THE RAY THEORY OF DOPPLER FREQUENCY SHIFTS

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Summary

A general expression is obtained according to ray theory for the Doppler frequency shift observed when both receiver and transmitter are in motion in a time-varying anisotropic but lossless medium. The result is obtained using Hamilton's optical method. The range of application of the resulting expression and its relationship to a number of published results are considered. The method of analysis is also applied to the calculation of the group path of a radio wave.

I. INTRODUCTION

One method of probing the ionosphere involves transmitting a sinusoidal radio signal from a vehicle (satellite or rocket) moving through the ionosphere and measuring the Doppler frequency shift of the received signal at a ground-based station. The results of such experiments have been analysed using ray theory.

Expressions for the Doppler shift in terms of the vehicle motion and the ionospheric structure have been obtained using various approximations. These may be classified as:

(i) straight ray paths, i.e. neglecting refraction;

(ii) isotropic ray theory; and

(iii) quasi-isotropic ray theory.

By (ii) we mean the ray theory appropriate to propagation in an isotropic inhomogeneous medium. The meaning of (iii) is discussed in Section X.

The corresponding general expression for the Doppler shift observed when both transmitter and receiver are moving in a time-dependent inhomogeneous anisotropic lossless medium will be obtained here by a method based on Fermat's principle and Hamilton's optical method.†

While we are particularly concerned with radio propagation in the ionosphere, the results apply to any wave propagation obeying Fermat's principle.

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† Since the present paper was prepared, two other papers (Kimura, Nishina, and Maeda 1967; Maeda and Kimura 1967) have come to the author's attention. In these, the major result of the present paper, equation (13), is obtained by an extension of Kelso's method. The term due to time variation of the medium is, however, not explicitly evaluated.

II. Doppler Shift

We assume that the Doppler frequency shift Δf is given by

$$\Delta f = -(f/c)\mathrm{d}P/\mathrm{d}t\,, \tag{1}$$

(e.g. Weekes 1958; Davies 1965). f is the operating frequency and P is regarded as a function of time t. The phase path P of a ray between points A and B is given by

$$P = \int_{\mathrm{A}}^{\mathrm{B}} \mu \cos \alpha \; \mathrm{d}s\,, \tag{2}$$

where μ is the refractive index at a point on the ray path, α is the angle between the wave normal and the ray direction, and the integration is carried out along the ray path. The minus sign on the right-hand side of (1) should be noted. It has been omitted by some authors. Increasing phase path corresponds to decreasing frequency.

III. Conditions

In this section we regard P as a function of both f and t and introduce a partial derivative notation. Equation (1) leads to useful results in connection with ionospheric radio propagation. However, it is not a simple matter to justify its use rigorously; the following assumptions are certainly necessary.

(i) The medium is slowly varying in space and lossless so that ray theory could be applied if the transmitter and receiver were fixed and the medium time stationary. Also we must have

$$fP/c > 1\,, \tag{3}$$

i.e. the phase path must be greater than several wavelengths.

(ii) The velocities of transmitter and receiver relative to the medium must be small in comparison with c.

(iii) The phase path does not change greatly during one cycle of the transmitted frequency, i.e.

$$\left|\frac{1}{fP}\frac{\partial P}{\partial t}\right| \ll 1\,. \tag{4}$$

This is necessary if the changes taking place at the signal frequency are to be treated separately from the much slower changes in the phase path.

(iv) In order that the undisplaced frequency may be used as an argument of P we must have

$$\left|\frac{f}{cP}\frac{\partial P}{\partial t}\frac{\partial P}{\partial f}\right| \ll 1\,. \tag{5}$$

Conditions (4) and (5) are related to the "pointwise" conditions (30.27) and (30.28) of Ginzburg (1964).

It is found in Section V that $\partial P/\partial t$ can be divided into two parts. Conditions (4) and (5) must be interpreted as applying separately to both.

Combining (3), (4), and (5) leads to

$$\left|\frac{1}{cP}\frac{\partial P}{\partial t}P'\right| \ll 1, \tag{6}$$

where $P' = \partial(Pf)/\partial f$ is the group path and the time of flight of a pulse of centre frequency f over the path is P'/c. The physical meaning of (6) is that the phase path must not change much during the time of flight of a pulse or element of the wave. Thus this important condition is implied by the preceding conditions. Also combining (1) with (3) and (4) leads to

$$|\Delta f/f| \ll 1,$$

i.e. the Doppler shift must be small.

