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Solution of Eight-vertex Lattice Model Without Elliptic Functions

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Abstract

Baxter's method of solving the eight-vertex model in lattice statistical mechanics is examined from an elementary point of view. It is shown that the algebraic operations in the method can be carried out without recourse to elliptic functions. These include: construction of certain subspaces invariant under the action of the transfer matrix; reduction of the transfer matrix eigenvalue problem to an equivalent ice-type problem and construction of certain matrices which commute with the transfer matrix and satisfy a functional matrix equation. The problem is examined in a somewhat more general context. It is shown that some of the identities which are crucial for constructing invariant subspaces no longer hold when the effect of external fields is included. The connection between the uniformization theory of algebraic functions and parameterization in terms of elliptic functions is pointed out in an appendix.

1. Introduction

In lattice statistical mechanics, exact solutions are known only for a small number of lattice models. The most important of these is the so-called eight-vertex model without external fields, which contains as special cases most other models on a plane square lattice. Detailed solutions for this model and the associated problem of a one-dimensional Heisenberg chain were given in a series of papers by Baxter (1972*a*, 1972*b*, 1973*a*, 1973*b*, 1973*c*). An important feature of Baxter's method is the appearance of elliptic functions in terms of which the vertex weights are parameterized. A number of algebraic identities among the elliptic functions facilitate the work throughout and the study of analytic properties of thermodynamic quantities is facilitated by the known analytic properties of elliptic functions.

The field of elliptic functions, like the field of the more general automorphic functions, is an algebraic function field of degree of transcendence one, i.e. an algebraic relation holds between any two members of the function field. In the eight-vertex model problem, elliptic functions are introduced to satisfy a basic polynomial relation. One can therefore say that all algebraic relations involving elliptic functions in this model are controlled by this basic identity. It has a sufficiently simple structure so that all these relations can be followed directly, without introducing elliptic functions. This is the main observation on which the present paper is based. It will be shown that one can construct the subspaces invariant under the action of the transfer matrix and reduce the eight-vertex model without fields to a generalized ice-type model without introducing elliptic functions. This in effect describes the structure of eigenvectors of the transfer matrix. The functional matrix equations for the eigenvalues of the transfer matrix are also derived without introducing the elliptic functions. In this way the purely algebraic aspects of the problem are dealt with in a more direct manner.

While the above features are sufficient justification for this work, a more general remark is also in order: Models in lattice statistical mechanics typically depend on a small number of parameters. The algebraic part of the problem is that part which is concerned with the transformation of the partition function, by introducing a transfer matrix or otherwise, to a form in which its evaluation can be performed by solving an eigenvalue type of equation in a small number of variables. Some properties of the eigenvalue equation may also be derived by purely algebraic methods. Analytic methods are needed only at the stage of solving the final equation and in the study of singularities of thermodynamic functions. Known solutions of lattice models have been obtained by methods invented to suit particular problems and a separation of the algebraic and analytic aspects of the problems has not been attempted. As the models become more complicated, however, such a separation becomes important, not only to obtain a better understanding of the structure of problems already solved but also because it may provide a more systematic point of view for attacking other problems.

While our main concern here is with some algebraic properties of the eight-vertex model, we start with a somewhat more general problem and state results for this problem where no advantage of brevity or clarity is to be gained by restricting attention to the eight-vertex model. In this context it becomes clear that the solution of the eight-vertex model for the field-free case has been obtained by a very special device which does not work even for the same model in the presence of fields.

In view of the limited scope of this paper, the model is defined directly in terms of its transfer matrix. Its physical interpretation and vertex diagrams are of no immediate concern here and are therefore not mentioned. For the eight-vertex model these are, of course, available in the papers of Baxter (1972*a*, 1972*b*, 1973*a*, 1973*b*, 1973*c*).

