s\mathfrak{F}^{(\nu^*)} = 0 \quad \text{except for} \quad s = 0,\ \mathbf{v}^* = (000). \tag{113}$$

(3) In view of the existence of the summational invariants, certain rows and columns of the matrix J vanish, and a procedure has to be specified to link this circumstance to the restrictions in (2) above and to devise a scheme for solving the equations.

(*b*) *Structure of the Collision Matrix*

Using (30) and (31) in (106) with $\mathbf{v} \equiv \nu, l, m$ and $\mathbf{v}' = \nu', l, m$ and suppressing some indices we get

$$(\mathbf{v}_3 \mid \mathsf{J} \mid \mathbf{v}_2, \mathbf{v}_1) = \sum_{\mathbf{N},\nu,\nu'} T(\mathbf{N}, \mathbf{v} \mid \mathbf{v}_3, \mathbf{0}) \left[T(\mathbf{N}, \mathbf{v}' \mid \mathbf{v}_1, \mathbf{v}_2) + T(\mathbf{N}, \mathbf{v}' \mid \mathbf{v}_2, \mathbf{v}_1)\right] V^l_{\nu\nu'}. \tag{114}$$

From equation (A3), the quantity in the square brackets in the summand may be multiplied by $(-)^l$ without changing its value, then using equation (A14) we get the more symmetric form

$$(\mathbf{v}_3 \mid \mathsf{J} \mid \mathbf{v}_2, \mathbf{v}_1) = \tfrac{1}{2} \sum_{\mathbf{N},\nu,\nu'} \left[T(\mathbf{N}, \mathbf{v} \mid \mathbf{v}_3, \mathbf{0}) + T(\mathbf{N}, \mathbf{v} \mid \mathbf{0}, \mathbf{v}_3)\right]$$
$$\times \left[T(\mathbf{N}, \mathbf{v}' \mid \mathbf{v}_1, \mathbf{v}_2) + T(\mathbf{N}, \mathbf{v}' \mid \mathbf{v}_2, \mathbf{v}_1)\right] V^l_{\nu\nu'}, \tag{115}$$

from which, following the discussion of Section II(*d*)(i), we recognize the existence of the summational invariant for a simple gas in the form

$$(\mathbf{v}^* \mid \mathsf{J} \mid \mathbf{v}_2, \mathbf{v}_1) = 0. \tag{116}$$

For other values of $\mathbf{v}_3$ this quantity in general does not vanish. In that case, considered as a matrix on $\mathbf{v}_1$ and $\mathbf{v}_2$ it is symmetric:

$$(\mathbf{v}_3 \mid \mathsf{J} \mid \mathbf{v}_1, \mathbf{v}_2) = (\mathbf{v}_3 \mid \mathsf{J} \mid \mathbf{v}_2, \mathbf{v}_1). \tag{117}$$

The matrix elements corresponding to $\mathbf{v}_1 = 0$ and $\mathbf{v}_2 = \mathbf{v}^*$ vanish by the same argument as for (116). Hence, for a given $\mathbf{v}_3 \neq \mathbf{v}^*$ we have the structure shown in Figure 1.

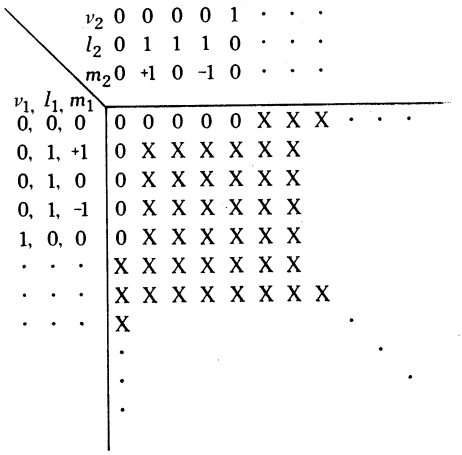

Fig. 1.—The quantity $(\mathbf{\nu}_3 \mid \mathsf{J} \mid \mathbf{\nu}_2, \mathbf{\nu}_1)$ considered as a matrix on $\mathbf{\nu}_1$ and $\mathbf{\nu}_2$ for $\mathbf{\nu}_3 \neq \mathbf{\nu}^*$. The matrix is infinite-dimensional. The elements that necessarily vanish because of the existence of summational invariants are marked 0. Other elements X may or may not vanish.

The special case in which one of the symmetric indices, say $\mathbf{\nu}_2$, is equal to zero plays an important role in solving the equations (109)–(111) of the Chapman–Enskog hierarchy. It is convenient to introduce a new symbol

$$\mathsf{J}^{\flat}_{\nu_3\nu_1} \equiv 2\,(\mathbf{\nu}_3 \mid \mathsf{J} \mid \mathbf{\nu}_1, \mathbf{0})\,. \tag{118}$$

From (115) this matrix is seen to be symmetric:

$$\mathsf{J}^{\flat}_{\nu_3\nu_1} = \mathsf{J}^{\flat}_{\nu_1\nu_3}\,; \tag{119}$$

and the five rows and columns corresponding to $\mathbf{\nu}_3 = \mathbf{\nu}_1 = \mathbf{\nu}^*$ vanish:

$$\mathsf{J}^{\flat}_{\nu^*\nu} \equiv 0\,. \tag{120}$$

It is shown in Appendix I that the products of type

$$T(\mathbf{N}, \mathbf{\nu} \mid \mathbf{\nu}_1, \mathbf{0})\, T(\mathbf{N}, \mathbf{\nu}' \mid \mathbf{\nu}_3, \mathbf{0})$$

yield a factor $\delta_{l_1 l_3}\, \delta_{m_1 m_3}$ when summed over the numbers M and m. Hence, from (118) and (115) we may write

$$\mathsf{J}^{\flat}_{\nu_3\nu_1} = J^{\flat}_{\nu_3 l_3, \nu_1 l_1}\, \delta_{l_3 l_1}\, \delta_{m_1 m_3}\,. \tag{121}$$

The matrix $\mathsf{J}^{\flat}$ thus breaks up in blocks along the diagonal characterized by values of l and m. Since the m-dependence is contained entirely in the δ functions, for a given l the same block of values is repeated along the diagonal $(2l+1)$ times, for $m = -l, -l+1, \ldots, l-1, l$. In each block the ν values range from 0 to ∞. In the block $l_3 = l_1 = 0$ the elements in the first two rows and columns corresponding to $\nu_3 = \nu_1 = 0$ and 1 vanish, and, in the blocks $l_3 = l_1 = 1$, $m = \pm 1$, 0, elements in the first rows and columns corresponding to $\nu_3 = \nu_1 = 0$ vanish. It is convenient to

collect all the rows and columns with vanishing elements at the beginning of the matrix and arrange only the nonvanishing elements in (l, m) blocks. The nonvanishing part of $\mathsf{J}^{\flat}_{\nu_3\nu_1}$ will be denoted by $\mathsf{J}_{\nu_3\nu_1}$. Then

$$\mathsf{J}_{\nu_3\nu_1} = J^{l_3}_{\nu_3\nu_1}\,\delta_{l_3l_1}\,\delta_{m_3m_1}, \tag{122}$$

where the matrices $J^{l}_{\nu\nu'}$ can be inverted with respect to the indices ν, to give finite results. The structure of these matrices and their relationship to each other are shown schematically in Figure 2.

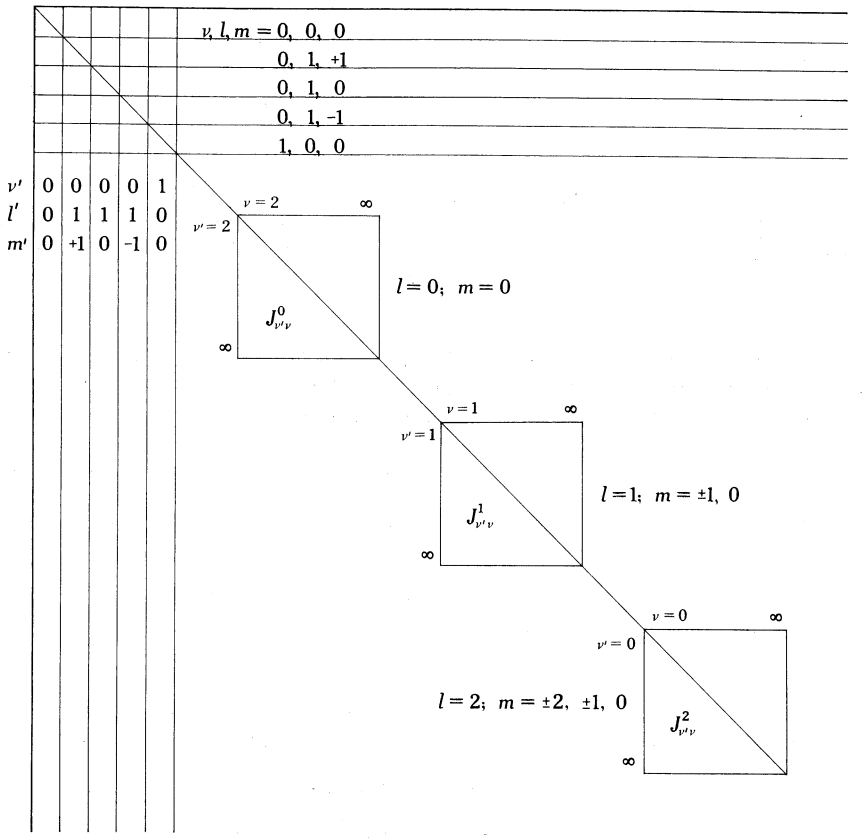

Fig. 2.—Structure of the matrix $\mathsf{J}^{\flat}_{\nu',\nu}$. The matrix $\mathsf{J}_{\nu'\nu}$ is obtained by omitting the first five rows and columns that have vanishing matrix elements. The nonvanishing elements are arranged in blocks along the diagonal. Each block is an infinite-dimensional matrix $J^{l}_{\nu'\nu}$.

From (114), (118), (121), (A14), and (A15)

$$J_{\nu_3l_3,\nu_1l_1}\,\delta_{l_3l_1} \equiv J^{l_3}_{\nu_3,\nu_1}\,\delta_{l_3l_1}$$

$$= 4\pi\,\delta_{l_3l_1} \sum_{NL,\nu l,\nu'} (-)^s\, 2^{-(P+\nu+\nu'+l)}\, \frac{\bar{N}^2_{NL}\,\bar{N}_{\nu l}\,\bar{N}_{\nu' l}}{\bar{N}_{\nu_1 l_1}\,\bar{N}_{\nu_3 l_3}}\,\sigma^2\,(lLl_1)\,V^{l}_{\nu\nu'}, \tag{123}$$

where $s = \nu+\nu'+\nu_1+\nu_3$ and the indices $NL, \nu l, \nu'$ are restricted by equations (A4)–(A7) of Appendix I according to the Talmi coefficients in (114).

From (100), (115), (118), and (121) we obtain the result that

$$[\phi^{(\nu_1)}, \phi^{[\nu_2]}] = 2\,(\mathbf{v}_1 \mid \mathsf{J} \mid \mathbf{v}_2, \mathbf{0})$$
$$= \mathsf{J}^{\flat}_{\nu_1\nu_2} = J^{\flat}_{\nu_1 l_1, \nu_2 l_2}\, \delta_{l_1 l_2}\, \delta_{m_1 m_2}\,. \tag{124}$$

(c) *First Approximation*

This is obtained by solving equation (108), in which the two factors ${}^0\mathfrak{F}^{(\nu)}$ may be regarded one as a row matrix and the other as a column matrix sandwiching between them the square matrix of Figure 1. Such a product can vanish only if

$${}^0\mathfrak{F}^{(\nu)} = \delta_{\nu,0}\,. \tag{125}$$

The normalization constant is chosen in accordance with (112) and (113). This, of course, corresponds to the Maxwell–Boltzmann distribution (see equations (25), (18), (15)):

$$f^{(0)} = n\bar{w}(\alpha, C) = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mC^2}{2kT}\right). \tag{126}$$

However, the functions n, α, and $\mathbf{c}_0$ are quite arbitrary at this stage and need not refer to the local equilibrium. They can be made appropriate to the local equilibrium only by imposing some further restrictions. Such restrictions arise when one goes to the equations of the next order.