Despite this fairly extensive list of conditions it is likely that they are not *sufficient* for two distinct reasons:

(i) because integral conditions have been given that are less restrictive than pointwise conditions to be satisfied at each point on the ray, and

(ii) because they are what might be called first-order conditions involving only first derivatives.

Higher order conditions will be necessary, for instance, the group path must not change greatly during the time of flight of a pulse, a condition involving $\partial^2 P/(\partial f\, \partial t)$.

Difficulties of type (i) occur at ionospheric reflection points in the vicinity of which the W.K.B. solutions of the wave equation (upon which ray theory may be regarded as being based) break down. However, such points do not lead to serious error in the calculated phase and direction of energy flow, provided that the points A and B are sufficiently remote from reflection points. For points at which $\mu = 0$ (these are reflection points for rays propagating with wave normals perpendicular to surfaces of constant refractive index in stratified media), the pointwise conditions corresponding to (4), (5), and (6) will be violated because of the occurrence of a μ^{-1} factor on the left-hand side.

Finally, we note that an observer in a laboratory frame, with respect to which the medium is in motion with a velocity v, will observe a refractive index μ given by

$$\mu_1 = \mu - (\mu^2 - 1)(v/c)\cos\Theta + O(v^2/c^2)$$

for $v/c \ll 1$. μ_1 is the refractive index in a frame at rest with respect to the medium and Θ is the angle between the wave normal and the velocity (Papas 1965).

We have neglected effects of this order. If μ_1 is small when the correction is large, we have another example where the assumptions apparently break down at an isolated point. As is found in the application of ray theory, it seems likely that isolated points of this type will not lead to serious error provided that they are not too near the transmitter or receiver.

IV. Fermat's Principle and Hamilton's Optical Method

It is convenient for our purposes to write P in the form

$$P = \int_{\mathrm{A}}^{\mathrm{B}} p_i \, \mathrm{d}q^i \,,$$

where $\mathbf{p} = (p_1, p_2, p_3)$ is a vector in the direction of the wave normal of magnitude μ, space coordinates are written $\mathbf{q} = (q_1, q_2, q_3)$, and a repeated index implies a summation over all possible values of the index. For the sake of generality in the choice of reference axes, tensor notation has been used. Note that the p_i are covariant components and $\mathrm{d}q^i$ contravariant components so that $p_i \, \mathrm{d}q^i$ is invariant (e.g. McConnell 1957). If u is an arbitrary parameter increasing monotonically along the ray from transmitter to receiver then

$$P = \int_{u_1}^{u_2} p_i(u) \, q'^i(u) \, \mathrm{d}u \,,$$

where $u = u_1$ at A and u_2 at B, and

$$q'^i = \mathrm{d}q^i/\mathrm{d}u \,.$$

We introduce the notation

$$P = \int_{u_1}^{u_2} F \, \mathrm{d}u \,,$$

where

$$F = p_i \, q'^i \,.$$

Fermat's principle may be stated, in the notation of the calculus of variations,

$$\delta \int_{u_1}^{u_2} F(q, q', \chi) \, \mathrm{d}u = 0 \,, \tag{7}$$

i.e. the first variation of the phase path with respect to variations of the path is zero. The end points are regarded as fixed and the parameter χ, introduced by Hamilton to represent the colour, is regarded as fixed since each ray must be characterized by a single colour.

It should be noted that in this form Fermat's principle may yield a true minimum or a type of saddle point (minimum for one class of variations and maximum for another class) but never a true maximum in the sense of the calculus of variations (e.g. Born and Wolf 1965).

