THE BEHAVIOUR OF DWBA STRIPPING AMPLITUDES NEAR THE BUTLER POLE

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Summary

Using integral representations of the bound state wave functions it is shown that, near the so-called Butler pole, DWBA amplitudes of (d, p) and certain (d, n) reactions depend only on the asymptotic parts of these wave functions. Furthermore, the zero-range approximation for the n–p interaction in the deuteron is exact at the pole. As well it is found that for (d, n) stripping to loosely bound levels of heavy nuclei the Butler pole disappears in the sense that the amplitude is no longer infinite there.

I. INTRODUCTION

The amplitude $M$ of a (d, p) or (d, n) reaction, at fixed incident deuteron energy, may be regarded as a function of the variable $Q^2$,

$$Q^2 = |Q|^2,$$

where

$$Q = k_1 - k_2, \quad k_2 = m_A/m_B k_y.$$

$k_1$ and $k_y$ are the momenta, in the centre of mass frame, of the incident deuteron and the outgoing nucleon $y$, while $m_A$ and $m_B$ are the masses of the target nucleus $A$ and the residual nucleus $B$, which consists of $A$ plus a captured nucleon $x$.

For (d, p) and (d, n) reactions the amplitude $M(Q^2)$ has a singularity at the point $Q^2 = -\kappa_x^2$, where

$$\kappa_x^2 = -2m_x B_x/\hbar^2, \quad m_x = m_x m_A/(m_x + m_A),$$

and $B_x$ is the binding energy of the captured nucleon. If Coulomb interactions are neglected, this singularity is a pole, the Butler pole (Schnitzer 1962), but when Coulomb interactions are included this singularity is a branch point. In the case of (d, p) reactions the cut is in the phase of the amplitude only and there is a simple pole in the modulus. The (d, n) case is somewhat more complicated.

In both cases the amplitude in a neighbourhood of the singularity, the so-called pole term, is of considerable interest. It represents the high partial waves in the partial wave expansion of the amplitude. It can therefore be used to aid the computation of these amplitudes (Bertram and Tassie 1968).

Another reason for current interest in the pole term is that the residue at the pole is proportional to the reduced width of the captured nucleon so that, at least

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in principle, model independent reduced widths may be obtained from the angular distribution of the differential cross section either by extrapolation (Amado 1959) or, in the case of sub-Coulomb stripping, by employing the method of Morinigo (1967).

In the distorted wave Born approximation (DWBA) the amplitude for deuteron stripping reactions is (see, for example, Glendenning 1963)

$$M = \int \psi^{(-)*}(\{m_A/m_B\} r - R) \phi_x^*(r) \phi_d(R) V_d(R) \psi^{(+)}(\{r + (m_B/m_d) R\} \, dr \, dR \, ,$$

where \(\psi^{(+)}\) and \(\psi^{(-)}\) are elastic scattering optical model wave functions of the incoming deuteron and the outgoing nucleon respectively, \(\phi_d\) is the internal wave function of the deuteron, \(\phi_x\) the bound state wave function of the captured nucleon, and \(V_d\) represents the neutron-proton interaction in the deuteron. The aim of this paper is to show that near the pole the amplitude \(M(Q^2)\) depends only on the asymptotic parts of the wave functions and that the zero-range approximation is exact at the pole. We shall neglect the spins of the particles involved and choose our units such that \(\hbar = 1\).

II. \((d,p)\) AND \((d,n)\) REACTIONS

If we introduce the Fourier transforms \(F_1(p_1)\) and \(F_2(p_2)\) of the scattered wave functions by the relations

$$\psi_1^{(+)}(r) = \int F_1(p_1) \exp(i p_1 \cdot r) \, dp_1 \, ,$$

$$\psi_1^{(-)*}(r) = \int F_2(p_2) \exp(-i p_2 \cdot r) \, dp_2 \, ,$$

then the amplitude (1) can be written as (Clement 1965)

$$M = -\frac{\hbar m_B}{4}(m_A/m_B) \int F_1(p_1) F_2(p_2) (\kappa_1^2 + \kappa_2^2) G_1(q_1) G_2(q_2) \, dp_1 \, dp_2 \, ,$$

where

$$q_1 = p_1 - (m_A/m_B) p_2 \, , \quad q_2 = (m_B/m_d) p_1 - p_2 \, ,$$

$$\frac{(m_B/m_A)(q_1^2 + \kappa_x^2)}{(m_A/m_B)(q_2^2 + \kappa_x^2)} = \frac{(m_d/m_y)(q_1^2 + \kappa_x^2)}{(m_y/m_d)(q_2^2 + \kappa_x^2)} \, ,$$

