ON THE VALIDITY OF THE TWO-TERM APPROXIMATION OF THE ELECTRON DISTRIBUTION FUNCTION

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Abstract

The Boltzmann equation for electrons moving in a neutral gas under the influence of an externally applied field is solved by expanding the electron distribution function in terms of Legendre and Sonine polynomials. The solution is given in terms of infinite matrices which have elements ordered by the Sonine polynomial index, and which are dependent upon the field strength. From the structure of the formulae, it is possible to infer that truncation of the Legendre polynomial expansion after two terms is a good approximation at all field strengths. This is supported by calculations of the electron drift velocity at low field strengths, which show that the error introduced by making the two-term approximation is small, even when the deviation from equilibrium is significant. The convergence of the Sonine polynomial expansion is shown to be strongly dependent upon field strength, and large matrices are required in the drift velocity formula at even small field strengths.

I. INTRODUCTION

Theoretical analyses of the mobility and diffusion of electrons in neutral gases under the influence of an applied electric field have traditionally relied upon the so-called two-term approximation (Lorentz 1916; Davydov 1935; Morse, Allis, and Lamar 1935; Margenau 1946; Allis 1956), in which the distribution function is approximated by the first two terms of an expansion in Legendre polynomials,

\[ f(c) = \sum_{l=0}^{\infty} f^l(c) P_l(\cos \theta), \quad \cos \theta = (E \cdot c)/Ec, \]

\[ \approx f^0(c) + f^1(c) \cos \theta. \]  

(1a)

The Boltzmann equation appropriate to this case, namely

\[ (e/m)E \cdot \frac{\partial f}{\partial c} = J(f_0) \]

(2)

with

\[ J(f_0) = \int \{f(c)f_0(c_0) - f(c')f_0(c_0)\} g\sigma(g, x) \, d\Omega \, dc_0, \]

(3)

is thus reduced to two coupled differential equations for \( f^0(c) \) and \( f^1(c) \). In equations (2) and (3) the 0 subscripts refer to the neutral gas while the other notation is standard. Solving for \( f^0(c) \) gives the Davydov distribution, which in the limiting case of rigid-sphere interaction and high fields becomes the Druyvesteyn distribution (Allis 1956).

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This approximation is usually justified on physical grounds (Chapman and Cowling 1970): because of their small mass, electrons suffer relatively large directional changes in velocity but lose only a small fraction (\(\sim 2 m/m_0\)) of their energy in collisions with the neutral atoms. (Collisions are assumed here to be elastic.) Thus, while the distribution of velocities may be non-Maxwellian, it would be expected to be nearly spherically symmetric in velocity space, that is, \(f^0\) would be the dominant term in (1). Although, as is shown in Section II below, the smallness of \(m/m_0\) provides the basic justification for using the two-term approximation, these arguments and others (Ginzburg and Gurevich 1960) do not fully justify the omission of higher order terms from (1).

In view of the accuracy of present day experiments (Crompton, Elford, and Robertson 1970), it is of interest to estimate the importance of \(f^2, f^3, \ldots\). A systematic method of approximating these terms is presented here and it is shown that they are insignificant. The amount of numerical work required increases with the field strength, but the structure of the formulae and the results for low fields, which are presented below, suggest that for the electrons the two-term approximation remains valid for all values of field strength.

II. Theory and Discussion

The neutral gas is assumed to be in equilibrium at temperature \(T_0\),

\[
f_0(c_0) = n_0 \bar{\omega}(x_0, c_0),
\]

and \(f^l(c)\) is expanded further in terms of Sonine polynomials as

\[
f^l(c) = n \bar{v}(x, c) \frac{4\pi}{2l+1} \sum_{\nu=0}^{\infty} f^{\nu l} R_{\nu l}(xc),
\]

where

\[
\bar{v}(x, c) = (x^2/2\pi)^{3/2} \exp(-x^2 c^2), \quad \alpha^2 = m/kT_0, \quad \alpha_0^2 = m_0/kT_0,
\]

\[
R_{\nu l}(c) = N_{\nu l}(c/\sqrt{2}) S_{l+\nu}(\sqrt{2} c^2), \quad N_{\nu l}^2 = 2\pi^{3/2} \nu! / \Gamma(\nu + l + 3/2).
\]