What will be called Hamilton's optical method here may be stated as

$$\delta \int_{u_1}^{u_2} p_i \, q'^i \, \mathrm{d}u = 0 \,, \tag{8a}$$

$$\Omega(\mathbf{p}, \mathbf{q}, \chi) = 0 \,, \tag{8b}$$

where $\mathbf{p}$ is arbitrary except that $\Omega = 0$ is satisfied, and again the end points of the ray and χ are regarded as fixed (Hamilton 1837, 1931; Synge 1954). Ω is written in the form

$$\Omega = (p_i \, p^i)^{\frac{1}{2}} \mu^{-1} - 1$$

and μ is regarded as a function of the variables indicated as

$$\mu = \mu(\mathbf{p}(p_i p^i)^{-\frac{1}{2}}, \mathbf{q}, \chi).$$

This form of Ω corresponds to that used by Hamilton, and not the more general form of Synge. While the equation $\Omega = 0$ can be regarded as the Hamilton–Jacobi equation for P, it is also recognizable as the equation to the refractive index surface. We take (8) as our starting point since the equation $\Omega = 0$ is given by magnetoionic theory, and since, following the work of Haselgrove (1955), ionospheric ray tracing is usually based on this approach. Equations (8) lead to the equations to the ray in the form (see Appendix)

$$q'^i = F\,\partial\Omega/\partial p_i, \tag{9a}$$

$$p'_i = -F\,\partial\Omega/\partial q^i, \tag{9b}$$

where

$$p'_i = \mathrm{d}p_i/\mathrm{d}u.$$

The seven equations (8b), (9a), and (9b) may be solved for the seven unknowns, F and the components of $\mathbf{p}$ and $\mathbf{q}$.

V. Differentiation

Consider the dependence of P on t. If the end points of the ray are changing with time then the path in space of the ray must change with time. Also, if the properties of the medium change with time, this can give rise to a change in the ray path. If we consider a particular point on the ray path at one instant then not only may the refractive index calculated in a given wave-normal direction change, but the direction of the wave normal may change with time. All these factors may contribute to the way P changes with time. Although the possible contributions to the time dependence of P are very complicated, a careful consideration of the various terms will show that most of them make no contribution to the time derivative of P.

Since each ray we are considering may be characterized by a single value of the variable t, the parameter χ that was introduced first in (7) as a colour index is now identified with t. Along a ray

$$P = \int_{u_1}^{u_2} p_i q'^i \,\mathrm{d}u.$$

Since $\Omega = 0$ at all points on a ray

$$P = \int_{u_1}^{u_2} \{p_i q'^i + \theta\Omega(\mathbf{p}, \mathbf{q}, t)\}\,\mathrm{d}u, \tag{10}$$

where θ is a function of u that is undetermined at this stage. Now in (10) $\mathbf{p}$, $\mathbf{q}$, $\mathbf{q}'$, as well as Ω are functions of t, and we allow θ also to depend on t. For clarity this dependence, and the dependence of $\mathbf{p}$, $\mathbf{q}$, $\mathbf{q}'$, and θ on u, is not shown explicitly.

The parameter u is now specialized so that it is independent of t and

$$\mathrm{d}u_1/\mathrm{d}t = \mathrm{d}u_2/\mathrm{d}t = 0.$$

If the trajectories of the end points of the ray lie in surfaces of constant u this condition will be satisfied for all t (see Fig. 1). With this choice of u we can differentiate (10) regarding $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{q}'$ as independent functions of t. Then

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \int_{u_1}^{u_2} \frac{\mathrm{d}}{\mathrm{d}t}\Big(p_i q'^i + \theta\Omega(\mathbf{p},\mathbf{q},t)\Big)\,\mathrm{d}u,$$

since u is independent of t. Using the chain rule of partial differentiation leads to

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \int_{u_1}^{u_2} \left\{q'^i\frac{\partial p_i}{\partial t} + p_i\frac{\partial q'^i}{\partial t} + \theta\left(\frac{\partial\Omega}{\partial p_i}\frac{\partial p_i}{\partial t} + \frac{\partial\Omega}{\partial q^i}\frac{\partial q^i}{\partial t} + \frac{\partial\Omega}{\partial t}\right)\right\}\mathrm{d}u,$$

since it follows from the fact that $\Omega = 0$ along a ray that

$$\int_{u_1}^{u_2} (\partial\theta/\partial t)\Omega\,\mathrm{d}u = 0.$$

Integrating $\int p_i(\partial q'^i/\partial t)\,\mathrm{d}u$ by parts and rearranging, yields

