SHORT COMMUNICATIONS

ON THE DRIFT VELOCITY OF ELECTRONS IN A GAS*

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In a recent paper, Crompton, Elford, and Robertson (1970; hereafter referred to as CER) considered certain questions concerning the steady-state distribution of electrons moving in a neutral gas under the influence of a uniform electric field \( E \). The present communication comments on some aspects of the error discussion in the Appendix of the paper by CER. The analysis will be restricted to the case of isotropic scattering. Two different definitions of the drift velocity are considered by CER, namely

\[
W_1 = \int dv f(v)v / \int dv f(v) \tag{1}
\]

and

\[
W_2 = \int dv g(v)\langle s\rangle^+(v) / \int dv g(v)\langle t\rangle^+(v), \tag{2}
\]

where \( f(v) \) is the conventional distribution function, \( g(v) \) is the distribution function immediately after collision, and

\[
\langle s\rangle^+(v_0) = w_0(v_0)\langle t\rangle^+(v_0)
\]

\[
= \frac{1}{2} \int_{-1}^{1} d\mu \int_{0}^{\infty} dv \frac{1}{2}at^2 + v_0 t\mu \exp \left( - \int_{0}^{t} \frac{dt'}{\tau(v(t'))} \right) \tag{3}
\]

and

\[
\langle t\rangle^+(v_0) = \frac{1}{2} \int_{-1}^{1} d\mu \int_{0}^{\infty} dt \frac{t}{\tau(v(t))} \exp \left( - \int_{0}^{t} \frac{dt'}{\tau(v(t'))} \right) \tag{4}
\]

are "path integral" expectation values. In equations (3) and (4)

\[
\mu = -v_0 \cdot E/v_0 E, \quad a = -|e/m_e| E, \quad v(t) = (v_0^2 + a^2t^2 + 2v_0\mu at)^{1/2}.
\]

It should be noted that some changes in nomenclature have been made; in particular, the quantities \( f, f_0, T, \lambda_m/c, \cos \theta, W, \) and \( W_e \) of CER correspond to \( f_0, g, \langle t\rangle^+, \tau, \mu, W_1, \) and \( W_2 \) of the present paper respectively. It can be seen that the relations (A2) and (A3) of CER are equivalent to the above relations (2) and (3).

The corresponding Boltzmann equation (see e.g. Paveri-Fontana 1970) is

\[
a \cdot \partial f / \partial v = g(v)T - f(v)\tau(v). \tag{5}
\]

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One should note the functional dependence of the initial distribution function \( g(v) \) on \( f(v) \):

\[
g(T = g(v, \{ f \})/T = \int dv' f(v')P(v' \rightarrow v)/\tau(v'). \tag{6} \]

A well-known method of approximation is to expand the function \( f(v) \) in terms of Legendre polynomials and then truncate the series after the second term. This leads to the equations

\[
\frac{a}{3v^2} \frac{d}{dv} (v^2f_1) = g(v, \{ f \})/T - \frac{f_0(v)}{\tau(v)}, \quad a \frac{df_0}{dv} = -\frac{f_1(v)}{\tau(v)}, \tag{7} \]

\[
f(v) \approx f_{\text{leg}}(v) \equiv f_0(v) + \mu f_1(v). \tag{8} \]

In order to solve these equations, one must assign an explicit form to \( g(v) \) as a functional of \( f(v) \). For instance, in order to derive the Davydov distribution that is employed in Section IV of CER, one replaces equation (5) by equations (7) and (8) and then assigns (see e.g. Bernstein 1969; Wannier 1971)

\[
g(v, \{ f \}) = \frac{f_0(v)}{\tau(v)} + \frac{\epsilon}{v^2} \frac{d}{dv} (v^3f_0(v)) \left( 1 + \frac{kT}{m_e v} \frac{d}{dv} f_0(v) \right), \tag{9} \]

where \( \epsilon \) is the ratio of the electron mass to the molecular mass. After evaluating \( f_0(v) \) and \( f_1(v) \), substitution back into equation (9) gives \( g(v, \{ f_{\text{leg}} \}) \) and then further substitution into equations (1) and (2) provides approximate expressions for the drift velocity. Let these approximations be \( W'_1 \) and \( W'_2 \). It should be noted that \( W'_1 \) is the same as \( W \) defined by relation (A1) of CER whenever \( v^2 < \lambda(v)f_0(v) \rightarrow 0 \) for \( v \rightarrow 0 \), while \( W'_2 \) is the approximation for \( W_e \) given on page 681 of CER (remembering that \( f \) and \( f_0 \) of CER are equivalent to the present \( f_0 \) and \( g \) respectively).