$$G_1(q_1) = \int \phi_x^*(r) \exp(i q_1 \cdot r) \, dr \, ,$$

$$G_2(q_2) = \int \phi_d(R) \exp(i q_2 \cdot R) \, dR \, .$$

Suppose that the potential \(V_d\) and the potential \(V_{xA}\), acting between \(A\) and the captured nucleon, can be expressed as superpositions of exponential or Yukawa potentials

$$V_d(r) = -\int_C C(\alpha) \exp(-\alpha r) \, d\alpha \, ,$$

$$V_{xA}(r) = -\int_C C(\alpha) \exp(-\alpha r) \, d\alpha \, .$$
\[ V_{x,A} = (2\eta \kappa_x) r^{-1} - \int_{\mu}^{\infty} B(x) r^{-1} \exp(-\alpha x) \, dx. \] (9)

The use of exponential potentials in one case and Yukawa potentials in the other is of no great significance as they can be related by partial integration. The first term of \( V_{x,A} \) represents the Coulomb interaction between \( x \) and \( A \) with

\[ Z_A e^{2\pi m_{xA}/\kappa_x} \quad \text{if } x \text{ is a proton} \]
\[ 0 \quad \text{if } x \text{ is a neutron.} \]

The wave function for the internal motion of the deuteron, which is assumed to be in a pure s-state, can be expressed as (Martin 1959)

\[ \phi_d(r) = (4\pi)^{-1} N_d R^{-1} \exp(-\kappa_d R) \left( 1 + \int_{\gamma}^{\infty} \rho(\alpha) \exp(-\alpha R) \, d\alpha \right), \] (10)

where \( \rho(\alpha) \) satisfies the equation

\[ \alpha(\alpha+2\kappa_d)\rho(\alpha) = C(\alpha) + \int_{0}^{\alpha-\gamma} C(\alpha-\beta) \rho(\beta) \, d\beta \]

and \( N_d \) is a normalization constant.

The bound state wave function of the captured particle, which is assumed to be captured into a state of definite angular momentum, may be written as (Andrews, to be published)

\[ \phi_x(r) = N_L Y_{LM}(r) r^L \exp(-\kappa_x r) \int_{0}^{\infty} \sigma(\alpha) \exp(-\alpha r) \, d\alpha. \] (11)

The constant \( N_L \) is related to the reduced width (see, for example, Dullemond and Schnitzer 1963) and \( \sigma(\alpha) \) satisfies the equation

\[ \alpha(\alpha+2\kappa_a)\sigma'(\alpha) - 2\{L(\alpha+\kappa_x)+\kappa_x \eta\}\sigma(\alpha) = \int_{0}^{\alpha-\mu} B(\alpha-\beta) \sigma(\beta) \, d\beta. \] (12)

The right-hand side does not contribute for \( 0 \leq \alpha \leq \mu \), and the solution can easily be seen to be

\[ \sigma(\alpha) = \sigma_0(\alpha) \quad 0 \leq \alpha \leq \mu, \] (13)

where

\[ \sigma_0(\alpha) = \alpha^{L+\eta}(\alpha+2\kappa_x)^{L-\eta}. \]

For all greater values of \( \alpha \) equation (12) can then be solved recursively.

If we define

\[ \sigma_1(\alpha) = \sigma(\alpha) - \sigma_0(\alpha), \]

equation (11) becomes

\[ \phi_x(r) = N_L Y_{LM}(r) r^L \exp(-\kappa_x r) \left( \int_{0}^{\infty} \sigma_0(\alpha) \exp(-\alpha r) \, d\alpha + \int_{\mu}^{\infty} \sigma_1(\alpha) \exp(-\alpha r) \, d\alpha \right). \] (14)
Substituting (14) in (6) and (10) in (7), we find

\[ G_1(q_1) = 8\pi (2i)^L (L+1)! q_1^L Y_{LM}^*(q_1) N_L \left( \int_0^\infty \sigma_0(x) \frac{\alpha + \kappa_x}{\{q_1 + (\alpha + \kappa_x)^2\}^{L+2}} dx \right. \]

\[ + \left. \int_\mu^\infty \sigma_1(x) \frac{\alpha + \kappa_x}{\{q_1 + (\alpha + \kappa_x)^2\}^{L+2}} dx \right), \]  

(15)

\[ G_2(q_2) = (4\pi)^L N_d \left( \frac{1}{q_2 + \kappa_d^2} + \int_\mu^\infty \frac{\rho(x)}{\{q_2 + (\alpha + \kappa_x)^2\}^{L+2}} dx \right). \]  