(d) *Structure of the Matrix* $(\mathbf{v} \mid \mathsf{D} \mid \mathbf{v}')$

In evaluating the expression (105) we first express the scalar operator $\mathscr{D}$ in terms of $\mathbf{c}_0$ and $\mathbf{C}$. Since $\mathbf{c} = \mathbf{c}_0+\mathbf{C}$,

$$\mathscr{D} \equiv \frac{\partial}{\partial t} + \mathbf{c}\cdot\frac{\partial}{\mathfrak{d}\mathbf{r}} + \mathbf{F}\cdot\frac{\partial}{\mathfrak{d}\mathbf{C}}$$
$$= \frac{\partial}{\partial t} + \mathbf{c}_0\cdot\frac{\partial}{\mathfrak{d}\mathbf{r}} + \mathbf{C}\cdot\frac{\partial}{\mathfrak{d}\mathbf{r}} + \mathbf{F}\cdot\frac{\partial}{\mathfrak{d}\mathbf{C}}\,. \tag{127}$$

Acting upon the functions n, w, and ϕ in (105) this operator generates the tensor functions of $\mathbf{r}$ and t, which were introduced in Section II(*c*). The tensor of the highest rank is generated by differentiation of $\mathbf{c}_0$ with respect to $\mathbf{r}$ and is in fact the tensor $\mathscr{S}^{(2)}{}_m$. The operator $\mathscr{D}$ then is a scalar product of tensor operators based on $\mathbf{C}$ and those based on $\mathbf{r}$ and t. The maximum rank of the tensors is 2.

From the Wigner–Eckart theorem (11) the integral (105) may be written in the form

$$(\mathbf{v} \mid \mathsf{D} \mid \mathbf{v}') = \sum_{\lambda=0}^{2} \sum_{\mu=-\lambda}^{\lambda} D^{(\lambda)}{}_{\mu}(\alpha, \mathbf{r}, t;\, \nu l, \nu' l')\, (lm \mid l'm'\, \lambda\mu)\,. \tag{128}$$

It is seen by the coupling rule (equation (9)) of irreducible tensors that the quantity $\Sigma_{\nu'}\, (\mathbf{v} \mid \mathsf{D} \mid \mathbf{v}')\, \mathfrak{F}^{(\nu')}$ is an irreducible standard tensor of rank l, as required.

Unlike the matrices J of Section III(b), the present matrix is neither symmetric nor does it break up into blocks along the diagonal. Its elements are, in general, functions of $\mathbf{r}$ and t and their differential operators. In view of the Wigner coefficient involved, the matrix elements vanish unless $|l-2| \leqslant l' \leqslant (l+2)$. Because of the orthogonality of ϕ functions the ν values are also restricted, but these restrictions cannot be stated as conveniently.

The explicit form of the matrix is obtained by a straightforward calculation. We give the calculation of the $\partial/\partial t$ part as an illustration and then outline the derivation of other parts.

Using the definitions from Section II(c)(i) we have

$$\int \phi^{(\nu)}(\alpha\mathbf{C}) \frac{\partial}{\partial t}\{n\bar{w}(\alpha, C)\, \phi^{[\nu']}(\alpha\mathbf{C})\}\, \mathrm{d}\mathbf{C}$$

$$= \left(\int n\bar{w}\phi^{(\nu)}\, \phi^{[\nu']}\, \mathrm{d}\mathbf{C}\right)\frac{\partial}{\partial t} + \int \phi^{(\nu)}\left\{\left(Nn\frac{\partial}{\partial n} + A\alpha\frac{\partial}{\partial \alpha} - \mathbf{S}\cdot\frac{1}{\alpha}\frac{\partial}{\mathfrak{d}\mathbf{C}}\right) n\bar{w}\phi^{[\nu']}\right\}\mathrm{d}\mathbf{C}, \qquad (129)$$

or, since

$$\alpha\frac{\partial}{\partial\alpha}\bar{w} = (3-\alpha^2C^2)\bar{w}, \qquad \frac{\partial}{\mathfrak{d}\mathbf{C}}\bar{w} = (-\alpha^2\mathbf{C})\bar{w}, \qquad (130)$$

the second term of (129) becomes

$$\int n\bar{w}\phi^{(\nu)}\left\{N + A\left(3-\alpha^2C^2+\alpha\frac{\partial}{\partial\alpha}\right) + \sum_{\mu} S_{\mu}{}^{(1)}\left(\alpha C_{\mu}{}^{[1]} - \frac{1}{\alpha}\partial_C{}^{[1]}{}_{\mu}\right)\right\}\phi^{[\nu']}\,\mathrm{d}\mathbf{C}.$$

From the Wigner–Eckart theorem (11) and the orthogonality relation (16) the whole integral (129) then becomes

$$n\left(\frac{\partial}{\partial t}+N\right)\delta_{\nu\nu'} + nA\,\delta_{ll'}\,\delta_{mm'}\,(\nu l \,||\, (3-\alpha^2C^2+\alpha\partial/\partial\alpha) \,||\, \nu' l)$$

$$+ \sum_{\mu} n\, S^{(1)}{}_{\mu}\,(lm \mid l'm'\ 1\ \mu)\,(\nu l \,||\, \alpha C^{[1]} - (1/\alpha)\partial_C{}^{[1]} \,||\, \nu' l'). \qquad (131)$$

The first two terms yield a part of $D^{(0)}{}_0$ and the last term a part of $D^{(1)}{}_{\mu}$. The quantities n, N, A, and $S^{[1]}{}_{\mu}$ are functions of $\mathbf{r}$ and t. The operator $\partial/\partial t$ acts on the quantities $\mathfrak{F}$, which must therefore stand to the right in the sum. The double-barred quantities are the reduced integrals defined in equation (11).

It is convenient to introduce the following velocity-dependent operators:

$$K^{[1]}{}_{\mu} = \alpha\, C^{[1]}{}_{\mu} - \frac{1}{\alpha}\partial_C{}^{[1]}{}_{\mu}, \qquad (132)$$

$$W^{[\lambda]}{}_{\mu} = \sum_{\nu} (\lambda\,\mu \mid 1\,\nu\ 1\ \mu-\nu)\,\alpha\, C^{[1]}{}_{\nu}\, K^{[1]}{}_{\mu-\nu}. \qquad (133)$$

By using explicit values of the coefficients it can be seen that

$$W^{[0]}{}_0 = \frac{1}{\sqrt{3}}\left\{\alpha^2C^2 - \alpha\mathbf{C}\cdot\frac{\partial}{\mathfrak{d}(\alpha\mathbf{C})}\right\} \qquad (134)$$

and
$$W^{[1]}{}_{\mu} = \frac{1}{\sqrt{2}}\left(\mathbf{C}\times\frac{\partial}{\mathfrak{d}\mathbf{C}}\right)^{[1]}{}_{\mu}. \tag{135}$$

Usually $W^{[0]}{}_0$ operates to the right upon $\phi^{[\nu]}(\alpha\mathbf{C})$, hence in (134) we may put $\alpha\mathbf{C}\cdot\partial/\mathfrak{d}(\alpha\mathbf{C}) = \alpha\,\partial/\partial\alpha = C\,\partial/\partial C$. The second-order tensor $W^{[2]}{}_{\mu}$ can be split into a spherical harmonic and a differential operator. However, except for purposes of orientation with respect to more familiar things, it is unnecessary to analyse the tensors in this way, and calculations can be performed directly with (133).

Using the definitions introduced in Section II(c) we can now write

$$(\mathbf{c}_0\cdot\mathfrak{d}_C\, n\bar{w}\phi) = n\bar{w}[\alpha\,\mathbf{c}_0\cdot\overline{\mathscr{N}} + \alpha\,\mathbf{c}_0\cdot\mathscr{A}(3-\sqrt{3}W^{[0]}) + \mathscr{C}\cdot\mathbf{K}]\phi, \tag{136}$$

$$(\mathbf{C}\cdot\mathfrak{d}_C\, n\bar{w}\phi) = n\bar{w}[\overline{\mathscr{N}}\cdot\alpha\mathbf{C} + \mathscr{A}\cdot\alpha\mathbf{C}(3-\sqrt{3}W^{[0]}) + \sum_{\lambda=0}^{2}\sum_{\mu=-\lambda}^{\lambda}\mathscr{S}^{(\lambda)}{}_{\mu}\,W^{[\lambda]}{}_{\mu}]\phi, \tag{137}$$

since, from (67) and (133),

$$\alpha\sum_{i,j}C_i K_j\,\partial_{C_i}c_{0_j} = \sum_{\lambda=0}^{2}\sum_{\mu=-\lambda}^{\lambda}\mathscr{S}^{(\lambda)}{}_{\mu}\,W^{[\lambda]}{}_{\mu}. \tag{138}$$

Similarly, due to external force (from the last term in (127)),

$$\mathbf{F}\cdot\mathfrak{d}_C(n\bar{w}\phi) = -n\bar{w}\alpha\,\mathbf{F}\cdot(\alpha\mathbf{C}-\mathfrak{d}_C)\phi. \tag{139}$$

In these equations the velocity-dependent parts are clearly separated, and the integrations can be performed as in the example of $\partial/\partial t$ terms. Collecting all the terms arising from the use of (129), (136), (137), and (139) in (105) and comparing with (128) the expressions for $D^{(\lambda)}{}_{\mu}$ are obtained:

$$\begin{aligned}D^{(0)}&(\alpha,\mathbf{r},t;\nu l,\nu' l')\\ &= \delta_{\nu\nu'}\,\delta_{ll'}\{N + \alpha\,\mathbf{c}_0\cdot\overline{\mathscr{N}} + 3(A + \alpha\,\mathbf{c}_0\cdot\mathscr{A}) + \partial/\partial t + \mathbf{c}_0\cdot\mathfrak{d}\}\\ &\quad - \sqrt{3}\,(\nu l\,||\,W^{[0]}\,||\,\nu' l')\{A + \alpha\,\mathbf{c}_0\cdot\mathscr{A} - \sqrt{\tfrac{1}{3}}\,\mathscr{S}^{(0)}\}.\end{aligned} \tag{140}$$

$$\begin{aligned}D^{(1)}{}_{\mu}&(\alpha,\mathbf{r},t;\nu l,\nu' l')\\ &= (\nu l\,||\,\alpha C^{[1]}\,||\,\nu' l')\{\mathscr{N}^{(1)}{}_{\mu} + 3\mathscr{A}^{(1)}{}_{\mu} + \alpha^{-1}\partial^{(1)}{}_{\mu}\}\\ &\quad - \sqrt{3}\,(\nu l\,||\,\alpha C^{[1]}\,W^{[0]}\,||\,\nu' l')\,\mathscr{A}^{(1)}{}_{\mu}\\ &\quad + (\nu l\,||\,K^{[1]}\,||\,\nu' l')\,(S^{(1)}{}_{\mu} + \mathscr{C}^{(1)}{}_{\mu} - \alpha F^{(1)}{}_{\mu})\\ &\quad + (\nu l\,||\,W^{[1]}\,||\,\nu' l')\,\mathscr{S}^{(1)}{}_{\mu}.\end{aligned} \tag{141}$$

$$\begin{aligned}D^{(2)}{}_{\mu}&(\alpha,\mathbf{r},t;\nu l,\nu' l')\\ &= (\nu l\,||\,W^{[2]}\,||\,\nu' l')\,\mathscr{S}^{(2)}{}_{\mu}.\end{aligned} \tag{142}$$

The velocity operators occurring above have been defined in such a way as to make their reduced integrals dimensionless and independent of α. The operators themselves are also dimensionless but depend upon α. The evaluation of the reduced

integrals is carried out in Appendix III. With the present arrangement of various quantities the tensors $D^{(\lambda)}{}_{\mu}$ depend on $\mathbf{r}$ and t entirely through the quantities introduced in Section II(c) and the external field vector $\mathbf{F}$. In addition they contain the differential operators $\partial/\partial t$ and $\mathfrak{d}$. The expressions given above are quite general and may be used also for solving the Boltzmann equation by methods other than that of Chapman and Enskog.