The present paper follows the earlier work of Kumar (1967) in notation and in the transformation to matrix form. Substituting for \(f(c)\) and \(f_0(c_0)\) in (2), we obtain an (infinite) homogeneous matrix equation for the expansion coefficients \(f^{\nu l}\). The existence of a summational invariant, corresponding to conservation of electron number, results in the vanishing of the determinant of the matrix of coefficients (Kumar 1967), and a non-trivial solution is assured. Using the normalization condition

\[
f^{00} = \sqrt{(4\pi)},
\]

the homogeneous matrix equation is then reduced to a solvable inhomogeneous matrix equation of the form

\[
[J - \epsilon D]f = -\epsilon d,
\]

where \(f\) is the unknown column vector, \(d\) is a known (constant) vector, and \(J\) and \(D\) are matrices labelled by \(\nu\) and \(l\) with \(\nu, l = 0, 1, \ldots\), except that \(\nu = 0 = l\) terms
do not occur. The field parameter $\epsilon$ is defined in equation (9) below. Equation (7) may be written as a series of coupled equations for matrices labelled by $\nu$ only,

\begin{align}
J^0 f^0 - \epsilon D^0_+ f^1 &= 0, \\
J^1 f^1 - \epsilon D^1_+ f^2 - \epsilon D^- f^0 &= -\epsilon d, \\
J^2 f^2 - \epsilon D^2_+ f^3 - \epsilon D^- f^1 &= 0, \\
& \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdOTS
(1) by the first \(L+1\) terms. The first three terms in the formal solution may be written as

\[
\begin{align*}
  f^0 &= \epsilon(J^0)^{-1}D_0^0 f^1, \\
  f^1 &= -\epsilon M_L^1 \cdot d, \quad L \geq 1, \\
  f^2 &= \epsilon N_L^{-1}D_L^2 f^1, \quad L \geq 2,
\end{align*}
\]

where

\[
M_L \equiv J^1 - \epsilon^2(D_1^1(J^0)^{-1}D_0^0 + D_1^1 N_L^{-1}D_L^2)
\]

and, for successive truncations,

\[
\begin{align*}
  N_1 &= 0, \\
  N_2 &= J^2, \\
  N_3 &= J^2 - \epsilon^2 D_1^2(J^3)^{-1}D_2^2, \\
  \cdots \cdots \cdots \cdots
\end{align*}
\]

It should be noted that as \(\epsilon \to 0\), \(M_L \to J^1\) so that high \(l\) contributions are not important in this situation. Furthermore, because of the smallness of \(m/m_0\), in the second term of (15) the contribution from \(J^0\) (defined in equation (11a)) always overshadows the higher \(l\) contributions which enter through \(N_L\). Hence, even for high fields, the contributions from \(l \geq 2\) will not be important. Then, from (11a) and (14a), we may conclude that \(|f^0|\) is always much larger than \(|f^1|\). The validity of the two-term expansion is thus seen to derive from the smallness of the mass ratio \(m/m_0\) and the form of the solutions (equations (14) and (15)).

In numerical calculations with the above equations, it is also necessary to truncate the infinite matrices \(J^l\) and \(D_L^l\) at some upper value, \(N\), of \(\nu\). The corrections introduced by increasing \(N\) are affected in a different way by the small mass ratio. They are found to be strongly dependent on the field strength, i.e. the parameter \(\epsilon\), although they do decrease if \(N\) is taken to be sufficiently large. The dimension of the matrices (determined by \(N\)) required to obtain convergence to a given accuracy increases rapidly with the field strength. (This is similar to previous experience in matrix calculations of related problems (Pekeris et al. 1962; Hochstim and Massel 1969).) The physical reason behind this is the well-known non-Maxwellian nature of the distribution function in this situation; although it is nearly isotropic, a large number of Sonine polynomials are required for a good representation of the velocity dependence.