(16)

Using (15) and (16) in equation (4), the amplitude becomes

\[ -2m_{np} M = M_1 + M_2 + \left( m_d m_A/m_y m_B\right)(M_3 + M_4). \]

Writing

\[ R(p_1, p_2) = (4\pi)^{3/2} (2i)^L (L+1)! 2N_d N_L q_1^L Y_{LM}^*(q_1) F_1(p_1) F_2(p_2), \]

we have

\[ M_1 = \int_0^\infty \sigma_0(x) dx \int R(p_1, p_2) \frac{\alpha + \kappa_x}{\{q_1 + (\alpha + \kappa_x)^2\}^{L+2}} dp_1 dp_2, \]  

(17a)

\[ M_2 = \int_\mu^\infty \sigma_1(x) dx \int R(p_1, p_2) \frac{\alpha + \kappa_x}{\{q_1 + (\alpha + \kappa_x)^2\}^{L+2}} dp_1 dp_2, \]  

(17b)

\[ M_3 = \int_0^\infty \sigma_0(x) dx \int_\gamma \rho(\beta) d\beta \int R(p_1, p_2) \frac{(\alpha + \kappa_x)(q_1^2 + \kappa_d^2)}{(q_2 + (\alpha + \kappa_x)^2)^{L+2}(q_2 + (\beta + \kappa_d)^2)} dp_1 dp_2, \]  

(17c)

\[ M_4 = \int_\mu^\infty \sigma_1(x) dx \int_\gamma \rho(\beta) d\beta \int R(p_1, p_2) \frac{(\alpha + \kappa_x)(q_1^2 + \kappa_d^2)}{(q_2 + (\alpha + \kappa_x)^2)^{L+2}(q_2 + (\beta + \kappa_d)^2)} dp_1 dp_2. \]  

(17d)

In Section III it is shown that the term \( M_1 \) is infinite at the point \( Q^2 = -\kappa_x^2 \) for \( \eta \leq 1 \) and that this singularity is an end-point singularity from the lower limit of the integration with respect to \( \alpha \). Since \( R(p_1, p_2) \) does not explicitly depend on \( \alpha, \kappa_d, \) and \( \kappa_x, \) and since \( \sigma_0(\alpha) = 0 \) at \( \alpha = 0, \) the singularity of \( M_1 \) can only come from the vanishing of the term \( q_1^2 + (\alpha + \kappa_x)^2 \) at \( \alpha = 0 \) in the integrand, i.e. the singularity arises from \( q_1^2 + \kappa_x^2 = 0 \) or equivalently from \( q_1^2 + \kappa_d^2 = 0. \)

Comparing equations (17b) and (17d) with (17a) it follows immediately that \( M_2 \) and \( M_4 \) in general have no singularities at \( Q^2 = -\kappa_x^2. \)

The term \( M_3 \) may be rewritten as

\[ M_3 = \int_\gamma \rho(\beta) d\beta \int \frac{R(p_1, p_2)}{q_2 + (\beta + \kappa_d)^2} dp_1 dp_2 \int_0^\infty \frac{\alpha + \kappa_x}{\{q_1 + (\alpha + 2\kappa_x)^L \eta \}^{L+1}} dx \]

\[ - \int_\gamma \rho(\beta) d\beta \int \frac{R(p_1, p_2)}{q_2 + (\beta + \kappa_d)^2} dp_1 dp_2 \int_0^\infty \frac{\alpha + \kappa_x}{\{q_1 + (\alpha + 2\kappa_x)^L \eta + 1 \}^{L+2}} dx. \]  

(18)
When \( q_1^2 = -\kappa_x^2 \) the integrands of both terms on the right of (18) behave like \( \alpha^{-1} \) as \( \alpha \to 0 \). Therefore, for \( \eta > 0 \), the lower limit of the integration with respect to \( \alpha \) does not cause any singularities in \( M_3 \) at \( Q^2 = -\kappa_x^2 \).

In the case of \( \eta = 0 \), the integration with respect to \( \alpha \) in (17c) can be carried out explicitly to yield

\[
M_3 = \int_\gamma^\infty \rho(\beta) \, d\beta \int R(p_1, p_2) \left\{ q_2^2 + (\beta + \kappa_d)^2 \right\}^{-1} \, dp_1 \, dp_2 .
\]

The integrand is finite at the point \( q_2^2 = -\kappa_3^2 \), and \( M_3 \) cannot have a singularity at \( Q^2 = -\kappa_x^2 \). Therefore, as \( Q^2 \to \kappa_x^2 \), \( M(Q^2) \rightarrow M_1(Q^2) \), except when the Coulomb parameter \( \eta \) of the captured particle is greater than unity.