(i) *General Form of the Hydrodynamic Equations*

The right-hand side of the Boltzmann equation (104) vanishes, according to (116), for $\mathbf{v}_3 = \mathbf{v}^*$. The equation then becomes

$$\sum_{\nu'} (\mathbf{v}^* \mid \mathsf{D} \mid \mathbf{v}_3') \, \mathfrak{F}^{(\nu_3')} = 0 . \tag{143}$$

The five equations corresponding to five values of $\mathbf{v}^*$ are the hydrodynamic equations. By using (140)–(142) and the explicit values of reduced integrals given in Appendix III (equations (A51)–(A67)) we get

(1) the continuity equation, $\mathbf{v}^* = (000)$,

$$N + \alpha \mathbf{c}_0 \cdot \bar{\mathcal{N}} + \sqrt{3}\, \mathscr{S}^{(0)} = 0 ; \tag{144}$$

(2) the equation of momentum transfer, $\mathbf{v}^* = (01m)$,

$$\begin{aligned} &S^{(1)}{}_m + \mathscr{C}^{(1)}{}_m - \alpha F^{(1)}{}_m + \alpha \rho^{-1} \{\partial^{(1)}{}_m p\} \\ &\quad = -\sqrt{\tfrac{10}{3}} (\alpha/\rho) [\partial^{(1)} \times \{p \mathfrak{F}^{(02)}\}]^{(1)}{}_m ; \end{aligned} \tag{145}$$

(3) the equation of energy transfer, $\mathbf{v}^* = (100)$,

$$\begin{aligned} &A + \alpha \mathbf{c}_0 \cdot \mathscr{A} - \sqrt{\tfrac{1}{3}}\, \mathscr{S}^{(0)} \\ &\quad = (1/3p) \sqrt{\tfrac{5}{2}} \sum_m [\partial^{[1]}{}_m \{(p/\alpha)\, \mathfrak{F}^{(11)}{}_m\}] - \sqrt{\tfrac{2}{9}} \sum_m \{\mathscr{S}^{[2]}{}_m \, \mathfrak{F}^{(02)}{}_m\} . \end{aligned} \tag{146}$$

By using the transformations (53), (69), (73), etc. these equations can be converted to the more usual Cartesian notation. In deriving these equations, apart from using the definitions given in Sections II(a)(i)(1), II(b), and II(c), we have used the fact that $\mathfrak{F}^{(00)}{}_0 \equiv 1$, $\mathfrak{F}^{(01)}{}_m \equiv \mathfrak{F}^{(10)}{}_0 \equiv 0$, which is the case only when the parameters n, α, and $\mathbf{c}_0$ are appropriate to the local equilibrium. If on the other hand this condition was not required then the equations would also contain the coefficients $\mathfrak{F}^{(\nu^*)}$ and their derivatives.

(ii) *Euler's Hydrodynamic Equations*

In the first approximation (Section III(c)) all coefficients of the expansion except the first, $\nu = l = m = 0$, vanish. Equation (144) remains unaltered while (145) and (146) become, respectively,

$$S^{(1)}{}_m + \mathscr{C}^{(1)}{}_m - \alpha F^{(1)}{}_m + \alpha \rho^{-1} \{\partial^{(1)}{}_m p\} = 0 , \tag{147}$$

and

$$A + \alpha \mathbf{c}_0 \cdot \mathscr{A} - \sqrt{\tfrac{1}{3}}\, \mathscr{S}^{(0)} = 0 . \tag{148}$$

Combining (148) and (144)

$$3(A + \alpha \mathbf{c}_0 \cdot \mathscr{A}) = (N + \alpha \mathbf{c}_0 \cdot \mathscr{N}), \tag{149}$$

from which the adiabatic law $(D/Dt)(nT^{3/2}) = 0$ follows in the usual way upon substituting for the symbols from Section II(c).

(e) *Second Approximation*

The equation for this approximation is

$$\sum_{\nu_3'} (\nu_3 \mid \mathsf{D} \mid \nu_3') \, {}^0\mathfrak{F}^{(\nu')} = -n \sum_{\nu_1,\nu_2} 2 (\nu_3 \mid \mathsf{J} \mid \nu_2 \nu_1) \, {}^0\mathfrak{F}^{(\nu_2)} \, {}^1\mathfrak{F}^{(\nu_1)} . \tag{109}$$

By using the first approximation ${}^0\mathfrak{F}^{(\nu)} = \delta_{\mathbf{0}\nu}$ and the matrix $\mathsf{J}^\flat$ discussed in Section III(b), the equation to be solved for ${}^1\mathfrak{F}^{(\nu)}$ becomes

$$(\nu_3 \mid \mathsf{D} \mid \mathbf{0}) = -n \sum_{\nu_1} \mathsf{J}^\flat_{\nu_3 \nu_1} \, {}^1\mathfrak{F}^{(\nu_1)} , \tag{150}$$

or, in matrix notation,

$$\{\mathsf{D}\} = -n [\mathsf{J}^\flat] \{\mathfrak{F}\} , \tag{151}$$

where $\{\mathsf{D}\}$ and $\{\mathfrak{F}\}$ are column matrices with infinite numbers of rows and $[\mathsf{J}^\flat]$ is an infinite-dimensional square matrix. As usual, in practice we work with truncated matrices of finite dimensions. It is assumed that by enlarging the dimensions of the matrix one converges to a certain desirable result; no attempt is made to prove this.

Now, from Figure 2 it is seen that, because of the vanishing of the first five rows of the matrix $\mathsf{J}^\flat$, the first five members of the column of equations (150) are

$$(\nu^* \mid \mathsf{D} \mid \mathbf{0}) = 0 . \tag{152}$$

In deriving the first-approximation solution in Section III(c) it was noted that the five parameters that define the polynomial expansion n, α (or T), and $\mathbf{c}_0$ remained undetermined at that stage. It is now seen that these may be determined by solving the five equations above, which are in fact the Euler hydrodynamic equations of subsection (d)(ii) above. The rest of the equations (150) can be represented as

$$(\nu \mid \mathsf{D} \mid \mathbf{0}) = -n \sum_{\nu_1} \mathsf{J}_{\nu \nu_1} \, {}^1\mathfrak{F}^{(\nu_1)} \qquad \nu, \nu_1 \neq \nu^* , \tag{153}$$

where the matrix J is related to $\mathsf{J}^\flat$ as shown in Figure 2. The restriction on ν and ν_1 corresponds to the vanishing of the rows and columns of $\mathsf{J}^\flat$.

These equations do not involve the first five members of the column $\{{}^1\mathfrak{F}\}$, i.e. the coefficients ${}^1\mathfrak{F}^{(\nu^*)}$. Hence, by solving only equation (109) the full column is not obtained. The first five members can be arbitrarily chosen without affecting the calculations. This corresponds to the well-known arbitrariness which arises in the solution of an integral equation when the kernel has vanishing eigenvalues. In the present discussions those theorems need not be invoked. The arbitrariness in the solution can be removed only by considering the matrix for the next approximation.

However, at this point it is already possible to calculate all of the coefficients ${}^1\mathfrak{F}^{(\nu)}$, for $\mathbf{v} \neq \mathbf{v}^*$, and to obtain results of physical interest.

By using the reduced integrals (A55)–(A67) in (140)–(142) the left-hand side of (153) is seen to be

$$(\mathbf{v} \mid \mathsf{D} \mid \mathbf{0}) = \sqrt{10}\, \mathscr{A}^{(1)}{}_m\, \delta_{\nu 1}\, \delta_{l1} + \sqrt{2}\, \mathscr{S}^{(2)}{}_m\, \delta_{\nu 0}\, \delta_{l2} \qquad \mathbf{v} \neq \mathbf{v}^*. \tag{154}$$

In view of the block structure of the matrix J (equation (122) and Figure 2), the set of equations (153) separate in the following groups:

$$0 = {}^1\mathfrak{F}^{(\nu 0)}{}_0 \qquad \nu \neq 1, 0, \tag{155}$$

$$\sqrt{10}\, \mathscr{A}^{(1)}{}_m\, \delta_{\nu 1} = -n \sum_{\nu'=1} J^1_{\nu\nu'}\, {}^1\mathfrak{F}^{(\nu' 1)}{}_m, \tag{156}$$

$$\sqrt{2}\, \mathscr{S}^{(2)}{}_m\, \delta_{\nu 0} = -n \sum_{\nu'=0} J^2_{\nu\nu'}\, {}^1\mathfrak{F}^{(\nu' 2)}{}_m, \tag{157}$$

$$0 = {}^1\mathfrak{F}^{(\nu l)}{}_m \qquad l \geqslant 3, \text{ all } \nu \text{ and } m. \tag{158}$$

The solution of (156) and (157) can be written formally as

$$-(1/n)\sqrt{10}\, (J^1)^{-1}{}_{\nu 1}\, \mathscr{A}^{(1)}{}_m = {}^1\mathfrak{F}^{(\nu 1)}{}_m, \tag{159}$$

$$-(1/n)\sqrt{2}\, (J^2)^{-1}{}_{\nu 0}\, \mathscr{S}^{(2)}{}_m = {}^1\mathfrak{F}^{(\nu 2)}{}_m. \tag{160}$$

It may be recalled that in the usual calculations of transport coefficients by the Chapman–Enskog method, as, for instance, in Chapman and Cowling (1939, 1952), the final equations are expressed in terms of the bracket expressions, which are evaluated in terms of certain determinants. Within the first approximation one recognizes various *orders* of calculations depending on the dimensions to which the infinite determinants are truncated. The integrals involving the intermolecular potential are then expressed in terms of the quantities $\Omega^l(s)$ (Chapman and Cowling 1939, 1952, p. 161). It is an important and time-consuming part of the usual theory to find the number of $\Omega^l(s)$ involved in the calculation of a given order and to determine their coefficients. The relationship of $\Omega^l(s)$ and our interaction integrals $V^l_{\nu\nu'}$ is shown in Appendix II. The advantage of the present method is seen in the fact that one is able to exhibit the complete structure of the matrix to be inverted in terms of irreducible quantities. For any given order it is an easy matter to determine which of the interaction integrals $V^l_{\nu\nu'}$ would be needed by referring to equation (123) and the properties of the Talmi coefficients (equations (A4)–(A7)). With this it might be said that the formal calculation has been done to all orders. This would also be a suitable form for machine calculations.

(i) *Viscosity*

In Cartesian notation the coefficient of viscosity μ is defined by the equation

$$P_{ij} - p\, \delta_{ij} = -2\mu S_{ij}. \tag{161}$$

In the present notation from equations (51) and (72) this becomes

$$p\,\mathfrak{F}^{(02)}{}_m = -\sqrt{2}\,\mu\,\mathscr{S}^{(2)}{}_m\,. \tag{162}$$

Comparing with (160) one gets

$$\mu = kT\,(J^2)^{-1}{}_{00}\,. \tag{163}$$

(ii) *Thermal Conductivity*

The defining equation for the thermal conductivity λ is

$$\mathbf{q} = -\lambda \frac{\partial}{\partial \mathbf{r}} T,$$

which in the present notation becomes, from (57),

$$\begin{aligned}\mathfrak{F}^{(11)}{}_m &= \sqrt{\frac{2}{5}}\frac{\lambda\alpha^3}{mn}\partial^{(1)}{}_m T \\ &= -2\sqrt{\frac{2}{5}}\lambda\frac{\alpha^2}{nk}\mathscr{A}^{(1)}{}_m\,.\end{aligned} \tag{164}$$

Hence, from (159),

$$\lambda = \frac{5k}{2\alpha^2}(J^1)^{-1}{}_{11}\,. \tag{165}$$

To establish the equivalence with the results of Section 4.7 of Chapman and Cowling, we note that their quantity **A** can be obtained in our notation by comparing the vector parts of the first-order function $f^{(1)} = f^{(0)}\,\Phi^{(1)}$. In our notation this vector corresponds to

$$\tfrac{1}{2}\sqrt{5}\sum_{\nu}(J^1)^{-1}{}_{\nu 1}\,\phi^{(\nu 1)}{}_m\,.$$

Hence, by using (124) we have

$$[\mathbf{A},\mathbf{A}]\quad\text{(Chapman–Cowling)} = \tfrac{15}{4}(J^1)^{-1}{}_{11}\,. \tag{166}$$

The equivalence of the viscosity formulae can be shown in a similar manner. The details of temperature and density dependence of these coefficients therefore need not be discussed here. It suffices to point out that apart from the already apparent factors such dependence is carried entirely in the functions $V^l_{\nu\nu'}$, which occur in $J^l_{\nu\nu'}$.

(f) Completion of the Second Approximation: Navier–Stokes Equations

As remarked earlier, the coefficients ${}^1\mathfrak{F}^{(\nu^*)}$ do not occur in the equations (150) of the second approximation and are, therefore, undetermined by it. For the first time they appear in the first five equations of the next, namely the third, approximation. From (110) these are

$$\sum_{\nu'}(\mathbf{v}^* \mid \mathsf{D} \mid \mathbf{v}')\,{}^1\mathfrak{F}^{(\nu')} = 0\,. \tag{167}$$

There are two ways of looking at equation (167).