Now it can be shown that in steady-state conditions \( W = W_1 = W_2 \) if both \( \tau(v)/v = O(v^{-a}) \) when \( v \rightarrow \infty \) (\( a > 0 \)) and exact solutions of the Boltzmann equation (5) are used (Cavalleri and Paveri-Fontana 1972). However, one cannot expect that generally \( W'_1 \) will be equal to \( W'_2 \), as they are both approximate expressions. As a matter of fact, the quantity \( (W'_1 - W'_2)/W'_1 \) could possibly be used as a measure of how suitable a Legendre truncation technique is for a particular problem.

In their analysis, however, CER obtained another expression \( W''_2 \) for the drift velocity by inserting into equation (2) the approximate relation

\[
T^{-1} g(v, \{ f \}) \langle \nabla \rangle f(v) \approx f_0(v). \tag{10} \]

They then concluded that this new expression should be better than \( W'_1 \). Such a claim appears questionable, as both \( W'_1 \) and \( W'_2 \) are obtained by applying one approximate technique (Legendre truncation) to the Boltzmann equation while \( W''_2 \) is obtained by applying a further approximation (relation (10)) to the results obtained by applying the previous approximation. It does not seem reasonable to expect that the second approximation will somehow cancel the effect of the first. Also note that, if the approximate values of \( f_0(v) \) and \( f_1(v) \) (based on approximation (7)) approach their exact values, then both \( W'_1 \) and \( W'_2 \) approach the exact \( W \) but \( W''_2 \) generally does not.
For these reasons, generally one should expect \( W_2^* \) to provide less reliable results than \( W_1' \) and \( W_2' \) unless the contrary can be proved for the specific case considered.

To the author's knowledge, only the value of \( W_1' \) is known for a large number of cases, and there are no published computations of \( W_2^* \), \( W_1^* \), or the exact \( W \). For this reason comparisons based on numerical results are not possible at present. The case \( \tau(v) = \text{const.} \) is well known, but one finds \( W_1' = W_2' = W_2^* = W = a \tau \) and no general conclusion can be drawn. Another special case is considered below.

**An Example**

The example introduced here is used to illustrate the previous considerations. Among the cases studied by the author, it is the only one that allows easy analytical manipulations.

Let \( f_{\text{leg}}(v) \) be the approximate solution obtained by replacing the Boltzmann equation (5) by the Legendre truncated relations (7). Then \( W_1' \) is obtained by substituting \( f_{\text{leg}}(v) \) into relation (1); the resulting expression is equal to relation (A1) of CER whenever \( v^3 \tau(v) f_0(v) \rightarrow 0 \) for \( v \rightarrow 0 \). Also, \( W_2' \) and \( W_2^* \) are obtained by substituting \( g(v, \{ f_{\text{leg}} \}) \) and \( f_0(v) T/\langle t \rangle^+(v) \) respectively into equation (2). Now consider the relations

\[
a \cdot \frac{\partial f'/\partial v + f'/\tau(v)}{\tau(v)} = g(v, \{ f_{\text{leg}} \})/T, \quad (11)
\]

\[
a \cdot \frac{\partial f''/\partial v + f''/\tau(v)}{\tau(v)} = f_0(v)/\langle t \rangle^+(v), \quad (12)
\]

remembering that \( f_0(v) \) and \( f_{\text{leg}}(v) \) are the approximate quantities given by the relations (7) and (8). By a technique similar to that of Cavalleri and Paveri-Fontana (1972) it can be shown that \( W_2' \) and \( W_2^* \) are equal to the value that \( W_1 \) takes when \( f' \) and \( f'' \) are substituted into equation (1). This is valid whenever \( \tau(v)/v = O(v^{-\alpha}) \) for \( v \rightarrow \infty \) (\( \alpha > 0 \)).