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \int_{u_1}^{u_2} \left\{\left(q'^i + \theta\frac{\partial\Omega}{\partial p_i}\right)\frac{\partial p_i}{\partial t} + \left(\theta\frac{\partial\Omega}{\partial q^i} - p_i'\right)\frac{\partial q^i}{\partial t} + \theta\frac{\partial\Omega}{\partial t}\right\}\mathrm{d}u + \left[p_i\frac{\partial q^i}{\partial t}\right]_{u_1}^{u_2}.$$

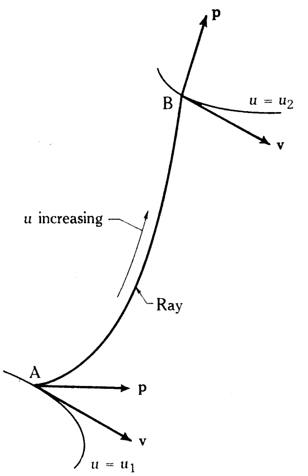

Fig. 1.—Parametric representation of ray path for calculation of Doppler shift.

Now, as can be seen from the Appendix, by taking as the variations of $\mathbf{p}$ and $\mathbf{q}$ the particular variations $\partial\mathbf{p}/\partial t$ and $\partial\mathbf{q}/\partial t$, θ can be chosen so that the first two terms of the integrand (which actually expand to the sum of 12 factors when written in full) make zero contributions to the integral along a ray. We therefore choose θ to satisfy equation (A6), that is, $\theta = -F$. Note that for this choice of θ the contribution from the first two terms is obviously zero because of (9) or (A7). Evaluating $\partial\Omega/\partial t$ leads to the result

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \int_{u_1}^{u_2} \frac{F}{\mu}\frac{\partial\mu}{\partial t}\,\mathrm{d}u + \left[p_i\frac{\partial q^i}{\partial t}\right]_{u_1}^{u_2}, \tag{11}$$

This may be rewritten in what may be more familiar notation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \int_{\mathrm{A}}^{\mathrm{B}} \frac{\partial\mu}{\partial t}\cos\alpha\,\mathrm{d}s + \Big[\mathbf{p}.\mathbf{v}\Big]_{\mathrm{A}}^{\mathrm{B}}, \tag{12}$$

where $\mathbf{v}$ represents the velocity of transmitter and receiver in turn.

VI. Importance of Introducing the Parameter u

A certain amount of confusion has arisen in the past through attempting to handle calculations related to that of Section V by using an integration variable that is a function of t (or whatever the corresponding variable may be). This confusion is avoided here and the calculation simplified since (i) the time dependence of the end points of the ray need not be separately considered, (ii) the dependence of $\mathbf{q}$, $\mathbf{q}'$, and $\mathbf{p}$ on u need not be explicitly considered, and (iii) all space variables are afforded equal treatment. This simplification should not be unexpected since it is well known that the treatment of geometrical problems in the calculus of variations, especially problems involving end points that are partly variable, is much simplified by the parametric approach, a method that is due to Weierstrass (see, for example, Forsyth 1927).

VII. Expressions for the Doppler Shift

Substitution from (12) into (1) yields

$$\Delta f = -\frac{f}{c}\left(\int_{\mathrm{A}}^{\mathrm{B}} \frac{\partial \mu}{\partial t}\cos\alpha\,\mathrm{d}s + \Big[\mathbf{p}.\mathbf{v}\Big]_{\mathrm{A}}^{\mathrm{B}}\right), \tag{13}$$

the major result of this paper. For a time-stationary medium, with moving transmitter and receiver,

$$\Delta f = -\frac{f}{c}\Big[\mathbf{p}.\mathbf{v}\Big]_{\mathrm{A}}^{\mathrm{B}}, \tag{14}$$

while, if the medium is isotropic, (13) reduces to

$$\Delta f = -\frac{f}{c}\left(\int_{\mathrm{A}}^{\mathrm{B}} \frac{\partial \mu}{\partial t}\,\mathrm{d}s + \Big[\mathbf{p}.\mathbf{v}\Big]_{\mathrm{A}}^{\mathrm{B}}\right), \tag{15}$$

where the vector $\mathbf{p}$ now lies in the ray direction since the wave-normal and ray directions correspond at all points in an isotropic medium. If transmitter and receiver are fixed, (13) gives

$$\Delta f = -\frac{f}{c}\int_{\mathrm{A}}^{\mathrm{B}} \frac{\partial \mu}{\partial t}\cos\alpha\,\mathrm{d}s. \tag{16}$$