(1) According to the first, which is related to the original Hilbert approach (Grad 1960), the quantities n, α, and $\mathbf{c}_0$ occurring in (167) are supposed to be those given by the solution of equations (152). Since these are correct only up to the zeroth order in ϵ, the restrictions (38) and (47) on $\mathfrak{F}$ are fulfilled only to that order; that is,

$$ {}^0\mathfrak{F}^{(00)}{}_0 \equiv 1, \qquad {}^0\mathfrak{F}^{(01)}{}_m \equiv {}^0\mathfrak{F}^{(10)}{}_0 \equiv 0 . \tag{168} $$

Equations (167) are, therefore, differential equations for ${}^1\mathfrak{F}^{(\nu^*)}$, to be solved by supplying suitable boundary and initial values. The solutions are in general nonvanishing. From these the first-order corrections to the quantities n, α, and $\mathbf{c}_0$ can be found.

(2) In the Chapman–Enskog method the conditions (38) and (47) on $\mathfrak{F}$ are supposed to be fulfilled as identities at each stage of calculation. In addition, since no loss of generality is involved, (168) are also to be satisfied, so that

$$ {}^1\mathfrak{F}^{(\nu^*)} \equiv 0 . \tag{169} $$

The equations (167) then have the same form as the hydrodynamic equations (144)–(146), with $\mathfrak{F}^{(11)}{}_m$ and $\mathfrak{F}^{(02)}{}_m$ replaced by the corresponding second-approximation quantities which have already been determined. Obviously both (152) and (167) cannot be satisfied by the same functions n, α, and $\mathbf{c}_0$; nor is this required. Equations (152) determine the zero-order quantities $n^{(0)}$, $\alpha^{(0)}$, and $\mathbf{c}_0{}^{(0)}$, and (167) determines the first-order corrections $n^{(1)}$, $\alpha^{(1)}$, and $\mathbf{c}_0{}^{(1)}$, whence the total quantities at this stage are $n = n^{(0)} + \epsilon n^{(1)}$, etc.

The equations (167), looked upon as equations for n, α, and $\mathbf{c}_0$, are the Navier–Stokes equations.

(g) *General Scheme of the Chapman–Enskog Method*

All elements of the method have already been illustrated. For the $(r+1)$th approximation the matrix equation from (111) is

$$ \begin{aligned} &\sum_{\nu_3'} (\nu_3 \mid \mathsf{D} \mid \nu_3')\, {}^{(r-1)}\mathfrak{F}^{(\nu_3')} \\ &+ n \sum_{\nu_1 \nu_2} (\nu_3 \mid \mathsf{J} \mid \nu_2, \nu_1) \big[{}^{(r-1)}\mathfrak{F}^{(\nu_2)}\, {}^1\mathfrak{F}^{(\nu_1)} + {}^{(r-2)}\mathfrak{F}^{(\nu_2)}\, {}^2\mathfrak{F}^{(\nu_1)} \\ &\qquad + \ldots + {}^1\mathfrak{F}^{(\nu_2)}\, {}^{(r-1)}\mathfrak{F}^{(\nu_1)} \big] \\ &\qquad = -n \sum_{\nu_2} 2\, (\nu_3 \mid \mathsf{J} \mid \nu_2, \mathbf{0})\, {}^r\mathfrak{F}^{(\nu_2)} . \end{aligned} \tag{170} $$

The first five of these equations are

$$ \sum_{\nu_3'} (\nu^* \mid \mathsf{D} \mid \nu_3')\, {}^{(r-1)}\mathfrak{F}^{(\nu_3')} = 0 . \tag{171} $$

With ${}^{(r-1)}\mathfrak{F}^{(\nu^*)} \equiv 0$ these become the equations for the corrections $n^{(r-1)}$, $\alpha^{(r-1)}$, and $\mathbf{c}_0{}^{(r-1)}$. The remaining equations (170) are for the rth-order terms ${}^r\mathfrak{F}^{(\nu)}$ ($\nu \neq \nu^*$). The matrix to be inverted is again $\mathsf{J}_{\nu,\nu'}$, which is the same in all orders. The only difference from order to order comes from the fact that the left-hand side involves tensors of higher and higher ranks. The maximum rank of tensors occurring in the $(r+1)$th approximation is $2r$, since the maximum rank of tensors in $(\nu \mid \mathsf{D} \mid \nu')$ is 2.

On the left-hand side of (170), the five quantities n, α, and $\mathbf{c}_0$ must be taken correct to the $(r-1)$th order. That is, in solving for ${}^r\mathfrak{F}$, (171) must be taken into account. The Chapman–Enskog prescription for doing this is to consider (171) as the identity to be satisfied between the time and space derivatives of n etc., and then to use this to eliminate the time derivatives of these quantities in the equations for ${}^r\mathfrak{F}$, which are consequently obtained as functionals of n, α, $\mathbf{c}_0$, and their space derivatives.

It is now possible to write a formal expression for ${}^r\mathfrak{F}$ that has a fairly straightforward structure in terms of the D matrix and the inverse of J. Since there is no immediate use for such an expression it will not be given here. It is evident that the method would be suitable if the space derivatives of n, α, and $\mathbf{c}_0$ are small.

In the present reformulation some formal features, especially those connected with the tensor character of the solutions, are brought out in clearer relief. Being a reformulation it cannot resolve any important fundamental difficulties. However, it may be noted that the Chapman–Enskog procedure does not appear to be very convenient for demonstrating the existence of different "time scales" for relaxation. The ideas about "contraction of description of the system" seem to be mere verbal accompaniments of the mathematical scheme, although they are perhaps useful in its physical interpretation. A proper clarification can come only by considering the original Boltzmann equation in some other way.

IV. Gas Mixtures

(a) *First and Second Approximations: Formal Structure*

The Boltzmann equation for the distribution function of the ith component is written as

$$\mathscr{D}_i f_i = -\sum_j J(f_i f_j)\,. \tag{172}$$

By using the expansion (32) for all functions f_i one gets

$$\sum_{\nu_3'} n_i(\mathbf{v}_3 \mid \mathsf{D}_i \mid \mathbf{v}_3')\,\mathfrak{F}^{(\nu_3')}(\alpha_i) = -\sum_{\nu_1\nu_2}\sum_j n_i n_j(\alpha_i\mathbf{v}_3 \mid \mathsf{J}_{ij} \mid \alpha_j\mathbf{v}_2,\, \alpha_i\mathbf{v}_1)\,\mathfrak{F}^{(\nu_2)}(\alpha_j)\;\mathfrak{F}^{(\nu_1)}(\alpha_i), \tag{173}$$

where

$$(\alpha_i\mathbf{v}_3 \mid \mathsf{J}_{ij} \mid \alpha_j\mathbf{v}_2,\, \alpha_i\mathbf{v}_1) = \int \phi^{(\nu_3)}(\alpha_i\mathbf{C}_i)\left[\phi^{[\nu_2]}(\alpha_j\mathbf{C}_j)\,\phi^{[\nu_1]}(\alpha_i\mathbf{C}_i)\right.$$
$$\left. - \phi^{[\nu_2]}(\alpha_j\mathbf{C}_j')\,\phi^{[\nu_1]}(\alpha_i\mathbf{C}_i')\right]\mathrm{d}\mathfrak{V}_{ij}\,. \tag{174}$$

Equation (173) can be linearized by substituting the form (36) for $\mathfrak{F}(\alpha_i)$. In the first approximation it is required that

$$\sum_{\nu_1\nu_2}\sum_{j}(\alpha_i\nu_3 \mid \mathsf{J}_{ij} \mid \alpha_j\nu_2, \alpha_i\nu_1)\,{}^0\mathfrak{F}^{(\nu_2)}(\alpha_j)\,{}^0\mathfrak{F}^{(\nu_1)}(\alpha_i) = 0\,. \tag{175}$$

Since

$$(\alpha_i\mathbf{0} \mid \mathsf{J}_{ij} \mid \alpha_j\nu_2, \alpha_i\nu_1) = 0\,, \tag{176}$$

it follows, exactly as in Section III(c), that

$${}^0\mathfrak{F}^{(\nu)}(\alpha_j) = \delta_{\mathbf{0},\nu}\,. \tag{177}$$

Although the right-hand side contains no reference to the index j it is nonetheless present in the manner in which f was expanded. As explained in Section II(b)(i), of the five parameters needed to specify the expansion of f_j only n_j and α_j bear the species index while $\mathbf{c}_0$ is common to all species. The requirement of common temperature and local mass-average velocity $\mathbf{c}_0$ imposes the condition (41) and (46) on the coefficients.

As before we obtain the linearized equations for the second approximation

$$\begin{aligned}(\nu_3 \mid \mathsf{D}_i \mid \mathbf{0}) = -\sum_{\nu_1}\sum_{j} n_j\big[&(\alpha_i\nu_3 \mid \mathsf{J}_{ij} \mid \alpha_j\nu_1, \alpha_i\mathbf{0})\,{}^1\mathfrak{F}^{(\nu_1)}(\alpha_j)\\ &+ (\alpha_i\nu_3 \mid \mathsf{J}_{ij} \mid \alpha_j\mathbf{0}, \alpha_i\nu_1)\,{}^1\mathfrak{F}^{(\nu_1)}(\alpha_i)\big]\,.\end{aligned} \tag{178}$$

Matrices on both sides can be calculated using the methods of previous sections. Following the work of Section III(b) it is seen that both matrices on the right-hand side give a factor $\delta_{l_3l_1}\delta_{m_3m_1}$ and consequently break up in blocks characterized by values of l and m, much as in Figure 2, with the exception that now only one row and one column has vanishing elements corresponding to (176). On the left-hand side there would now be more terms than in (154), because the conservation equations are no longer as simply related to these terms as in the case of the simple gas. However, as before, the terms separate in tensors characterized by (l, m), and therefore equation (178) breaks up into a number of tensorial equations analogous to equations (156)–(158). The chief difference between these equations and those for the simple gas arises from the fact that (178) is a matrix not only with respect to the indices ν but also with respect to the indices i and j denoting the species. In order to solve for $\mathfrak{F}(\alpha_i)$ the matrices have to be inverted with respect to both indices. As for the simple gas this can be done for each block. The process can be schematically indicated as follows. Each tensorial equation of (178) can be written in the form ($\mathfrak{F}^{(\nu l)}(\alpha_k) \equiv \mathfrak{F}_k{}^{(\nu l)}$)

$$\mathsf{d}^{(l)}(\nu k) = \sum_{\nu' k'} \mathscr{J}^{l}{}_{\nu k,\nu' k'}\,\mathfrak{F}_{k'}{}^{(\nu' l)}\,, \tag{179}$$

where the definitions of the relevant quantities can be extracted from (178). Then the solution is

$$\mathfrak{F}_k{}^{(\nu l)} = \sum_{\nu',k'} (\mathscr{J}^{l})^{-1}{}_{\nu k,\nu' k'}\,\mathsf{d}^{(l)}(\nu' k')\,. \tag{180}$$

The transport coefficients are obtained by picking out the coefficients of appropriate flux quantities on the right-hand side.

(i) *Conservation Equations*

From (173) and (176) we have

$$\sum_{\nu'} (\mathbf{0} \mid \mathsf{D}_i \mid \nu') \, \mathfrak{F}_i^{(\nu')} = 0 .$$

From (140)–(142) and Appendix III this gives

$$N_i + \alpha_i \mathbf{c}_0 \cdot \overline{\mathscr{N}}_i + \sqrt{3}\, \mathscr{S}^{(0)} + \sqrt{3} \sum_m (\mathscr{N}_i^{[1]}{}_m - \mathscr{A}_i^{[1]}{}_m + \alpha_i^{-1} \partial^{[1]}{}_m) \, \mathfrak{F}^{(01)}{}_m$$

$$= N_i + \alpha_i \mathbf{c}_0 \cdot \overline{\mathscr{N}}_i + \sqrt{3}\, \mathscr{S}^{(0)} + \frac{\sqrt{3}}{n_i} \mathfrak{d} \cdot \left\{ \frac{n_i}{\alpha_i} \mathfrak{F}^{(01)} \right\} = 0 . \quad (181)$$

This is the usual equation of conservation of the number of molecules for each species. The equation for conservation of mass for the mixture as a whole is obtained by performing the mass average of the above equation and using the condition (46) on $\mathfrak{F}^{(01)}{}_m(i)$:

$$\sum_i n_i \alpha_i \, \mathfrak{F}^{(01)}{}_m(i) = 0 . \quad (46)$$

Other conservation equations can be written down following the equations in Section 8.1 of Chapman and Cowling.

