VIBRATIONAL STABILITY OF PURE HELIUM STARS SURROUNDED BY PURE HYDROGEN ENVELOPES

By R. Van der Borght*

[*Manuscript received April* 14, 1969]

Summary

This paper investigates the influence of a pure hydrogen envelope on the vibrational stability of pure helium stars and establishes the limiting mass as a function of the fractional mass of such an envelope. It is found that an envelope with a mass fraction of the order of 10^{-4} will make the star vibrationally stable, even when the total mass of the star is well above the limiting mass for pure helium stars.

I. Introduction

The vibrational stability of pure helium stars has already been investigated by Boury and Ledoux (1965) in the case of constant opacity and by Noels-Grötsch (1967) when the temperature dependence of the opacity is taken into account. In the first case the limiting mass is found to lie between 7 and $8\,M_\odot$, whereas in the case of variable opacity the limiting mass is of the order of $9\cdot2\,M_\odot$. This should be compared to the limiting mass of about $55\,M_\odot$ found by Schwarzschild and Härm (1959) in the case of massive stars with normal composition.

Considerable mass loss from one component of a binary system to its companion is possible (e.g. Paczynski 1967) leaving the primary at one stage with a large pure helium core surrounded by a hydrogen-rich envelope containing only a small fraction of the total mass. Therefore it would seem of interest to investigate the vibrational stability of such models and to find out how much hydrogen a pure helium star, of mass larger than the limiting mass, can carry before becoming again vibrationally stable.

II. Stellar Models

Models of pure helium stars surrounded by hydrogen-rich envelopes have already been constructed by Cox and Salpeter (1961) and Giannone (1967). The stellar models used in the present paper are obtained by numerical integration of the following structure equations:

$$\frac{\mathrm{d}q}{\mathrm{d}x}=\frac{\pi x^2 C\beta\tau^3\mathcal{M}^2\lambda^3\alpha}{48(1-\beta)}. \tag{I}$$

In the envelope

$$\frac{\mathrm{d}\tau}{\mathrm{d}x}=-\frac{\beta}{1-\beta}\frac{1-\beta_R}{x^2}\frac{\eta\alpha}{\lambda}, \tag{IIe}$$

* Department of Applied Mathematics, Monash University, Clayton, Vic. 3168.

$$\frac{\mathrm{d}\beta}{\mathrm{d}x} = \frac{4\beta(1-\beta)}{\tau\lambda x^2}\left(\eta\frac{1-\beta_R}{1-\beta}-q\right)\alpha. \tag{IIIe}$$

In the core

$$\frac{\mathrm{d}\tau}{\mathrm{d}x} = -\frac{8(4-3\beta)\alpha\beta q}{\lambda\Delta x^2}, \tag{IIc}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}x} = \frac{12\beta^3(1-\beta)q\alpha}{x^2\tau\lambda\Delta}. \tag{IIIc}$$

These equations are in fact a modified version of the equations used by Van der Borght (1964*a*) in a study of the evolution of massive stars initially composed of pure hydrogen. The modifications were necessary to avoid an exponential overflow during the numerical integrations.

In order to ensure this, the temperature variable $\tau = \bar{t}/\bar{t}_e$ was normalized to unity at the outer boundary of the star, $\bar{t}$ being defined by

$$\bar{t} = R\mu_e T/M_\odot,$$

where R, μ_e, and T are the radius, molecular weight in the outer layers, and temperature respectively, and $\bar{t}_e$ being the value of $\bar{t}$ near the outer boundary, given by

$$\bar{t}_e = \beta_R G\mathscr{M}(x_e^{-1}-1)/4\mathscr{R}.$$

In this expression β_R is the value at the surface of the ratio of gas pressure to total pressure β, $\mathscr{M} = \mu_e^2 M/M_\odot$, M being the mass of the star, and x_e is the value of x at the start of the numerical integrations, in this case $0\cdot9975$.

In the structure equations the following notations have been used.

$$C = aM_\odot^2 G^3/\mathscr{R}^4, \qquad q = M(r)/M, \qquad x = r/R,$$

$$\lambda = \beta_R(x_e^{-1}-1), \qquad \Delta = 32-24\beta-3\beta^2,$$

$$\alpha = \mu/\mu_e, \qquad \eta = \kappa/\kappa_e.$$

Both α and η are equal to unity in the outer layer. The opacity κ is assumed to be constant in the present models and is given by

$$\kappa = 0\cdot2(1+X),$$

X being the abundance by weight of hydrogen.

The system of differential equations given above was integrated numerically, β_R at the surface being chosen in such a way as to satisfy the condition $q = 0$ at the centre.