The amplitude \( M_1 \) when expressed in terms of wave functions is

\[
M_1 = \int \psi^{(-)}^* (m_A|m_B) r \, \phi^*(r) \psi^{(+)}(r) \, dr,
\]

with

\[
\phi^0(r) = N_L \, Y_{LM}(r) \exp(-\kappa_x r) \, \int_0^\infty x^{L+\eta} (x+2\kappa_x)^{-\eta} \exp(-x r) \, dx .
\]

Thus \( M_1 \) is the DWBA amplitude for the stripping reaction when the zero-range approximation for the deuteron is made and with the bound state wave function of the captured nucleon replaced by its asymptotic part.

After carrying out the integration in (20) we find that for \( (d, p) \) reactions \( (\eta = 0) \) the correct asymptotic form of the bound state wave function is

\[
\phi^0(r) = N_L \, h_{LM}(i\kappa_x r) \, Y_{LM}(r) ,
\]

where \( h_{LM}(z) \) is the spherical Hankel function of the first kind, and for \( (d, n) \) reactions \( (\eta > 0) \)

\[
\phi^0(r) = N_L r^{-1} \, W_{LM}(r) ,
\]

where \( W_{LM}(x) \) is the Whittaker function.

III. THE TERM \( M_1 \)

In this section we shall show that the term \( M_1 \) in the amplitude is infinite at \( Q^2 = -\kappa_x^2 \), the singularity arising from the lower limit of the integration with respect to \( \alpha \) in equation (17a).

The scattering wave functions \( \psi^{(+)}(r) \) and \( \psi^{(-)}(r) \) in (19) can be expanded as Born series; the leading terms in these expansions are given as (Mott and Massey 1965)

\[
\psi^{(+)}(r) = \exp(-\frac{1}{2} \pi \eta_1) \, \Gamma(1+i\eta_1) \exp(i k_1. r) F\{i \eta_1; 1; i(k_1 - k_1. r) \} ,
\]

\[
\psi^{(-)}(r) = \exp(-\frac{1}{2} \pi \eta_2) \, \Gamma(1+i\eta_2) \exp(-i k_2. r) F\{-i \eta_2; 1; i(k_2 + k_2. r) \} ,
\]

where

\[
\eta_1 = (Z_A e^2 m_A d)/k_1
\]

and

\[
\eta_2 = Z_B e^2 m_B y/k_y
\]

for \( (d, p) \) reactions

\[
\eta_2 = 0
\]

for \( (d, n) \) reactions.
The amplitude $M_1$ is then

$$M_1 = K \int_0^\infty \sigma_0(\alpha) \, d\alpha \, \int \exp(iQ \cdot r) r^L Y_{LM}^\star(r) \exp\{-((\alpha + \kappa_x) r)\} F\{-i\eta_1; 1; i(k_1 r - k_1 \cdot r)\}
\times F\{-i\eta_2; 1; i(k_2 r + k_2 \cdot r)\} \, dr. \tag{23}$$

All the constant factors that occur in $M_1$ have been absorbed in the quantity $K$. In order to keep the expressions in the remainder of this section as concise as possible, any other constant factors that may appear as the result of our manipulations shall be automatically included in $K$.

If we assume that $\eta_1$ and $\eta_2$ contain a small imaginary part $i \epsilon$ ($\epsilon > 0$), which we can put equal to zero after the calculations, we may use the integral representation of the hypergeometric functions (Erdélyi 1953) and write

$$M_1 = K \int_0^1 \int_0^1 s^{-1-\eta_1} (1-s)^{\eta_1} t^{-1-\eta_2} (1-t)^{\eta_2} \, ds \, dt \int_0^\infty \sigma_0(\alpha) \, d\alpha \times \int r^L \exp\{-((\alpha + a) r)\} Y_{LM}^\star(r) \exp(iQ \cdot r) \, dr, \tag{24}$$

where

$$a = k_x - i(k_1 s + k_2 t), \tag{25}$$

$$q = Q - k_1 s + k_2 t. \tag{26}$$

Integration with respect to $r$ yields

$$M_1 = K \int_0^1 \int_0^1 s^{-1-\eta_1} (1-s)^{\eta_1} t^{-1-\eta_2} (1-t)^{\eta_2} \, ds \, dt \times q^L Y_{LM}(q) \int_0^\infty \sigma_0(\alpha) (\alpha + a)(q^2 + (\alpha + a)^2)^{-\frac{L+2}{2}} \, d\alpha. \tag{27}$$

For large $\alpha$ the integrand goes as $\alpha^{-4}$ and thus the integral converges at the upper limit of the $\alpha$-integration. Hence, if $M_1$ is to have a singularity at $Q^2 = -\kappa_x^2$, the only way this can happen is when the term $\{q^2 + (\alpha + a)^2\}$ in the integrand vanishes at $Q^2 = -\kappa_x^2$. From equations (25) and (26) we find that this occurs only when $\alpha = s = t = 0$. Hence we have shown that if $M_1$ has a singularity at $Q^2 = -\kappa_x^2$ it must arise from the lower limit of the integration with respect to $\alpha$.