(*b*) *Structure of* $(\nu \mid \mathsf{D}_i \mid \mathbf{0})$ *and the Role of Subsidiary Conditions*

In calculating the left-hand side of (178) by (140)–(142) and Appendix III, the first approximation to the conservation equation (181) may be used to simplify the expression for use in developing the second approximation. The result is

$$\begin{aligned} (\nu lm \mid \mathsf{D}_i \mid 000) = {} & \delta_{\nu 1} \delta_{l0} \delta_{m0} \sqrt{6} \{ A_i + \alpha_i \mathbf{c}_0 \cdot \mathscr{A}_i - (1/\sqrt{3}) \, \mathscr{S}^{(0)} \} \\ & + \delta_{\nu 0} \delta_{l1} [\mathscr{N}_i^{(1)}{}_m - 2 \mathscr{A}_i^{(1)}{}_m + \mathscr{S}_i^{(1)}{}_m + \mathscr{C}_i^{(1)}{}_m - \alpha_i F_i^{(1)}{}_m] \\ & + \delta_{\nu 1} \delta_{l1} \sqrt{10} \, \mathscr{A}_i^{(1)}{}_m \\ & + \delta_{\nu 0} \delta_{l2} \sqrt{2} \, \mathscr{S}^{(2)}{}_m . \end{aligned} \quad (182)$$

For the simple gas the first and second terms vanished because of the conservation equation. In the present case this does not happen, but the existence of the conservation equations implies that the quantities occurring in these terms are not linearly independent. Hence, the solutions (180) obtained from these will also not be linearly independent. Such a lack of linear independence is also required by the subsidiary conditions (equations (38), (41), and (46)) which the solutions must satisfy. As in the treatments based upon the integral equations the number of linear relations between the quantities in (182), which arise from conservation equations, is exactly balanced by the number of subsidiary conditions ((38), (41), and (46)) which the solution is required to satisfy. This balancing of constraints can be achieved within the framework of any polynomial expansion that requires at least five parameters for its full specification. The special choice of polynomials here employed has the additional advantage of simplifying the calculation of the collision term.

(c) *Transport Coefficients: Example of Binary Mixtures*

The formulation of the previous subsections is suitable for showing the symmetry properties of the solutions and therefore of the fluxes and the associated transport coefficients. In actual computation of these coefficients it is necessary to keep clearly separated the two matrix inversion problems, namely, inversion in the indices (i,j) referring to the species and the indices ν,ν' referring to the polynomial system. This can be done in various ways. Perhaps the most practical method would be first to invert the matrices on the indices ν by truncating them to the desired order and then to invert them with respect to indices of species.

As an illustration consider the vector part of the first-order equation for a binary mixture. On introducing various straightforward abbreviations we obtain from (178) and (182)

$$\{\mathsf{d}_1\} = [\mathsf{J}_1]\{\mathfrak{F}_1\} + [\mathsf{J}_{12}]\{\mathfrak{F}_2\}, \tag{183}$$

$$\{\mathsf{d}_2\} = [\mathsf{J}_2]\{\mathfrak{F}_2\} + [\mathsf{J}_{21}]\{\mathfrak{F}_1\}, \tag{184}$$

where

$$\mathsf{d}_i = d_i^{(1)}\,\delta_{\nu 0} + \mathscr{A}_i^{(1)}\,(\sqrt{10}\,\delta_{\nu 1} - 2\,\delta_{\nu 0})\,. \tag{185}$$

Hence,

$$\{\mathfrak{F}_1\} = \left[[\mathsf{J}_2]^{-1}[\mathsf{J}_1] - [\mathsf{J}_{12}]^{-1}[\mathsf{J}_{21}]\right]^{-1}\left[[\mathsf{J}_2]^{-1}\{\mathsf{d}_1\} - [\mathsf{J}_{12}]^{-1}\{\mathsf{d}_2\}\right]. \tag{186}$$

There is a similar equation for $\mathfrak{F}_2$. The quantities in the square brackets are square matrices with respect to ν, and the superscript -1 represents the corresponding inverse.

The coefficients of diffusion and thermal diffusion are obtained by separating the parts depending on $d_i^{(1)}$ and $\mathscr{A}_i^{(1)}$ in (186) and using the basic definitions of these coefficients. The discussion of the linear relations between $\{\mathsf{d}_1\}$ and $\{\mathsf{d}_2\}$ and between the diffusion constants follows exactly as in the usual theory by making verbal changes appropriate to the present notation.

V. Concluding Remarks

The chief purpose of the present paper has been to demonstrate that the calculations in kinetic theory can be simplified in some respects by using certain algebraic devices. The example of the Chapman–Enskog method was chosen because it is the most familiar and, from the point of view of transport theory, the most useful. It was possible to reformulate this method in such a way that it can be understood simply in terms of the usual matrix operations and the algebra of irreducible tensors. In particular, the theorems on integral equations are not required, and the conservation, compatibility, and subsidiary conditions can be understood in a more direct manner.

Further, the tensorial properties are also expressed more clearly. At the level of the lowest-order calculations of viscosity and diffusion this advantage may appear only marginal, but the difference will be seen if calculations involving matrices or tensors of higher orders are contemplated. The main point is that in the Cartesian

notation, on going to a higher-order tensorial equation, a fresh investigation of the symmetries has to be made and the notation augmented to represent new tensors, whereas with the irreducible tensors in spherical polar coordinates no such changes are required, tensors of any order being characterized by only two indices. Transformations between Cartesian and spherical tensors of arbitrary order were considered by Kumar (1966*a*). With the help of that analysis higher-order tensors based on differentials of n, α, and $\mathbf{c}_0$ may be calculated to extend the work of Section II(*c*)(i).

With respect to the collision term also, it is possible that some method specially designed for a particular lower-order calculation might be as efficient as ours, or perhaps even more so, but the advantage of our method is seen in that no change of technique is required in going from order to order. The collision integral has been resolved into parts that can be calculated separately and then put together with the help of a small number of rules.

For generalization in a different direction it may be noted that a non-spherical potential can be accommodated by changing $V^l_{\nu\nu'}$ to $V_{\nu,\nu'}$ and replacing $\nu' lm$ in the appropriate Talmi coefficient by $\nu' l' m'$. The present form of the collision integral also suggests a generalization to accommodate certain types of three-body interactions, but the advantages to be gained from such a generalization are still not clear to us and therefore it has not been presented here.

The Boltzmann equation is complicated because of the interplay of its tensor parts and because of its nonlinearity. One of the ways of understanding the associated problems is to construct model equations that are easier to solve and in which one or the other characteristic of the equation is emphasized. The present reformulation could perhaps provide some further scope to such model making inasmuch as it provides a new perspective in terms of irreducible tensors and matrices. The polynomial expansion about the local equilibrium may be looked upon as a description of the distribution function in a function space with moving coordinates. The movement of the coordinates is governed in this theory by physical phenomena taking place in the velocity space on which the basic functions, the basic vectors of the function space, are defined.

When linearized in a different way the Boltzmann equation yields an equation very similar to the equations of quantum mechanics. One can then write down formal solutions of the linearized equation by analogy. The rules of interpretation in the two theories are, of course, very different. In particular, there is no counterpart of the moving coordinates of the function space in quantum mechanics. This is only to be expected because kinetic theory has the more detailed structure.

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Appendix I

The Coefficients of the Talmi Transformation (Equation (29))

As indicated in the Introduction, these coefficients arose out of studies on the harmonic oscillator shell model of nuclear physics, where many of the properties were established in terms of the quantum mechanical conservation laws of energy and angular momentum. This interpretation is, of course, not applicable in kinetic theory, but the relations between the indices still hold.

An introduction to the use of these transformations in kinetic theory and an algebraic derivation of the elementary properties have been given elsewhere (Kumar 1966*a*). The correspondence between the nuclear physics terminology and kinetic theory requirements has also been indicated in that paper. General formulae for the Talmi coefficients have been given by Smirnov (1961) and Kumar (1966*b*).

We list, without derivation, the properties needed for the purposes of the present paper following the earlier work (Kumar 1966*a*, 1966*b*).

By multiplying equation (29) by

$$\bar{w}(\Gamma, G)\,\bar{w}(\gamma, g)\,\phi^{(\mathbf{N})}(\Gamma\mathbf{G})\,\phi^{(\nu)}(\gamma\mathbf{g})$$

and integrating over $\mathbf{G}$ and $\mathbf{g}$ we get from the orthogonality of ϕ's

$$T\left(\begin{matrix}(\Gamma)\ NLM \\ (\gamma)\ \nu lm\end{matrix}\,\middle|\,\begin{matrix}(\alpha_1)\ \nu_1 l_1 m_1 \\ (\alpha_2)\ \nu_2 l_2 m_2\end{matrix}\right) = \int \bar{w}(\Gamma, G)\,\bar{w}(\gamma, g)\,\phi^{(NL)}{}_M(\Gamma\mathbf{G})\,\phi^{(\nu l)}{}_m(\gamma\mathbf{g})$$

$$\times\,\phi^{[\nu_1 l_1]}{}_{m_1}(\alpha_1\mathbf{C}_1)\,\phi^{[\nu_2 l_2]}{}_{m_2}(\alpha_2\mathbf{C}_2)\,\mathrm{d}\mathbf{G}\,\mathrm{d}\mathbf{g}\,. \qquad \text{(A1)}$$

The integral remains unchanged under a uniform change of scale of $\mathbf{C}_1$ and $\mathbf{C}_2$, so that the coefficient depends only on the mass ratio $\alpha_1/\alpha_2 \equiv (m_1/m_2)^{\frac{1}{2}}$ and is independent of temperature. For the case in which the two masses are equal (Talmi 1952; Thieberger 1957; Ford and Konopinski 1958; Brody 1959; Moshinsky 1959; Balashov and Eltekov 1960; Brody, Jacob, and Moshinsky 1960; Brody and Moshinsky 1960; Moshinsky and Brody 1960; Arima and Terasawa 1960; Lawson and Goeppert-Mayer 1960; Kaufman and Noak 1965) the coefficient is completely independent of mass. This is the case most often studied in nuclear physics, where it is convenient to express the coefficient in the form ($\Gamma = \gamma^{-1} = \sqrt{2}$)

$$T(\Gamma\mathbf{N}, \gamma\mathbf{v} \mid 1\mathbf{v}_1, 1\mathbf{v}_2)$$

$$= \sum_{\lambda} (lm\ LM \mid \lambda\mu)\,(\lambda\mu \mid l_1 m_1\ l_2 m_2)\,\mathrm{i}^{l_1+l_2-l-L}\,\langle \nu l,\ NL,\ \lambda \mid \nu_1 l_1,\ \nu_2 l_2,\ \lambda\rangle\,. \qquad \text{(A2)}$$

The quantities in angular brackets, called simply the bracket expressions, have been tabulated by Brody and Moshinsky (1960). Other related tables were given by Balashov and Eltekov (1960) and by Kaufman and Noak (1965).

The quantities

$$\phi^{[\nu_1]}(\alpha_1\mathbf{C}_1)\,\phi^{[\nu_2]}(\alpha_2\mathbf{C}_2) + \phi^{[\nu_1]}(\alpha_2\mathbf{C}_2)\,\phi^{[\nu_2]}(\alpha_1\mathbf{C}_1) \quad \text{and} \quad \phi^{[\mathbf{N}]}(\Gamma\mathbf{G})$$

remain unchanged by an interchange of particles $C_1 \leftrightarrow C_2$, $\alpha_1 \leftrightarrow \alpha_2$, but the relative velocity changes sign, $\mathbf{g} \to -\mathbf{g}$. Hence,

$$\phi^{[\nu]}(\gamma\mathbf{g}) \to (-)^l \phi^{[\nu]}(\gamma\mathbf{g}) .$$

Therefore, by (A1), if

$$[\ ] = [T(\Gamma N, \gamma\mathbf{v} \mid \mathbf{v}_1, \mathbf{v}_2) + T(\Gamma N, \gamma\mathbf{v} \mid \mathbf{v}_2, \mathbf{v}_1)],$$

then

$$[\ ] = (-)^l [\ ] . \tag{A3}$$

Thus, the symmetric sum vanishes unless l is even (Talmi 1952).