The models obtained in this way consist of three regions:

(1) an inner convective core composed of pure helium,

(2) an intermediate radiative layer also composed of pure helium, and

(3) an outer envelope composed of pure hydrogen but also in radiative equilibrium.

In Table 1 the values of β_R and β_c are given for stellar models of different total mass M in the case when the mass of the outer layer of pure hydrogen is just high enough to ensure vibrational stability. We see from this table that there is a marked decrease of β_R in the outer layers of the star corresponding to an increase in mass. That this should be so follows quite clearly from equation (IIIe). At the boundary between the hydrogen and helium radiative layers the value of the ratio $\eta = \kappa/\kappa_e$ changes suddenly from 1 to $0\cdot5$ resulting in a sudden jump in the value of $d\beta/dx$. From that point onwards the value of β will first increase while the integration proceeds inwards and after reaching a maximum decrease again in the normal way to its central value β_c.

Table 1

VALUES OF β_R AND β_c FOR STELLAR MODELS IN WHICH FRACTIONAL MASS OF HYDROGEN ENVELOPE HAS ITS CRITICAL VALUE Q_c

$M/M_\odot$	β_R	β_c	$M/M_\odot$	β_R	β_c
10	0·6136	0·6711	25	0·2243	0·4894
12·5	0·5257	0·6261	30	0·1436	0·4558
15	0·4493	0·5891	35	0·0764	0·4295
20	0·3235	0·5318			

The discontinuity in the derivative $d\beta/dx$ at the boundary between the two radiative layers is not the only one. From the equation of state and from the requirement of continuity in temperature and pressure it follows that there must be a discontinuity in the density to compensate for the discontinuity in the molecular weight. This in turn results in a discontinuity in the values of dp/dx and dT/dx across the boundary.

Equations (I)–(III) were derived on the assumptions that

(1) the opacity is constant in each radiative region,

(2) the luminosity L is constant outside the core, and

(3) the mass–luminosity relation is given by

$$\kappa L = 4\pi c G M(1-\beta_R)\,. \tag{1}$$

It is well known that the first two assumptions enable us to eliminate any explicit reference to the radius, reducing the problem of constructing the stellar model to the solution of the system of three ordinary differential equations given above.

The radius can then be found using the formula

$$R = (\tau_c/\mu_e T_8^c) M_\odot (G\mathscr{M}/4\mathscr{R})\lambda \times 10^{-8}\,, \tag{2}$$

where τ_c is the value of τ at the centre of the star, given by the previous integration, and T_8^c is the central temperature of the star divided by 10^8.

The central temperature is obtained from the energy equation

$$L = \int_0^R 4\pi r^2 \rho\epsilon \, dr\,, \tag{3}$$

where L is the luminosity of the star and ϵ the amount of nuclear energy generated per gram per second in the central regions of the star. In this case, for the 3α-reaction, we have

$$\epsilon = 3\cdot46\times10^{11}\times\rho^2 T_8^{-3}\exp(-43\cdot2\,T_8^{-1})\,.$$

Substituting this value of ϵ in the energy equation (3), together with the following expression for the luminosity obtained from the mass–luminosity relation (1)

$$L = 4\pi cG\mathscr{M}(1-\beta_R)M_{\odot}/\kappa_e\,\mu_e^2\,, \tag{4}$$

we obtain

$$\frac{1-\beta_R}{\kappa_e} = \frac{A\mathscr{M}^8\lambda^9}{R^6\mu_e^4}\int_0^1 x^2\tau^9\alpha^3\frac{\beta^3}{(1-\beta)^3}\frac{1}{T_8^3}\exp\left(-\frac{43\cdot 2}{T_8}\right)\mathrm{d}x\,, \tag{5}$$

where

$$A = \frac{3\cdot 46\times 10^{11}\times a^3G^8M_{\odot}^8}{27\times 4^9\times\mathscr{R}^{12}R_{\odot}^6\,c}\,. \tag{6}$$

Substituting the value of the radius given in (2) into equation (5) we obtain an equation that can be solved for the central temperature T_8^{c}. All physical quantities are then known for our models.