2. Transfer Matrix

(*a*) *Definition*

We consider transfer matrices defined in terms of 16 parameters $t^a{}_b\,(a,b = 1,2,3,4)$. The eight-vertex model without external fields depends on only four parameters and corresponds to the case

$$t^a{}_b = t_a\,\delta_{ab}\,, \tag{1}$$

where δ_{ab} is the Kronecker symbol. With the help of 2×2 matrices σ^i, given by

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2}$$

and ρ_i, which are copies of σ^i but act independently of them, we define the objects

$$t{:} = t^a{}_b\,\sigma^a\,\rho_b \equiv t^{\alpha\beta}{}_{\lambda\mu} \qquad (\alpha,\beta,\lambda,\mu = 1,2 \text{ or } \pm)\,. \tag{3}$$

The transfer matrix T is then given by

$$\begin{aligned} T &= t^{\alpha_1\beta_1}{}_{\lambda_1\lambda_2}\, t^{\alpha_2\beta_2}{}_{\lambda_2\lambda_3} \dots t^{\alpha_N\beta_N}{}_{\lambda_N\lambda_1} \\ &\equiv \mathrm{tr}_\rho\Big(t{:} \overset{\sigma}{\otimes} t{:} \overset{\sigma}{\otimes} \dots \overset{\sigma}{\otimes} t{:}\Big) \qquad (N \text{ terms}). \end{aligned} \tag{4}$$

Here the dot superscript indicates an object (matrix or vector) referring to σ-type 2×2 matrices and the dot subscript an object referring to ρ-type matrices. The transfer matrix as defined by (4) is a $2^N\times 2^N$ matrix. It acts on column vectors with 2^N components which may be formed from cross products of N two-component vectors $\phi_l^{\bullet}$ on which σ matrices act. In the second line of equation (4) there is a cross product of σ-type matrices and an ordinary product of ρ-type matrices on which the trace is finally taken. Apart from this clarification of the operations to be performed, the ρ and σ matrices play no explicit part in the following work.

(b) Invariance of trace

If we take ρ-type nonsingular matrices $M_{\bullet J}$ $(J = 1,2,\dots,N+1)$ such that

$$M_{\bullet N+1} = M_{\bullet 1} \tag{5}$$

and replace $t_{\bullet}^{\bullet}$ in the Jth position inside the trace in equation (4) by

$$\tilde{t}_{\bullet J}^{\bullet} = M_{\bullet J}^{-1}\, t_{\bullet}^{\bullet}\, M_{\bullet J+1}, \tag{6}$$

then the trace remains unaltered. Hence the transfer matrix also remains unaltered, although this operation introduces some extra parameters in it. By introducing suitable choices of the same parameters in the vectors on which the transfer matrix acts, it is possible to simplify the action of the transfer matrix on these vectors.

In general we may write

$$M_{\bullet J} = \begin{bmatrix} r'_J q_J & r_J p_J \\ r'_J & r_J \end{bmatrix}, \tag{7}$$

$$\Delta_J = \det M_{\bullet J} = r_J r'_J (q_J - p_J), \tag{8}$$

so that from equation (6)

$$\tilde{t}_{\bullet J}^{\bullet} \equiv \begin{bmatrix} A_J^{\bullet} & B_J^{\bullet} \\ C_J^{\bullet} & D_J^{\bullet} \end{bmatrix}, \tag{9}$$

with

$$A_J^{\bullet} = \Delta_J^{-1} r_J r'_{J+1}(q_{J+1} t^{\bullet}_{11} + t^{\bullet}_{12} - p_J q_{J+1} t^{\bullet}_{21} - p_J t^{\bullet}_{22}), \tag{10}$$

$$B_J^{\bullet} = \Delta_J^{-1} r_J r_{J+1}(p_{J+1} t^{\bullet}_{11} + t^{\bullet}_{12} - p_J p_{J+1} t^{\bullet}_{21} - p_J t^{\bullet}_{22}), \tag{11}$$

$$C_J^{\bullet} = -\Delta_J^{-1} r'_J r'_{J+1}(q_{J+1} t^{\bullet}_{11} + t^{\bullet}_{12} - q_J q_{J+1} t^{\bullet}_{21} - q_J t^{\bullet}_{22}), \tag{12}$$

$$D_J^{\bullet} = -\Delta_J^{-1} r'_J r_{J+1}(p_{J+1} t^{\bullet}_{11} + t^{\bullet}_{12} - q_J p_{J+1} t^{\bullet}_{21} - q_J t^{\bullet}_{22}). \tag{13}$$