VIII. Physical Significance of the Result (13)

From (13) it can be seen that, in this approximation, the Doppler shift can be split into two components. The first is due to time changes in refractive index of the medium along the ray, which is regarded as fixed in space. The change in refractive index at each point of the ray is calculated for a constant wave-normal direction. This is the Doppler shift that would be observed if transmitter and receiver were stationary.

The second contribution is due to movements of the end points of the ray. We notice that the Doppler shift is completely specified by the velocities of the end points of the ray and the value of $\mathbf{p}$ at the ends of the ray. This is the Doppler shift that would be observed in a time-stationary medium.

The fact that it is the wave-normal direction, and not the ray direction, that is important in determining the Doppler shift means that in an anisotropic time-stationary medium a Doppler shift may be observed if the ray and velocity vectors are orthogonal. This should be contrasted with the case of an isotropic medium when a transverse Doppler shift is only possible in the relativistic case.

IX. Comparison with Other Results

The $-(\mathbf{p}.\mathbf{v})f/c$ terms occurring in (13), (14), and (15) are the ray-theory equivalent of the general form $-\nabla\Phi.\mathbf{v}$, where Φ is the phase of the wave variable (Tisher 1959; Lee and Papas 1963), since according to ray theory $\nabla\Phi = \mathbf{p}f/c$.

For a transmitter moving with velocity $\mathbf{v}$ in a homogeneous anisotropic medium the Doppler shifted frequency f, for an observer stationary with respect to the time-invariant medium, satisfies

$$(f/c)\mathbf{p}.\mathbf{v} = f - f_0, \tag{17}$$

where $\mathbf{p} = \mathbf{p}(f)$, $f_0 = f_0^*(1-|v/c|^2)^{\frac{1}{2}}$, and f_0^* is the proper frequency of the transmitter (e.g. Barsukov 1959, 1962). Under the assumptions of Section III it can be seen that (17) reduces to the result given by (13) or (14) with the point B regarded as fixed. It should be noted that the assumptions of Section III preclude the occurrence of Čerenkov radiation or the complex Doppler effect (Frank 1960; Barsukov 1962), which are results that arise from (17).

Returning to the consideration of an inhomogeneous medium, expression (15) for the isotropic case has been obtained by Kelso (1960, 1964) by making use of Fermat's principle. Equation (16) has been obtained by a technique based on Hamilton's principle (Bennett 1967*a*). However, the handling of the function Ω in that paper is not very satisfactory.

X. Quasi-isotropic Expression for the Doppler Shift

Kelso (1960) obtained (15) as a special case of a result obtained using a quasi-isotropic ray theory. His result, stated for a moving transmitter only, is

$$\Delta f = -\frac{f}{c}\left[\mu_v v_{11} + \int \frac{\partial\mu}{\partial t}\,\mathrm{d}s + \left\{\left(\frac{\partial\mu}{\partial x'}\frac{\partial x}{\partial t} + \frac{\partial\mu}{\partial y'}\frac{\partial y}{\partial t}\right)\left(1+(x')^2+(y')^2\right)^{\frac{1}{2}}\right\}_{z=\zeta}\right], \tag{18}$$

where v_{11} is the component of the transmitter velocity in the (negative) ray direction, $x' = \partial x/\partial z$ and $y' = \partial y/\partial z$ refer to the ray, and μ_v is the refractive index at the transmitter, calculated in the ray direction. The term in braces is to be evaluated at the transmitter end of the ray, subject to the condition $z = \zeta$, so that $\partial x/\partial t$ and $\partial y/\partial t$ are the components of the velocity of the point on the ray having a z component ζ.

In general they are not components of the transmitter velocity. A minus sign has been inserted for consistency with (1). Also Kelso has regarded the ray as being traversed from the receiver to the transmitter. Thus his ray direction is, in the present notation, the negative ray direction.