In the numerical example considered in Section III, because of the slow convergence the whole experimentally available range of energy was not covered, but it appears that results of any accuracy could be obtained provided one were prepared to use sufficiently large matrices.

* This conclusion is expected to be valid for the types of cross sections encountered in the present problem. The hard-sphere interaction considered in Section III has been used in a case where it agrees most satisfactorily with experiment and the results may be considered typical for this type of problem.
III. Numerical Example

The drift velocity $W$ and the effective electron temperature $T$, as conventionally defined, are given here by

$$W = (3/4\pi)^1 \{ f^{01}(E)/\alpha E \} E$$

and

$$T/T_0 = 1 - f^{10}(E)/(6\pi)^1.$$  \hfill (18)

The neutral gas was taken to be helium and a hard-sphere interaction potential of radius $r$ was assumed. In this case, the partial cross sections (13) simplify to

$$\sigma_i(g) = \pi r^2 \cdot \delta_{i0}. \hfill (19)$$

The values of parameters chosen were $\pi r^2 = 5.4 \times 10^{-16}$ cm$^2$, $m/m_0 = 1.371 \times 10^{-4}$, and $T_0 = 77$ K. Figure 1 shows the effect of increasing the order $N$ of the $\nu$ truncation in the two-term approximation ($L = 1$) and also provides a comparison with the usual two-term calculation by numerical solution of differential equations. Clearly, the convergence becomes worse for higher values of $E/n_0$, indicating that very large matrices would be required to cover the whole of the experimental range. Also shown in Figure 1 is a plot of the ratio of the electron temperature to the gas temperature for $N = 19$. It shows that with increasing field strengths the electron distribution must deviate strongly from the Maxwellian appropriate to $T_0$. 

![Figure 1](image-url)

Fig. 1.—Drift velocity $W$ of electrons in helium at $T_0 = 77$ K calculated from truncated matrices for the indicated values of $N$ (dashed curves) and compared with the usual two-term differential equation calculation (solid $W$ curve). All curves coincide for low values of $E/n_0$. The calculated values of electron temperature $T$ for $N = 19$ are also shown. (Note that 1 Townsend (Td) $= 10^{-17}$ V cm$^2$.)
Because of the slow convergence with \( N \), the present work was limited to low values of \( E/n_0 \) in calculations showing the effect of convergence with \( L \). Table 1 shows the variation of the drift velocity with \( L = 1, 2, \) and 3 for fixed \( N \) and different \( E/n_0 \) in a region where the deviation from equilibrium is significant. It is seen that the relative magnitude of the correction from higher \( l \) terms is even smaller than may have been anticipated, i.e. less than \( m/m_0 \). The values change somewhat for different values of \( N \) but the pattern remains the same. Similar results have been obtained for many different sizes of matrices and for different parameters. On this basis, it can be conjectured that, although very large matrices in the Sonine polynomial index \( \nu \) are needed at high fields, as far as the Legendre polynomial expansion (equation (1)) is concerned, inclusion of terms beyond the first two produces corrections of a relative order of magnitude not exceeding \( m/m_0 \).

### Table 1

**Variation of Drift Velocity with Successive Orders of Truncation in Legendre Polynomial Index**

<table>
<thead>
<tr>
<th>Order of Truncation</th>
<th>Drift Velocity ( W ) (10(^5) cm s(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L )</td>
<td>( E/n_0 = 0.01 )</td>
</tr>
<tr>
<td>1</td>
<td>0.4080835</td>
</tr>
<tr>
<td>2</td>
<td>0.4080756</td>
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<tr>
<td>3</td>
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### IV. Acknowledgments

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### V. References