In order to show that $M_1$ is in fact infinite at $Q^2 = -\kappa_x^2$ we make use of a result due to Cejpek (1966)

$$A(Q^2, \beta) \equiv \int_0^1 s^{-1-\eta_1} (1-s)^{\eta_1} \, ds \int_0^1 t^{-1-\eta_2} (1-t)^{\eta_2} \, dt \times \int r^{L-1} Y_{LM}^\star(r) \exp[i(Q - k_1 s + k_2 t) \cdot r] \exp[-(\beta - i(k_1 s + k_2 t))] \, dr$$

$$= K \frac{(1+Q^{2l})(B)^{2l}L^l}{(Q^2 + \beta^2)^{L+1}} \sum_{i=0}^{L-l} \sum_{m=-l}^{l} \left( \frac{4\pi}{2l+1} \right)^i \left( \frac{2L+1}{2l} \right)^i \binom{L-l, M-m, l, m}{LM} \times$$

$$\times$$
\[
\times k_1^{L-l}(-k_2)^l Y_{L-l,M-m}(k_1) Y_{l m}(k_2) \frac{1}{\Gamma(l+1+\imath \eta)} \Gamma(l+1) \\
\times \sum_{s=0}^{l} \frac{\Gamma(-\imath \eta + s)}{\Gamma(-\imath \eta)} \frac{1}{(L-l+s+1) \Gamma(l-s+1) \Gamma(s+1)} \frac{(-1)^s C^s}{(1+C)^s} \\
\times \sum_{j=0}^{L-l} \frac{(1+B)^{-j}}{(L-l-j+1) \Gamma(l-j+1) \Gamma(j+1)} \\
\times {}_2F_1(-\imath \eta_2 + j; -\imath \eta_1 + s; L-l+s+1; H), \quad (28)
\]

with

\[
B = \frac{-2ik_2 \beta + k_1^2 - k_2^2 - Q^2}{Q^2 + \beta^2}, \quad C = \frac{-2ik_1 \beta - k_1^2 + k_2^2 - Q^2}{Q^2 + \beta^2}, \quad H = 1 - \frac{Q^2 + \beta^2}{\beta^2 + (k_1 - k_2)^2}.
\]

Comparing this formula with equation (24) it follows immediately that, with \( \beta = \alpha + \kappa_x \),

\[
M_1 = K \int_0^\infty \sigma_0(\alpha) \frac{d \sigma}{d \beta}(A(Q^2, \beta)) \, d\alpha. \quad (29)
\]

When \( Q^2 = -\kappa_x^2 \), a typical term in the integrand of (29) behaves like \( r^{\eta+j-2} \) as \( \alpha \to 0 \). Therefore, due to the terms with \( j = 0 \) in (28), \( M_1 \) is infinite at \( Q^2 = -\kappa_x^2 \) only if \( \eta \leq 1 \).

**IV. Conclusions**

We have shown that the DWBA amplitude \( M(Q^2) \) for stripping reactions can be expressed as a sum of four terms, the first of which \( M_1(Q^2) \) corresponds to DWBA when the zero-range approximation is used for the deuteron, and with the bound state wave function of the captured nucleon replaced by its asymptotic part. The term \( M_1 \) is infinite at \( Q^2 = -\kappa_x^2 \), this singularity being known as the Butler pole. Since the other terms are finite at the Butler pole, \( M(Q^2) \to M_1(Q^2) \) as \( Q^2 \to -\kappa_x^2 \).

There is, however, one exception. In the case of \((d,n)\) reactions with \( \eta > 1 \), that is, when the proton is captured into a loosely bound level of a highly charged nucleus, \( M_1 \) is no longer infinite at the point \( Q^2 = -\kappa_x^2 \) and therefore in this case \( M(Q^2) \) does not approach \( M_1(Q^2) \) as \( Q^2 \to -\kappa_x^2 \).

**V. References**


Erdélyi, A. (1953).—“*Higher Transcendental Functions.*” Vol. 1. (Bateman manuscript project.) (McGraw-Hill: New York.)