It is interesting to compare the complicated analytical form of (18), derived using the calculus of variations in its nonparametric form, with the much simpler equation (13).

The assumptions underlying the derivation of (18) are the following. The refractive index is calculated as if the wave-normal and ray directions coincided and the expression for the phase path appropriate to isotropic propagation is used, but the refractive index is allowed to depend upon direction. The path of the ray is determined by applying Fermat's principle.

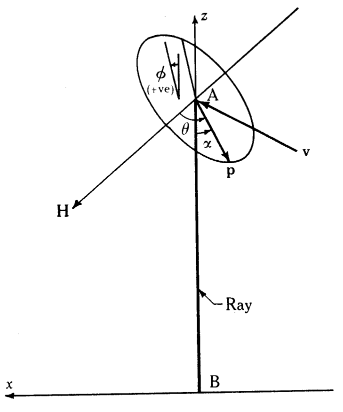

Fig. 2.—An example used in the comparison of the anisotropic and quasi-isotropic expressions. The refractive index surface is shown superimposed at one end of the ray.

It is of considerable interest to determine if, and under what circumstances, (18) is an approximation to (13). The assumptions would be expected to be most nearly valid when the refractive index calculated for a wave normal in the supposed ray direction is approximately equal to the refractive index in the corresponding true wave-normal direction. In this case the angle α will be small and the integral terms of (13) and (18) will be approximately equal. It is difficult to compare the terms quantitatively since slightly different ray paths will be calculated in the two cases unless the medium should be homogeneous.

In order to focus attention on the remaining terms of (13) and (18) consider the simple example illustrated in Figure 2. A sinusoidal signal of frequency f is transmitted from a transmitter moving with velocity $\mathbf{v}$ through a homogeneous anisotropic plasma, the properties of which do not change with time. Axes are chosen so that the observer is at the origin and the ray lies along the $(0,z)$ axis. Since a magnetoionic medium is assumed if the intensity vector of the applied magnetic field $\mathbf{H}$ (or in the general case an axis of rotational symmetry of the refractive index surface) is chosen to lie in the (x,z) plane, $\mathbf{p}$ also lies in that plane (e.g. Budden 1961).

Consider first Kelso's expression (18). At the transmitter

$$x' = \partial x/\partial z = 0\,, \qquad y' = \partial y/\partial z = 0\,,$$

$$(\partial x/\partial t)_z = v_x\,, \qquad (\partial y/\partial t)_z = v_y\,, \qquad v_{11} = v_z\,.$$

If the medium is approximately isotropic

$$\partial\mu/\partial x' \simeq \partial\mu/\partial\phi\,, \qquad \partial\mu/\partial y' = 0\,,$$

where ϕ is defined by $\partial x/\partial z = \tan\phi$ and increases in the sense indicated in Figure 2. Therefore we have approximately

$$\Delta f = -(f/c)\{\mu v_z + (\partial\mu/\partial\phi)v_x + 0v_y\}\,.$$

The v_y term is included to emphasize that a y component of velocity makes no contribution to the Doppler shift.

Now from (13) we obtain

$$\Delta f = -(f/c)\{(p\cos\alpha)v_z + (p\sin\alpha)v_x + 0v_y\}\,, \qquad \tan\alpha = p^{-1}\partial p/\partial\psi\,,$$

where ψ is the angle between the wave normal and **H** (e.g. Budden 1961) and angles are positive when measured in a clockwise sense from **H**. The refractive index is written p to emphasize that it is calculated in the wave-normal direction. Since **H** remains constant in direction we have

$$\partial p/\partial\phi = \partial p/\partial\psi \qquad \text{and} \qquad \partial p/\partial\phi = p\tan\alpha\,.$$

Now, making use of the facts that α is small and $p \simeq \mu$, we find that (13) leads to

$$\Delta f = -(f/c)\{\mu v_z + (\partial\mu/\partial\phi)v_x + 0v_y\}\,,$$

showing that in this case at least (18) is an approximation to (13). In a more general case either (18) could be rewritten in terms of the arc length s as parameter, or the z axis could be chosen to lie in the calculated direction of the ray at its end point. Thus it can be seen that, in general, (18) is an approximation to (13) which is valid when the anisotropy is small, provided the quasi-isotropic ray theory leads to a sufficiently accurate estimate of the final ray direction.