The chief reason for the usefulness of the coefficients is that by virtue of the properties of the polynomials only a small set of indices is connected by nonvanishing coefficients. The coefficient always vanishes unless the following relations are satisfied:

(i) $m_1+m_2 = M+m$; (A4)

(ii) $(-)^{l_1+l_2} = (-)^{L+l}$; (A5)

(iii) $(l_1+l_2) \geqslant \lambda \geqslant |l_1-l_2|$, and $(L+l) \geqslant \lambda \geqslant |L-l|$; (A6)

(iv) $2\nu_1+l_1+2\nu_2+l_2 = 2\nu+l+2N+L$. (A7)

The general formula is (Kumar 1966*b*)

$$T\left(\begin{matrix}(\Gamma)\ NLM \\ (\gamma)\ \nu lm\end{matrix} \middle| \begin{matrix}(\alpha_1)\ \nu_1 l_1 m_1 \\ (\alpha_2)\ \nu_2 l_2 m_2\end{matrix}\right)$$

$$= \frac{(-)^{\nu_1+\nu_2+\nu+N}\, \hat{l}_1 \hat{l}_2 \hat{l} \hat{L}}{\bar{N}_{\nu_1 l_1} \bar{N}_{\nu_2 l_2} \bar{N}_{\nu l} \bar{N}_{NL}} \left(\frac{\alpha_1}{\Gamma}\right)^{\frac{1}{2}(p_1+p_2+p+P)}$$

$$\times \sum \Big[(-)^{l''} (\alpha_2/\alpha_1)^{P''+p'} \left[\bar{N}_{\nu' l'} \bar{N}_{\nu'' l''} \bar{N}_{N' L'} \bar{N}_{N'' L''}\right]^2$$

$$\times \sigma(l'L'l_1)\, \sigma(l''L''l_2)\, \sigma(l'l''l)\, \sigma(L'L''L)\, X\begin{pmatrix} l' & l'' & l \\ L' & L'' & L \\ l_1 & l_2 & \lambda \end{pmatrix}$$

$$\times (lm\ LM \mid \lambda\mu)\, (\lambda\mu \mid l_1 m_1, l_2 m_2) \Big] , \tag{A8}$$

where

$$\bar{N}_{\nu l}^2 = N_{\nu l}^2/(\nu!)^2 = \frac{2\pi^{3/2}}{\Gamma(\nu+1)\, \Gamma(\nu+l+\frac{3}{2})}, \tag{A9}$$

$$\sigma(l_1 l_2 l_3) = \mathrm{i}^{l_1+l_2-l_3} \frac{\hat{l}_1 \hat{l}_2}{\hat{l}_3\, (4\pi)^{\frac{1}{2}}} (l_1 0\ l_2 0 \mid l_3 0) , \tag{A10}$$

$$\hat{l} = (2l+1)^{\frac{1}{2}}, \tag{A11}$$

and the $X(\ )$ is the so-called 9-j symbol of the theory of angular momentum (Edmonds 1957; Fano and Racah 1959; Brink and Satchler 1962). The other bracket symbols are the Wigner coefficients defined in Section II(a)(ii). The sum in (A8) is over all

allowed values of the dashed quantities, e.g. L'', l', etc. and λ. There is no sum over μ since $\mu = m_1+m_2 = m+M$. The restrictions on the values that the variable indices l' etc. can take are imposed by the properties of the Wigner coefficients contained in the σ's. In addition the following equations are to be satisfied:

$$\left.\begin{aligned} P'+p' &= p_1, \qquad & P''+p'' &= p_2, \\ P'+P'' &= P, & p'+p'' &= p. \end{aligned}\right\} \tag{A12}$$

The P's are defined in all cases following the pattern of $2\nu+l = p$. The equations (A12) determine three out of the four quantities ν', ν'', N', N''.

The formula simplifies enormously when one set of indices vanishes:

$$\begin{aligned} &T(\Gamma\,\mathbf{N}, \gamma\,\boldsymbol{\nu} \mid \alpha_1\,\boldsymbol{\nu}_1, \alpha_2\,\mathbf{0}) \\ &= (4\pi)^{\frac{1}{2}}(-)^{N+\nu+\nu_1}\frac{\bar{N}_{NL}\bar{N}_{\nu l}}{\bar{N}_{\nu_1 l_1}}\left(\frac{\alpha_1}{\Gamma}\right)^{P+p}\left(\frac{\alpha_2}{\alpha_1}\right)^{p}\sigma(lLl_1)\,(lm\;LM \mid l_1 m_1). \end{aligned} \tag{A13}$$

Further,

$$T(\Gamma\mathbf{N}, \gamma\boldsymbol{\nu} \mid \alpha_1\boldsymbol{\nu}_1, \alpha_2\mathbf{0}) = (-)^{l}\left(\frac{\alpha_1}{\alpha_2}\right)^{P-p} T(\Gamma\mathbf{N}, \gamma\boldsymbol{\nu} \mid \alpha_1\mathbf{0}, \alpha_2\boldsymbol{\nu}_1). \tag{A14}$$

From the orthogonality of the Wigner coefficients

$$\sum_{m,M}(lm\;LM \mid l_1 m_1)\,(lm\;LM \mid l_3 m_3) = \delta_{m_3 m_1}\,\delta_{l_1 l_3},$$

we have, for $\mathbf{N}' = N'LM$, $\boldsymbol{\nu}' = \nu' lm$,

$$\begin{aligned} &\sum_{M,m} T(\Gamma\mathbf{N}, \gamma\boldsymbol{\nu} \mid \alpha_1\boldsymbol{\nu}_1, \alpha_2\mathbf{0})\,T(\Gamma\mathbf{N}', \gamma\boldsymbol{\nu}' \mid \alpha_1\boldsymbol{\nu}_3, \alpha_2\mathbf{0}) \\ &\quad = \delta_{l_3 l_1}\,\delta_{m_3 m_1}(-)^{N+\nu+N'+\nu'+\nu_1+\nu_3}\frac{\bar{N}_{NL}\bar{N}_{\nu l}\bar{N}_{N'L'}\bar{N}_{\nu' l}}{\bar{N}_{\nu_1 l_1}\bar{N}_{\nu_3 l_1}} \\ &\qquad \times\left(\frac{\alpha_1}{\Gamma}\right)^{P+P'+p+p'}\left(\frac{\alpha_2}{\alpha_1}\right)^{p+p'}\sigma^2(lLl_1). \end{aligned} \tag{A15}$$

Similarly, the sum

$$\sum_{M,m} T(\Gamma\mathbf{N}, \gamma\boldsymbol{\nu} \mid \alpha_1\boldsymbol{\nu}_1, \alpha_2\mathbf{0})\,T(\Gamma\mathbf{N}', \gamma\boldsymbol{\nu}' \mid \alpha_1\mathbf{0}, \alpha_2\boldsymbol{\nu}_3')$$

is proportional to $\delta_{l_1 l_3}\,\delta_{m_1 m_3}$.

Appendix II

The Integrals Involving the Intermolecular Interaction and Their Relation to the Usual Integrals of Kinetic Theory

In the collision integrals discussed in Sections II(d) and III(b) the effect of the intermolecular interaction appears through the term $V^{l}_{\nu\nu'}$, which is an integral involving the function $\sigma_l(g)$ (equations (86), (77), and (78))

$$\sigma_l(g) = 2\pi\int\sigma(g,\chi)\,P_l(\cos\chi)\,\mathrm{d}(\cos\chi). \tag{78}$$

The quantity $\sigma(g,\chi)$ is the differential cross section for scattering of particles of relative velocity of magnitude g when the angle between the initial and final velocities is χ. This form arises from the spherical symmetry of the intermolecular potential. If this were not the case then it would be necessary to describe the orientation of the plane of initial and final relative velocities with respect to the plane containing the intrinsic directions of the molecules. Then the differential cross section would be $\sigma(g,\chi,\epsilon)$ and would have to be expanded in terms of spherical harmonics of χ and ϵ rather than the Legendre polynomials of χ. No special change of point of view is needed to accommodate that case, hence it is sufficient to confine ourselves to the case of spherical symmetry.

According to Waldmann (1960, Sec. 26),

$$\sigma(g,\chi) = \frac{1}{\sin\chi}\left|b\frac{\mathrm{d}b}{\mathrm{d}\chi}\right|, \tag{A16}$$

where b is the impact parameter. Hence,

$$\sigma_l(g) = 2\pi\int_0^\infty P_l(\cos\chi)\, b\,\mathrm{d}b\,. \tag{A17}$$

In this expression χ is to be substituted as a function of g and b from the well-known relation

$$\tfrac{1}{2}(\pi-\chi) = \int_0^{s^*}\left(1-s^2-\frac{2\Phi}{mg^2}\right)^{-1}\mathrm{d}s\,, \tag{A18}$$

where Φ is the intermolecular potential, and the upper limit of the integration, s^*, is obtained from the equation

$$1-s^{*2}-2\frac{\Phi(b/s^*)}{mg^2} = 0\,. \tag{A19}$$

In usual kinetic theory one uses the function (Waldmann 1960, equation (50.8))

$$Q^{(l)} = 2\pi\int_0^\infty (1-\cos^l\chi)\, b\,\mathrm{d}b \tag{A20}$$

and the integral (equation (50.5) of Waldmann 1960)

$$\Omega^{(l,s)} = \frac{1}{\gamma(2\pi)^{\frac{1}{2}}}\int_0^\infty \mathrm{e}^{-x^2}x^{2s+3}Q^{(l)}(\sqrt{2}\,x/\gamma)\,\mathrm{d}x\,, \tag{A21}$$

where $\gamma^{-2} = \alpha_1^{-2}+\alpha_2^{-2}$ and $\sqrt{2}\,x = \gamma g$. The quantities $\Omega^{(l,s)}$ are the same as $\Omega^{(l)}(s)$ of Chapman and Cowling (1939, 1952, p. 157). If

$$P_\lambda(\cos\chi) = \sum_l a_l^\lambda\cos^l\chi, \qquad \sum_l a_l^\lambda = 1\,, \tag{A22}$$

then, by (A17) and (A20),

$$\sigma_0(g)-\sigma_\lambda(g) = \sum_l^\lambda a_l^\lambda\, Q^{(l)}(g)\,. \tag{A23}$$

In terms of the variable x, equation (86) becomes

$$V^{\lambda}_{\nu\nu'} = \frac{\sqrt{2}}{\gamma}\int w(x)\, R_{\nu\lambda}(\sqrt{2}\,x)\, R_{\nu'\lambda}(\sqrt{2}\,x)\{\sigma_0(\sqrt{2}\,x/\gamma) - \sigma_\lambda(\sqrt{2}\,x/\gamma)\}\, x^3\, \mathrm{d}x\,, \tag{A24}$$

with $w(x) = \pi^{-3/2}\exp(-x^2)$.

Now, $\{w(x)\}^{\frac{1}{2}} R_{\nu\lambda}(\sqrt{2}\,x)$ is the same as the radial part of a harmonic oscillator wavefunction, hence, by expanding these according to Brody, Jacob, and Moshinsky (1960) and Brody and Moshinsky (1960), the integral may be expressed as

$$V^{\lambda}_{\nu\nu'} = (\sqrt{2}/\gamma)\sum_s B(\nu\lambda, \nu'\lambda, s)\, I^{\lambda}_s\,, \tag{A25}$$

with

$$I^{\lambda}_s = \frac{2}{\Gamma(s+\frac{3}{2})}\int_0^{\infty} \mathrm{e}^{-x^2} x^{2s+3}(\sigma_0 - \sigma_\lambda)\, \mathrm{d}x \tag{A26}$$

$$= \frac{2\gamma(2\pi)^{\frac{1}{2}}}{\Gamma(s+\frac{3}{2})}\sum_l^{\lambda} a^{\lambda}_l\, \Omega^{(l,s)}\,. \tag{A27}$$

The coefficients a^{λ}_l are standard, while the coefficients $B(\nu l, \nu' l', s)$ were introduced in the work of Moshinsky (1959) and of Brody, Jacob, and Moshinsky (1960) and have since been tabulated (Brody and Moshinsky 1960). These formulae are given only to show the relationship; it is not suggested that it is necessary to evaluate the integrals in the form $\Omega^{(l,s)}$. With a machine program it may be just as convenient to evaluate (A17) and (A24) directly.

The integrals $\Omega^{(l,s)}$ and I^{λ}_s are of the same type. In nuclear physics literature the latter are known as Talmi integrals.

Appendix III

Evaluation of the Reduced Integrals

The reduced integrals occurring in equations (140)–(142) are defined according to the Wigner–Eckart theorem (11)

$$\int \bar{w}(\alpha, C)\, \phi^{(\nu l)}{}_m(\alpha\mathbf{C})\, \mathscr{T}^{[\lambda]}{}_\mu\, \phi^{[\nu' l']}{}_{m'}(\alpha\mathbf{C})\, \mathrm{d}\mathbf{C} = (lm\,|l'm'\lambda\mu)\,(\nu l\,||\,\mathscr{T}^{[\lambda]}\,||\,\nu' l')\,. \tag{A28}$$

According to (12), the left-hand side is the coefficient of

$$\bar{\chi}^{[\nu]}(\mathbf{a})\,\bar{\chi}^{(\nu')}(\mathbf{b})$$

in the integral

$$\int \bar{w}(\alpha, C)\, G(\mathbf{a}, \alpha\mathbf{C}/\sqrt{2})\, \mathscr{T}^{[\lambda]}{}_\mu\, G(\mathbf{b}, \alpha\mathbf{C}/\sqrt{2})\, \mathrm{d}\mathbf{C}\,, \tag{A29}$$

where we have set

$$\bar{\chi}^{(\nu)}(\mathbf{a}) = N_{\nu l}\,\chi^{(\nu)}(a) = (-)^{\nu}\,\bar{N}_{\nu l}\, a^{2\nu+l}\, \mathfrak{Y}^{(l)}{}_m(\hat{\mathbf{a}})\,. \tag{A30}$$

The notation here is chosen so as not to conflict with that in an earlier work (Kumar 1966*b*). The coefficient $\bar{N}_{\nu l}$ was defined in (A9).