III. Vibrational Stability

In order to investigate the vibrational stability of the stellar models derived in the previous section we need the relative amplitude $\xi = \delta r/r$ throughout the star, δr being the Lagrangian displacement from the equilibrium position. This is obtained through integration of the following system of differential equations.

$$x\frac{\mathrm{d}\xi}{\mathrm{d}x} = -3\xi-\frac{p_1}{\Gamma}, \qquad x\frac{\mathrm{d}p_1}{\mathrm{d}x} = Vp_1+V\left(\omega^2\frac{x^3}{q}+4\right)\xi\,, \tag{7}$$

where

$$p_1 = \delta p/p\,, \qquad \omega^2 = \frac{R^3}{GM}\left(\frac{2\pi}{\mathscr{T}}\right)^2,$$

$$\Gamma = \beta+\frac{2}{3}\frac{(4-3\beta)^2}{\beta+8(1-\beta)}, \qquad V = -\frac{x}{p}\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{4\beta\alpha q}{x\tau\lambda}\,,$$

$\mathscr{T}$ being the period of oscillation.

A first guess of the value of ω^2, corresponding to the fundamental mode of oscillation, was obtained from the formula

$$\omega^2 = \left(\int_0^1 \alpha\tau^3\frac{\beta}{1-\beta}(3\Gamma-4)q(x)\,x\,\mathrm{d}x\right)\div\left(\int_0^1 \alpha\tau^3\frac{\beta}{1-\beta}x^4\,\mathrm{d}x\right). \tag{8}$$

Equations (7) were integrated numerically, adjusting the eigenvalue ω^2 in such a way as to satisfy the boundary condition

$$p_1+(4+\omega^2)r_1 = 0$$

at $x = 1$.

The stability coefficient σ' is given by (e.g. Ledoux and Walraven 1958)

$$\sigma' = -\{D-(S_1+S_2)\}/2E\,, \tag{9}$$

where

$$D = \int_0^{x_{\mathrm{f}}} \epsilon\frac{\delta T}{T}\frac{\delta\epsilon}{\epsilon}\,\mathrm{d}\bar{q}\,, \qquad S_1 = \frac{L}{M_{\odot}}\int_{x_{\mathrm{f}}}^1 \frac{\delta T}{T}\,\mathrm{d}\left(\frac{\delta L}{L}\right),$$

$$S_2 = \left(\frac{\delta L}{L}\right)_{\mathrm{f}} \frac{L}{M_\odot} \frac{1}{2}\left\{\left(\frac{\delta T}{T}\right)_{\mathrm{c}} + \left(\frac{\delta T}{T}\right)_{\mathrm{f}}\right\}, \qquad E = \frac{GM}{R}\omega^2 \int_0^1 x^2 \xi^2 \,\mathrm{d}\bar{q},$$

and

$$\bar{q} = \frac{M}{M_\odot} q(x), \qquad \frac{\delta\rho}{\rho} = -\left(3\xi + x\frac{\mathrm{d}\xi}{\mathrm{d}x}\right), \qquad \frac{\delta T}{T} = -\left(\Gamma_3 - 1\right)\frac{\delta\rho}{\rho},$$

$$\frac{\delta\epsilon}{\epsilon} = \frac{2\delta\rho}{\rho} + \nu\frac{\delta T}{T}, \qquad \frac{\delta L}{L} = 4\xi + \frac{4\delta T}{T} + \left(\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}m}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}m}\left(\frac{\delta T}{T}\right),$$

$$\nu = 43\cdot 2 T_8^{-1} - 3, \qquad \Gamma_3 - 1 = (4 - 3\beta)/(12 - 10\cdot 5\beta),$$

x_{f} being the boundary between the convective core and the radiative envelope.

Negative values of σ' correspond to vibrational instability, positive values to vibrational stability, i.e. to the case when radial oscillations, here in the fundamental mode, are gradually damped out by non-adiabatic effects.

Our problem therefore is solved in five successive steps:

(1) Integration of the structure equations (I)–(III) for a given value of $Q = M_{\mathrm{H}}/M$, that is, for a given fractional mass of pure hydrogen surrounding a pure helium star of given total mass M.

(2) Determination of the radius and central temperature by means of the energy equation.

(3) Determination of the relative amplitude of oscillation in the fundamental mode by solving the appropriate eigenvalue problem for radial adiabatic oscillations.

(4) Determination of the coefficient of vibrational stability σ'.

(5) Gradually modifying Q for a given M in order to find the critical value Q_{c} for which $\sigma' = 0$. This gives then the fractional mass Q_{c} of pure hydrogen required to make a star of mass M, larger than the critical mass, vibrationally stable again.

IV. Conclusions

The main results of this rather complicated sequence of numerical integrations are summarized in Table 2. This table shows that the critical fractional mass Q_{c} at first increases from zero at 7–8 $M_\odot$, reaching a maximum at $M = 15\,M_\odot$, and then decreases, gradually levelling off at the value of $0\cdot 05\times 10^{-3}$.