XI. Application to Calculation of Group Path

The main step in the preceding derivation was essentially a differentiation of P with respect to χ. The expression for the derivative therefore applies to the differentiation of P with respect to any parameter that can be formally identified with χ. For instance χ may be identified with the frequency f, which is perhaps more natural than the identification with t made in Section V.

The expression

$$P' = \mathrm{d}(Pf)/\mathrm{d}f \tag{19}$$

arises naturally if the group path of a pulse is defined in terms of the time of flight of a pulse along the ray. Equation (19) leads to

$$P' = \int_{\mathrm{A}}^{\mathrm{B}} \mu' \cos a \, \mathrm{d}s \tag{20}$$

for a slowly varying lossless medium (Gautier 1966; Bennett 1967*b*). The pulse travels between points A and B, which are regarded as fixed.

Titheridge (1965) has pointed out that if a ray is regarded as being divided into a number of segments by intersecting surfaces then the group path for each section is not obtained by applying (19) to the expression for $P(f)$ obtained for each segment since the end point of the segments are, in general, functions of frequency. To investigate under what circumstances (19) can be used in this case we make use of (11) and find

$$\frac{\mathrm{d}(Pf)}{\mathrm{d}f} = \int_{u_1}^{u_2} \mu' \cos a \, \mathrm{d}s + \left[p_i \frac{\partial q^i}{\partial f} f\right]_{u_1}^{u_2}, \tag{21}$$

where P is the phase path of the section of the path lying between surfaces defined by $u = u_1$ and $u = u_2$. $\mu' = \partial(\mu f)/\partial f$ is known as the group refractive index. However, it should be noted that the group velocity is of magnitude $c/(\mu' \cos a)$ and lies in the ray and not the wave-normal direction. The integral term on the right-hand side of (21) arises when the end points of the segment are held constant during the differentiation and can be recognized as the group path of the segment of the ray (see equation (20)). Thus, formally (19) can be applied to the calculation of the group path if $\mathbf{p}$ and $\partial \mathbf{q}/\partial f$ are orthogonal at both ends of the segment, i.e. if the terminating surfaces are tangential to surfaces of *constant phase*, a condition making excellent sense physically.

An example where this condition applies is the case of vertical incidence on a spherically stratified ionosphere when the ray is divided by a number of concentric spheres, a procedure often adopted in analysing ionograms. The phase path of a section of the ray lying between two consecutive spheres is frequency dependent. Although the points of intersection are also frequency dependent, because the wave normal remains vertical and orthogonal to the intersecting surfaces, (19) may be applied directly to the calculation of the group path of each segment of the ray.

XII. Conclusions

An expression (13) has been found according to ray theory for the Doppler shift which is present when both source and transmitter are moving in a time-varying medium. This expression should help to remove the confusion that arises in considering the various contributions to the Doppler shift. The Doppler shift can be split into two parts, one due to movement of transmitter and receiver and the other due to changes in the refractive index taking place along the ray, the ray being regarded as fixed in space, with the change in refractive index at each point being calculated for a wave-normal direction that is regarded as unchanging.

The result may be of practical application since Kelso (1960) has suggested that the isotropic formula (15) be used for analysis of satellite Doppler experiments, while (16) has been successfully used in computing theoretical Doppler shifts for a path with fixed transmitter and receiver (Jones 1968).

The method can also be applied to the calculation of the group path.

Note added in proof. Because all derivatives have been implicitly assumed to exist and be continuous the argument of this paper cannot be applied directly to a ray having corners (where $\mathbf{q}'$ in particular is discontinuous). For a ray with corners it is natural to apply (13) to the segments lying between corners and sum the several results to obtain the total Doppler shift.

At a corner lying in a continuous medium $\mathbf{p}$ is continuous (Weierstrass–Erdmann corner condition, e.g. Forsyth 1927) and there is no contribution associated with the corner. On the other hand, at a corner lying on a surface of discontinuity of refractive index $\mathbf{p}$ is discontinuous ($\mathbf{p}$ satisfies the law of reflection or refraction). There is thus, in general, a contribution to the Doppler shift associated with such a corner.