The operators $\mathscr{T}^{[\lambda]}{}_{\mu}$ we need to integrate are

$$\alpha C^{[1]}{}_{\mu}; \qquad K^{[1]}{}_{\mu} = \alpha C^{[1]}{}_{\mu} - (1/\alpha)\,\partial_{C}{}^{[1]}{}_{\mu}; \qquad \alpha C^{[1]}{}_{\mu}\, W^{[0]};$$

$$W^{[\lambda]}{}_{\mu} = \sum_{\nu} (\lambda\mu \mid 1\nu\, 1\, \mu{-}\nu)\, \alpha C^{[1]}{}_{\nu} K^{[1]}{}_{\mu-\nu} \qquad (\lambda = 0, 1, 2).$$

Since, further,

$$\bar{w}(\alpha, C) = (\alpha^2/2\pi)^{3/2} \exp(-\tfrac{1}{2}\alpha^2 C^2),$$

the integrals (A28) and (A29) are seen to be independent of α, which may therefore be chosen equal to $\sqrt{2}$ in order to simplify the expressions of all quantities involved. Thus, we have

$$\bar{w}(\sqrt{2}, C) \equiv w(C) = \pi^{-3/2} \exp(-C^2), \tag{A31}$$

$$\alpha C^{[1]}{}_{\mu} = \sqrt{2}\, C^{[1]}{}_{\mu}, \qquad K^{[1]}{}_{\mu} = \sqrt{2}\,(C^{[1]}{}_{\mu} - \tfrac{1}{2}\,\partial_{C}{}^{[1]}{}_{\mu}), \tag{A32}$$

$$W^{[\lambda]}{}_{\mu} = 2 \sum (\lambda\mu \mid 1\nu\, 1\, \mu{-}\nu)\, C^{[1]}{}_{\nu}\, (C^{[1]}{}_{\mu-\nu} - \tfrac{1}{2}\,\partial_{C}{}^{[1]}{}_{\mu-\nu}), \tag{A33}$$†

and (A29) becomes

$$I(\mathscr{T}^{[\lambda]}{}_{\mu}) = \pi^{-3/2} \int \mathrm{e}^{-C^2 - a^2 + 2\mathbf{a}\cdot\mathbf{C}}\, \mathscr{T}^{[\lambda]}{}_{\mu}\, \mathrm{e}^{-b^2 + 2\mathbf{b}\cdot\mathbf{C}}\, \mathrm{d}\mathbf{C}. \tag{A34}$$

Consider first the integral

$$I(\mathbf{C}) = \pi^{-3/2} \int \mathbf{C} \exp\{-C^2 - a^2 - b^2 + 2(\mathbf{a}+\mathbf{b})\cdot\mathbf{C}\}\, \mathrm{d}\mathbf{C}$$

$$= \pi^{-3/2}\, \mathrm{e}^{2\mathbf{a}\cdot\mathbf{b}} \int \mathbf{C}\, \mathrm{e}^{-(\mathbf{C}-\mathbf{a}-\mathbf{b})^2}\, \mathrm{d}\mathbf{C},$$

or, after shifting the origin and integrating,

$$I(\mathbf{C}) = (\mathbf{a}+\mathbf{b})\, \mathrm{e}^{2\mathbf{a}\cdot\mathbf{b}}. \tag{A35}$$

From (A32)

$$K^{[1]}{}_{\mu}\, G(\mathbf{b}, \mathbf{C}) = \sqrt{2}\,(C^{[1]}{}_{\mu} - b^{[1]}{}_{\mu})\, G(\mathbf{b}, \mathbf{C}).$$

The integration can be done as for (A35)

$$I(K^{[1]}{}_{\mu}) = \sqrt{2}\, a^{[1]}{}_{\mu}\, \mathrm{e}^{2\mathbf{a}\cdot\mathbf{b}}. \tag{A36}$$

Similarly, from (A33),

$$I(W^{[\lambda]}{}_{\mu}) = 2\pi^{-3/2} \sum_{\nu} (\lambda\mu \mid 1\nu\, 1\, \mu{-}\nu) \exp(2\mathbf{a}\cdot\mathbf{b})$$

$$\times \int (C^{[1]}{}_{\nu} + a^{[1]}{}_{\nu} + b^{[1]}{}_{\nu})(C^{[1]}{}_{\mu-\nu} + a^{[1]}{}_{\mu-\nu})\, \mathrm{e}^{-C^2}\, \mathrm{d}\mathbf{C}. \tag{A37}$$

† Note that the ν used inside the Wigner coefficients is a different type of index from the ν used in $\phi^{[\nu l]}{}_{m}$ or in $(\nu l \,||\, \mathscr{T}^{[\lambda]} \,||\, \nu' l')$.

The terms linear in $\mathbf{C}$ in the coefficient of the exponential vanish on integration. The terms independent of $\mathbf{C}$ have the factor $\int \exp C^2 \, \mathrm{d}\mathbf{C} = \pi^{3/2}$. The terms bilinear in $\mathbf{C}$ vanish unless $\mu = 0$, since

$$\int C^{[1]}{}_{\mu} \, C^{[1]}{}_{\nu} \, \mathrm{e}^{-C^2} \, \mathrm{d}\mathbf{C} = (-)^{1-\nu} \tfrac{1}{2}\pi^{3/2} \, \delta_{\mu,-\nu} ,$$

so that

$$I(W^{[\lambda]}{}_{\mu}) = 2 \sum_{\nu} (\lambda\mu \mid 1\nu \; 1 \; \mu{-}\nu) \left[(a^{[1]}{}_{\nu} + b^{[1]}{}_{\nu}) a^{[1]}{}_{\mu-\nu} + \tfrac{1}{2}(-)^{1-\nu} \delta_{\mu 0} \right] \mathrm{e}^{2\mathbf{a}\cdot\mathbf{b}} . \quad \text{(A38)}$$

Since $\sqrt{3}\, W^{[0]} = (\alpha^2 C^2 - \mathbf{C}\cdot\mathfrak{d}_C)$,

$$\sqrt{3}\, W^{[0]} G(\mathbf{b}, \mathbf{C}) = (2C^2 - 2\mathbf{C}\cdot\mathbf{b}) \, G(\mathbf{b}, \mathbf{C}) = \mathfrak{d}_b \cdot \mathbf{C} \, G(\mathbf{b}, \mathbf{C}) . \quad \text{(A39)}$$

On substituting in (A34) for $C_i \, W^{[0]}$ the differentiation with respect to $\mathbf{b}$ may be performed after the integration. Therefore,

$$\sqrt{3}\, I(C_i \, W^{[0]}) = \pi^{-3/2} \mathfrak{d}_b \cdot \int (\mathbf{C}{+}\mathbf{a}{+}\mathbf{b})(C_i{+}a_i{+}b_i) \, \mathrm{e}^{-C^2+2\mathbf{a}\cdot\mathbf{b}} \, \mathrm{d}\mathbf{C} .$$

Using the relation

$$\int C_i \, C_j \, \mathrm{e}^{-C^2} \, \mathrm{d}\mathbf{C} = \tfrac{1}{2}\pi^{3/2} \, \delta_{ij}$$

we get

$$\sqrt{3}\, I(C_i \, W^{[0]}) = [a_i + (a_i + b_i)\{4 + 2\mathbf{a}\cdot(\mathbf{a}+\mathbf{b})\}] \, \mathrm{e}^{2\mathbf{a}\cdot\mathbf{b}} ,$$

which may be put in the required form

$$I(\sqrt{2}\, C^{[1]}{}_{\mu} \, W^{[0]}) = \sqrt{\tfrac{2}{3}} \left[a^{[1]}{}_{\mu} \, \mathrm{e}^{2\mathbf{a}\cdot\mathbf{b}} + \left\{ (4 + 2a^2 + \partial/\partial\beta) \left(a^{[1]}{}_{\mu} + b^{[1]}{}_{\mu} \right) \mathrm{e}^{2\mathbf{a}\cdot\mathbf{b}\beta} \right\}_{\beta=1} \right] . \quad \text{(A40)}$$

The main part of this calculation is in the picking out of appropriate coefficients to find the reduced integral of (A28). Here we need the algebra of the quantum mechanical angular momentum theory (Edmonds 1957; Rose 1957; Fano and Racah 1959; Brink and Satchler 1962).

We require the relation

$$\mathrm{e}^{2\mathbf{a}\cdot\mathbf{b}} = \sum_{\nu l m} \bar{N}^2_{\nu l} \, (ab)^{2\nu+l} \, \mathcal{Y}^{[l]}{}_{m}(\hat{\mathbf{a}}) \, \mathcal{Y}^{(l)}{}_{m}(\hat{\mathbf{b}}) \quad \text{(A41)}$$

and the coupling rule for spherical harmonics in the form

$$a^{[1]}{}_{\mu} \, \mathcal{Y}^{[l]}{}_{m}(\hat{\mathbf{a}}) = a \sum_{l'} \bar{\sigma}(l1l') \, (lm \; 1\mu \mid l'm') \, \mathcal{Y}^{[l']}{}_{m'}(\hat{\mathbf{a}}) , \quad \text{(A42)}$$

where

$$\bar{\sigma}(l1l') = (l'0 \; 10 \mid l0) \, \mathrm{i}^{(l-l'+3)} .$$

With these the left-hand side of (A36) becomes

$$\sqrt{2}\sum_{\nu'l'm',l}\bar{N}^2_{\nu'l'}a^{2\nu'+l'+1}b^{2\nu'+l'}\bar{\sigma}(l'1l)(l'm'\,1\,\mu\mid lm)\,\mathcal{Y}^{[l]}{}_m(\hat{\mathbf{a}})\,\mathcal{Y}^{(l')}{}_{m'}(\hat{\mathbf{b}}).$$

Using (A30) the coefficient of $\bar{\chi}^{[\nu]}(\mathbf{a})\,\bar{\chi}^{(\nu')}(\mathbf{b})$ can be picked up, which by comparing with (A28) gives

$$(\nu l\,||\,K^{[1]}\,||\,\nu'l') = (-)^{\nu'+\nu}\bar{\sigma}(l'1l)\sqrt{2}\,\delta_{p'+1,p}\frac{\bar{N}_{\nu'l'}}{\bar{N}_{\nu l}}. \tag{A43}$$

As in Appendix I, the symbols p are formed after the pattern of $p = 2\nu+l$. In calculation with (A35) the same process is repeated with due account taken of the relation (5) between the standard and contrastandard tensors. The result is

$$(\nu l\,||\,\alpha C^{[1]}\,||\,\nu'l') = (-)^{\nu'+\nu}\bar{\sigma}(l'1l)\sqrt{2}\left(\delta_{p'+1,p}\frac{\bar{N}_{\nu'l'}}{\bar{N}_{\nu l}}+\delta_{p'-1,p}\frac{\bar{N}_{\nu l}}{\bar{N}_{\nu'l'}}\right). \tag{A44}$$

In (A40) the first term is the same as in (A36), and the tensor part of its second term is the same as in (A35). Since the variable β is scalar, the tensor separation can be carried out as for (A44), and then the operator $(4+2a^2+\partial/\partial\beta)$ can be taken into account. The result is

$$\begin{aligned}(\nu l\,||\,\alpha C^{[1]}\,W^{[0]}\,||\,\nu'l') &= (-)^{\nu'+\nu}\sqrt{\tfrac{2}{3}}\,\bar{\sigma}(l'1l)\Bigg[\left\{2\delta_{p'+3,p}+(p'+5)\,\delta_{p'+1,p}\right\}\frac{\bar{N}_{\nu'l'}}{\bar{N}_{\nu l}}\\ &\quad+2\delta_{p'+1,p}\frac{\bar{N}^2_{\nu-1,l}}{\bar{N}_{\nu l}\bar{N}_{\nu'l'}}+(p'+3)\,\delta_{p'-1,p}\frac{\bar{N}_{\nu l}}{\bar{N}_{\nu'l'}}\Bigg].\end{aligned} \tag{A45}$$