Now consider the special case \( 1/\tau(v) = H(v-v*)/\tau_0 \), where \( H \) is the Heaviside step function. It can be considered a fairly good approximation for certain cases where the electron–molecule cross section presents a significant Ramsauer dip. Also assume that the Davydov representation (9) of the collision source is valid. Multiplying equations (5), (11), and (12) by \( v \, dv \) and integrating leads to a relation of the form

\[
\int_{v=v*} \, dv \, v \, y(v) = a \tau_0 \int \, dv \, y(v), \quad (13)
\]

where \( y(v) \) can equal any of the quantities \( f(v), f'(v), \) or \( f''(v) \). The absence of a collision source for \( v < v* \) in equations (5) and (11) guarantees that, for reasons of symmetry, both \( f(v, \mu) = f(v, -\mu) \) and \( f'(v, \mu) = f'(v, -\mu) \) for \( v < v* \). However, it is easy to show that in equation (12)

\[
f''(v, \mu) > f''(v, -\mu) \quad \text{for} \quad 0 < \mu \leq 1, \quad 0 < v < v*.
\]

For this reason, from the form of the relation (13), one finds that the exact electron drift velocity is given by

\[
W = \int \, dv \, v \, f(v) \bigg/ \int \, dv \, f(v) = a \tau_0
\]
and that also
\[ W'_2 = \int dv \, v f'(v) / \int dv \, f'(v) = a\tau_0 = W, \]
while
\[ |W'_2| = \left| \int dv \, v f^*(v) / \int dv \, f^*(v) \right| > a\tau_0. \]

Let us now consider \( W'_1 \). Substituting \( 1/\tau(v) = H(v-v^*)/\tau_0 \) into the well-known Davydov distribution function (see e.g. equation (1) of CER) gives
\[ f_0(v) = C \]
\[ = C \exp\left( -\int_{v^*}^v \frac{c \, dc}{kT/m_e + (a\tau_0)^2/3e} \right) \]
for \( v < v^* \),
\[ v > v^* \]
Then, using the relations (1) and (7), we once again find
\[ W'_1 = \frac{a}{3a} \int dv \, v^3 f_1(v) / \int dv \, v^3 f_0(v) = a\tau_0. \]

For one specific case we have thus computed the exact \( W \) and the approximations \( W'_1, W'_2, \) and \( W''_2 \). In this example \( W'_1 \) and \( W'_2 \) yield the exact value while \( W''_2 \) does not. This result would seem to confirm the earlier conclusions.

Actually, in this problem, the approximation (10) is an extremely poor representation of \( g(v, \{j\}) \) as it yields a nonzero \( g \) even for \( v < v^* \). Indeed the exact result \( W''_2 = a\tau_0 \) could be obtained by postulating
\[ g(v)/T = \{f_0(v)/\langle j \rangle \} H(v-v^*) \]
(14)
instead of (10). However, this choice would have other shortcomings. Whereas equation (11) conserves the number of collision events in the sense that
\[ \int dv \, f'(v)/\tau(v) = \int dv \, f_{\text{leg}}(v)/\tau(v), \]
and equation (12), based on (10), conserves the number of electrons in the sense that
\[ \int dv \, f^*(v) = \int dv \, f_{\text{leg}}(v), \]
it can be shown that the representation (14) fails to satisfy either conservation property. Furthermore, it seems impossible to generalize (14) to a general \( \tau(v) \).

Now, it could be claimed that the present example violates the condition of validity for the approximation (10), which, as shown by Cavalleri (1969, relation (13)) and Braglia (1970, relation (31)), can be stated as
\[ a\tau(v)/v \ll 1. \]
(15)
This is indeed true. However, it should be noted that the experimental example
considered by CER also violates the condition (15), since, for \( v \to 0 \), \( \lambda \) approaches a finite value and therefore \( a\tau(v)/v = a\lambda/v^2 \to \infty \).

Thus, on the basis of the example presented here and the preceding discussion, we may conclude that:

1. After a Legendre truncation has been applied, it would probably be better to follow the procedure that leads to the evaluation of \( W'_1 \) and \( W'_2 \) instead of \( W'_1 \) and \( W'_2 \).

2. The discrepancy found by Crompton, Elford, and Robertson (1970) in the region \( v \to 0 \) between the integrands appearing in the evaluations of \( W'_1 \) and \( W'_2 \) most probably arises more from the shortcomings of the assumption (10) than from the inadequacy of the Legendre truncation technique.

3. In the region \( v \to 0 \), the condition of validity of approximation (10) (namely condition (15)) is not met.

Further clarification of this subject could be obtained, for instance, by calculating \( W'_2 \) numerically from the data of Crompton, Elford, and Robertson. The resulting values of \( W'_1 \) and \( W'_2 \) could then be compared. Further comparisons could also be made between the values of \( W'_1 \), \( W'_2 \), and \( W'_2 \) obtained from the formulae given here with computations of \( W_2 \) based on the numerical solution of Cavalleri and Sesta’s (1968, 1969) integral equation for \( g(v) \).

References