We see that the values of Q_{c} are surprisingly small and that a mass fraction of the order of 10^{-4} is sufficient to make a star vibrationally stable again, even if its mass is well above the limiting mass for pure helium stars.

A second point is that these values of Q_{c} do not increase uniformly with mass, as one would at first expect, but in fact level off at the value of about $0\cdot 05\times 10^{-3}$ for higher masses after passing through a maximum at $15\,M_\odot$.

The small value of the critical mass fraction Q_{c} is perhaps not surprising when one realizes that the pure hydrogen layer, although containing a small fraction of the total mass, is nevertheless quite extended. This can be seen from Table 3 in which the position x_{h} of the inner boundary of the hydrogen layer is given as a function of the total mass. For the larger masses considered in this paper the hydrogen layer extends over 20% of the radius and has a considerable influence on the run of the relative amplitude ξ in the outer layers and therefore on the stabilizing factor.

Using an expansion of the form

$$\xi = \xi_0+\xi_{01}y+\xi_{02}y^2+\ldots, \qquad p_1 = p_0+p_{01}y+p_{02}y^2+\ldots$$

near the outer boundary, where $y = 1-x$, we get through substitution in equations (7) and neglecting the mass variations in the outer layers

$$\xi/\xi_0 = 1+\{3-(4+\omega^2)/\Gamma\}y(1+2y). \tag{10}$$

TABLE 2

Q_c AS FUNCTION OF MASS

$M/M_\odot$	$Q_c \times 10^3$	$M/M_\odot$	$Q_c \times 10^3$
10	0·088	25	0·081
12·5	0·155	30	0·055
15	0·176	35	0·056
20	0·139		

TABLE 3

x_h AS FUNCTION OF MASS FOR $Q = Q_c$

$M/M_\odot$	x_h	$M/M_\odot$	x_h
10	0·8975	25	0·8275
12·5	0·8750	30	0·8125
15	0·8575	35	0·8050
20	0·8350		

In Table 4 are given, for various masses, the values of the relative amplitude at the outer boundary ξ_0, the eigenvalue ω^2, the stability coefficient σ', β_R, and the corresponding values of Γ and $(4+\omega^2)/\Gamma$ in the case of a mass fraction $Q = 0\cdot15\times10^{-3}$. This table illustrates the importance of the value of β_R in the outer layers. As usual (e.g. Van der Borght 1964*b*) ω^2 decreases when the mass increases and would therefore decrease the stability of the model since, as can be seen from (10), it results in a decrease of the slope of the relative displacement curve in the outer layers. However, β_R decreases as well and through Γ has the opposite effect. This results in an increase with mass of the stability coefficient since the changes with mass of β_R are more pronounced than those of ω^2. For masses larger than 15 $M_\odot$ the variation of β_R dominates and this explains the levelling off of the critical value Q_c.

TABLE 4

VALUES OF PARAMETERS AS FUNCTION OF MASS FOR $Q = 0\cdot15\times10^{-3}$

$M/M_\odot$	ξ_0	β_R	Γ	$(4+\omega^2)/\Gamma$	ω^2	$\sigma'\times10^{12}$
10	2·662	0·6136	1·450	5·70	4·253	+4·636
12·5	2·553	0·5257	1·433	5·58	3·993	−0·403
15	2·522	0·4493	1·419	5·54	3·863	−2·515
20	2·627	0·3235	1·394	5·62	3·8407	+1·383

A stability analysis, similar to the one described above, has been carried out on a stellar model constructed by Professor R. Kippenhahn of the University of Göttingen (personal communication). The star, after having lost a substantial fraction of its mass to a companion, is in the helium burning and hydrogen shell burning stage. It has a mass of 8·96 $M_\odot$, larger than the stability limit for pure helium stars, and a radius of $3 \cdot 443\, R_{\odot}$ which is much larger than the corresponding radius $0 \cdot 74\, R_{\odot}$ for pure helium stars. As expected for such an inflated model, the relative amplitude ξ_0 at the outer boundary is very large and equal to $4 \cdot 668 \times 10^3$. This has a very marked damping effect on the oscillations and the star is exceedingly stable for oscillations in the fundamental mode.

It appears from the analysis described in the present paper that stars which have reached the helium burning stage will only become vibrationally unstable if they are able to lose almost all their outer layers, even if their mass is substantially higher than $8\, M_{\odot}$.

V. Acknowledgment

The work reported here was carried out while the author was staying at the Institut d'Astrophysique of the Université de Liège and he wishes to thank Professor P. Ledoux for his kind hospitality.

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