XIII. Acknowledgments

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APPENDIX

A Derivation of the Hamiltonian Equation to the Ray by a Lagrangean Method

The following method is an interesting way of obtaining the Hamiltonian equations to the ray. However, it is presented here because it parallels the differentiation of P in Section V, and because it is convenient in considering ionospheric radio propagation to start from equation (8) rather than (7), since the equation $\Omega = 0$ is given by magnetoionic theory. We wish to solve

$$\delta \int_{u_1}^{u_2} p_i q'^i \, \mathrm{d}u = 0, \qquad \Omega(\mathbf{p}, \mathbf{q}, \chi) = 0, \tag{A1}$$

where

$$\Omega = (p_i p^i)^{\frac{1}{2}} \mu^{-1} - 1 \qquad \text{and} \qquad \mu = \mu(\mathbf{p}(p_i p^i)^{-\frac{1}{2}}, \mathbf{q}, \chi).$$

The end points and χ are regarded as fixed. By the method of Lagrange, we seek a solution of

$$\delta \int_{u_1}^{u_2} \{p_i q'^i + \theta \Omega(\mathbf{p}, \mathbf{q}, \chi)\} \, \mathrm{d}u = 0, \tag{A2}$$

i.e.

$$\delta J = 0,$$

where θ is a function of u to be determined, $\mathbf{p}$ and $\mathbf{q}$ may now be chosen freely, and J is written for the integral. If $\mathbf{p}$ is varied by $\delta\mathbf{p}$, $\mathbf{q}$ by $\delta\mathbf{q}$, and $\mathbf{q}'$ by $\delta\mathbf{q}'$ then

$$\delta J = \int_{u_1}^{u_2} \{q'^i \delta p_i + p_i \delta q'^i + \theta(\partial\Omega/\partial p_i)\delta p_i + \theta(\partial\Omega/\partial q^i)\delta q^i\} \, \mathrm{d}u = 0. \tag{A3}$$

Integrating $\int p_i \delta q'^i \, \mathrm{d}u$ by parts and rearranging yields

$$\delta J = \int_{u_1}^{u_2} [\{q'^i + \theta(\partial\Omega/\partial p_i)\}\delta p_i + \{\theta(\partial\Omega/\partial q_i) - p'^i\}\delta q^i] \, \mathrm{d}u. \tag{A4}$$

Now $\delta\mathbf{p}$ and $\delta\mathbf{q}$ are arbitrary except for mild restrictions on their smoothness. Therefore we must have

$$q'^i = -\theta\,\partial\Omega/\partial p_i, \tag{A5a}$$

$$p'_i = \theta\,\partial\Omega/\partial q^i \tag{A5b}$$

(see Synge 1954).

The six equations (A5), together with $\Omega = 0$, determine the seven unknowns, the components of $\mathbf{p}$ and $\mathbf{q}$ and the function θ. θ can be determined as follows. From (A5a) we find

$$p_i q'^i = -\theta p_i \partial\Omega/\partial p_i.$$

However, $\Omega+1$ is homogeneous of degree one in $\mathbf{p}$. Therefore

$$p_i \partial\Omega/\partial p_i = p_i \mathrm{d}(\Omega+1)/\mathrm{d}p_i = \Omega+1 = 1$$

and

$$-\theta = p_i q'^i = F. \tag{A6}$$

Thus (A5) becomes

$$q'^i = F\,\partial\Omega/\partial p_i, \qquad p'^i = -F\,\partial\Omega/\partial q^i. \tag{A7}$$

If we choose $u = u^*$ say, so that $F = 1$, then

$$p_i \,\mathrm{d}q^i/\mathrm{d}u^* = 1 \qquad \text{or} \qquad p_i\,\mathrm{d}q^i = \mathrm{d}u^*.$$

For this particular parameter we find

$$q'^i = \partial\Omega/\partial p_i, \qquad p'_i = -\partial\Omega/\partial q^i, \tag{A8}$$

which may be identified with the canonical form of the equations of dynamics.