To evaluate the reduced integral of $W^{[\lambda]}$ consider the successive terms of (A38). The tensor part of the first term involves

$$\sum_{\nu}(\lambda\mu\mid 1\nu\,1\,\mu-\nu)\,a^{[1]}{}_{\nu}\,a^{[1]}{}_{\mu-\nu}\,\mathcal{Y}^{[l']}{}_{m'}(\hat{\mathbf{a}}).$$

By applying (A42) successively this becomes

$$a^2\sum_{\nu,l'',m'',lm}\bar{\sigma}(l'1l'')\,\bar{\sigma}(l''1l)\,(\lambda\mu\mid 1\nu\,1\,\mu-\nu)\,(l'm'\,1\nu\mid l''m'')\,(l''m''\,1\,\mu-\nu\mid lm)\,\mathcal{Y}^{[l]}{}_m(\hat{\mathbf{a}}). \tag{A46}$$

The sum over ν and m'' can be performed and the result expressed in terms of the 6-j symbol $\overline{W}$ (e.g. equations (11.19) and (10.13) of Fano and Racah (1959))

$$a^2\sum_{l'',l}\hat{\lambda}\hat{l}''\,\bar{\sigma}(l'1l)\,\bar{\sigma}(l''1l)\,\overline{W}\begin{pmatrix}1 & 1 & \lambda\\ l' & l & l''\end{pmatrix}(l'm'\,\lambda\mu\mid lm)\,\mathcal{Y}^{[l]}{}_m(\hat{\mathbf{a}}). \tag{A47}$$

In the second term of (A38) we have

$$\sum_{\nu}(-)^{l''-m''}(\lambda\mu\mid 1\nu\,1\mu-\nu)\,b^{[1]}{}_{\nu}\,a^{[1]}{}_{\mu-\nu}\,\mathcal{Y}^{[l'']}{}_{m''}(\hat{\mathbf{a}})\,\mathcal{Y}^{[l'']}{}_{-m''}(\hat{\mathbf{b}}).$$

On applying (A42) this gives

$$ab \sum_{\nu,l,m,l'm'} (-)^{l''-m''-l'+m'} (\lambda\mu \mid 1\nu\, 1\mu{-}\nu)\,(l''{-}m''\, 1\nu \mid l'{-}m')\,(l''m''\, 1\mu{-}\nu \mid lm)$$
$$\times\, \bar{\sigma}(l''1l')\,\bar{\sigma}(l''1l)\,\mathcal{Y}^{[l]}{}_m(\hat{\mathbf{a}})\,\mathcal{Y}^{(l')}{}_{m'}(\hat{\mathbf{b}}),$$

where the sign of m', all values of which are to be summed over, was taken negative in the Wigner coefficient for convenience. On using a symmetry property of the Wigner coefficient (e.g. Rose 1957, equation (3.17a)) and noting that $m'+\nu = m''$ and, because of the $\bar{\sigma}$ factor, $l''-l'+1$ is even, we get

$$ab \sum_{\nu,l,m,l'm'} (\hat{l}'/\hat{l}'')\,\bar{\sigma}(l''1l')\,\bar{\sigma}(l''1l)\,(\lambda\mu \mid 1\nu\, 1\mu{-}\nu)\,(l'm'\, 1\nu \mid l''m'')$$
$$\times\, (l''m''\, 1\ \mu{-}\nu \mid lm)\,\mathcal{Y}^{[l]}{}_m(\hat{\mathbf{a}})\,\mathcal{Y}^{(l')}{}_{m'}(\hat{\mathbf{b}}).$$

This expression, of course, occurs in the expansion of (A38) in which all indices except 1, λ, and μ are to be summed over. Therefore, once again a sum over the indices m'' and ν is indicated which gives, as in (A47),

$$ab \sum \hat{\lambda}\hat{l}'\,\bar{\sigma}(l''1l')\,\bar{\sigma}(l''1l)\,\overline{W}\begin{pmatrix}1 & 1 & \lambda\\ l' & l & l''\end{pmatrix}(l'm'\,\lambda\mu \mid lm)\,\mathcal{Y}^{[l]}{}_m(\hat{\mathbf{a}})\,\mathcal{Y}^{(l')}{}_{m'}(\hat{\mathbf{b}}). \qquad \text{(A48)}$$

By using specific values or the orthogonality property of the Wigner coefficients the sum in the third term of (A38) can be shown to reduce as follows:

$$\sum_{\nu} (\lambda 0 \mid 1\nu\, 1{-}\nu)\,(-)^{1-\nu} = \sqrt{3}\,\delta_{\lambda 0}. \qquad \text{(A49)}$$

In equations (A47), (A48), and (A49) we have reduced the tensor parts of the expansion to the desired form; with the help of (A30) the products of $\bar{\chi}^{[\nu]}(\mathbf{a})\,\bar{\chi}^{(\nu')}(\mathbf{b})$ can be formed from these tensors and the coefficients collected from (A38) to give

$$\sqrt{2}\,(\nu l \,||\, W^{[\lambda]} \,||\, \nu' l')$$
$$= \sqrt{3}\,\delta_{\lambda 0}\,\delta_{\nu\nu'}\,\delta_{ll'}\,\delta_{mm'}$$
$$+ (-)^{\nu'+\nu} 2\Bigg[\sum_{l''} \hat{\lambda}\hat{l}''\,\bar{\sigma}(l''1l)\,\bar{\sigma}(l'1l'')\,\overline{W}\begin{pmatrix}1 & 1 & \lambda\\ l' & l & l''\end{pmatrix}$$
$$\times \left\{\delta_{p'+2,p}\,\frac{\bar{N}_{\nu'l'}}{\bar{N}_{\nu l}} + \delta_{p',p}\,\frac{\bar{N}^2_{\nu''l''}}{\bar{N}_{\nu l}\bar{N}_{\nu'l'}}\right\}\Bigg], \qquad \text{(A50)}$$

where

$$2\nu''+l''+1 = 2\nu+l \quad \text{or} \quad \nu'' = \tfrac{1}{2}(2\nu+l-l''-1). \qquad \text{(A51)}$$

The sum is over the values of l'' allowed by the functions in the summand. The condition (A51) arises from comparing the powers of a; ν'' would always be an integer because $l+l''+1$ is required to be even, in view of the properties of $\bar{\sigma}$.

In evaluating these formulae it is to be noted that the restrictions on p and p' imposed by the δ-function are to be taken into account first, then the restrictions on the l's imposed by the $\bar{\sigma}$-functions, and finally those by the $\overline{W}$-functions wherever they occur. No contribution arises when the values of the indices p, ν, l, etc. become negative. We recall that the symbols $\bar{\sigma}$ and $\bar{N}$ were defined in equations (A42) and (A9). For the $\overline{W}$-function and specific values of $\bar{\sigma}$ reference has to be made to books on the quantum mechanical angular momentum theory (Rose 1957; Fano and Racah 1959; Brink and Satchler 1962). Our choice of phases of $\mathfrak{Y}^{[l]}{}_m$ is different from the one used in most books (Condon and Shortley 1953; Edmonds 1957; Rose 1957; Brink and Satchler 1962) but leads to more uniform signs in the formulae according to the recommendation of Fano and Racah (1959).

These reduced integrals are not symmetrical in the indices νl and $\nu' l'$ because the operators involve differentiations. In deriving the hydrodynamic equations the reduced integrals corresponding to $(\nu l) = (00)$, (01), and (10) are needed for arbitrary $(\nu' l')$; and in the second approximation of the Chapman–Enskog procedure one needs the reduced integrals for $(\nu' l') = (00)$ and arbitrary (νl). A list of integrals for these special cases, as obtained from equations (A43), (A44), (A45) and (A50), (A51) is given below. Mr. S. C. Gupta, of our Department, has derived these formulae in a different form by the more usual methods of nuclear theory. I am thankful to him for checking the present calculations.

$$(00 \,||\, K^{[1]} \,||\, \nu' l') \equiv 0 \,; \tag{A52}$$

$$(01 \,||\, K^{[1]} \,||\, \nu' l') = \delta_{\nu' 0}\, \delta_{l' 0} \,; \tag{A53}$$

$$(10 \,||\, K^{[1]} \,||\, \nu' l') = \sqrt{2}\, \delta_{\nu' 0}\, \delta_{l' 1} \,; \tag{A54}$$

$$(\nu l \,||\, K^{[1]} \,||\, 00) = \delta_{\nu 0}\, \delta_{l 1} \,; \tag{A55}$$

$$(00 \,||\, \alpha C^{[1]} \,||\, \nu' l') = \sqrt{3}\, \delta_{\nu' 0}\, \delta_{l' 1} \,; \tag{A56}$$

$$(01 \,||\, \alpha C^{[1]} \,||\, \nu' l') = \delta_{\nu' 0}\, \delta_{l' 0} - \sqrt{\tfrac{2}{3}}\, \delta_{\nu' 1}\, \delta_{l' 0} + \sqrt{\tfrac{10}{3}}\, \delta_{\nu' 0}\, \delta_{l' 2} \,; \tag{A57}$$

$$(10 \,||\, \alpha C^{[1]} \,||\, \nu' l') = \sqrt{2}\, \delta_{\nu' 0}\, \delta_{l' 1} + \sqrt{5}\, \delta_{\nu' 1}\, \delta_{l' 1} \,; \tag{A58}$$

$$(\nu l \,||\, \alpha C^{[1]} \,||\, 00) = \delta_{\nu 0}\, \delta_{l 1} \,; \tag{A59}$$

$$(00 \,||\, \alpha C^{[1]}\, W^{[0]} \,||\, \nu' l') = 4\, \delta_{\nu' 0}\, \delta_{l' 1} \,; \tag{A60}$$

$$(01 \,||\, \alpha C^{[1]}\, W^{[0]} \,||\, \nu' l') = \sqrt{\tfrac{25}{3}}\, \delta_{\nu' 0}\, \delta_{l' 0} - \tfrac{5}{3}\sqrt{2}\, \delta_{\nu' 1}\, \delta_{l' 0} + \tfrac{5}{3}\sqrt{10}\, \delta_{\nu' 0}\, \delta_{l' 2} \,; \tag{A61}$$

$$(10 \,||\, \alpha C^{[1]}\, W^{[0]} \,||\, \nu' l') = -3\sqrt{6}\, \delta_{\nu' 0}\, \delta_{l' 1} + 2\sqrt{15}\, \delta_{\nu' 1}\, \delta_{l' 1} \,; \tag{A62}$$

$$(\nu l \,||\, \alpha C^{[1]}\, W^{[0]} \,||\, 00) = \sqrt{\tfrac{25}{3}}\, \delta_{\nu 0}\, \delta_{l 1} - \sqrt{\tfrac{10}{3}}\, \delta_{\nu 1}\, \delta_{l 1} \,; \tag{A63}$$

$$(00 \,||\, W^{[\lambda]} \,||\, \nu' l') = \sqrt{3}\, \delta_{\nu' 0}\, \delta_{l' 0}\, \delta_{\lambda 0} \,; \tag{A64}$$

$$(01 \,||\, W^{[\lambda]} \,||\, \nu' l') = \sqrt{3}\, \delta_{\nu' 0}\, \delta_{l' 1}\, \delta_{\lambda 0} + (-)^{\lambda} \{\tfrac{1}{3}(2\lambda+1)\}^{\frac{1}{2}}\, \delta_{\nu' 0}\, \delta_{l' 1} \,; \tag{A65}$$

$$(10 \,||\, W^{[\lambda]} \,||\, \nu' l') = \sqrt{\tfrac{25}{3}}\, \delta_{\nu' 1}\, \delta_{l' 0}\, \delta_{\lambda 0} + \sqrt{2}\, \delta_{\nu' 0}\, \delta_{l' 0}\, \delta_{\lambda 0} - 2\sqrt{\tfrac{5}{3}}\, \delta_{\nu' 0}\, \delta_{l' 2}\, \delta_{\lambda 2} \,; \tag{A66}$$

$$(\nu l \,||\, W^{[\lambda]} \,||\, 00) = \sqrt{3}\, \delta_{\nu 0}\, \delta_{l 0}\, \delta_{\lambda 0} - \sqrt{2}\, \delta_{\nu 1}\, \delta_{l 0}\, \delta_{\lambda 0} + \sqrt{2}\, \delta_{\nu 0}\, \delta_{l 2}\, \delta_{\lambda 2} \,. \tag{A67}$$