In equations (10)–(13) dotted quantities are σ-type matrices (see equation (3)). It is seen that the ratios p_J and q_J of elements in columns of $M_{\bullet J}$ determine these matrices, apart from the normalization which is determined by the choice of the factors r_J and r'_J.

It will be convenient to give here another transformation of $t_{\bullet}^{\bullet}$, namely

$$\hat{t}_{\bullet}^{\bullet} = M_{\bullet J+1}^{-1}\, t_{\bullet}^{\bullet}\, M_{\bullet J} \equiv \begin{bmatrix} \bar{A}_J^{\bullet} & \bar{B}_J^{\bullet} \\ \bar{C}_J^{\bullet} & \bar{D}_J^{\bullet} \end{bmatrix}, \tag{14}$$

with

$$\bar{A}^{\bullet}_J = \Delta^{-1}_{J+1} r_{J+1} r'_J (q_J t^{\bullet}{}_{11} + t^{\bullet}{}_{12} - p_{J+1} q_J t^{\bullet}{}_{21} - p_{J+1} t^{\bullet}{}_{22}), \tag{15}$$

$$\bar{B}^{\bullet}_J = \Delta^{-1}_{J+1} r_{J+1} r_J (p_J t^{\bullet}{}_{11} + t^{\bullet}{}_{12} - p_{J+1} p_J t^{\bullet}{}_{21} - p_{J+1} t^{\bullet}{}_{22}), \tag{16}$$

$$\bar{C}^{\bullet}_J = -\Delta^{-1}_{J+1} r'_{J+1} r'_J (q_J t^{\bullet}{}_{11} + t^{\bullet}{}_{12} - q_{J+1} q_J t^{\bullet}{}_{21} - q_{J+1} t^{\bullet}{}_{22}), \tag{17}$$

$$\bar{D}^{\bullet}_J = -\Delta^{-1}_{J+1} r'_{J+1} r_J (p_J t^{\bullet}{}_{11} + t^{\bullet}{}_{12} - q_{J+1} p_J t^{\bullet}{}_{21} - q_{J+1} t^{\bullet}{}_{22}). \tag{18}$$

The transfer matrix is also unchanged by the replacement $t^{\bullet}_{\bullet} \to \bar{t}^{\bullet}_{\bullet}$, if the condition (5) is satisfied.

(c) Action of transfer matrix on vectors

Let $\phi^{\bullet}_J$ be a two-component vector on which the σ matrices act,

$$\phi^{\bullet}_J = \begin{bmatrix} \phi_{1J} \\ \phi_{2J} \end{bmatrix}, \tag{19}$$

and consider the action of the transfer matrix T on the cross-product vector

$$\psi = \phi^{\bullet}_1 \otimes \phi^{\bullet}_2 \otimes \ldots \otimes \phi^{\bullet}_N. \tag{20}$$

From now on the superscript σ is omitted from the cross-product symbol where it is clear what matrices are referred to. We have

$$T\psi = \mathrm{tr}_{\rho}\{(\bar{t}^{\bullet}_{\bullet 1} \phi^{\bullet}_1) \otimes (\bar{t}^{\bullet}_{\bullet 2} \phi^{\bullet}_2) \otimes \ldots \otimes (\bar{t}^{\bullet}_{\bullet N} \phi^{\bullet}_N)\}, \tag{21}$$

where the quantities in parentheses are the 2×2 matrices

$$\bar{t}^{\bullet}_{\bullet J} \phi^{\bullet}_J = \begin{bmatrix} A^{\bullet}_J \phi^{\bullet}_J & B^{\bullet}_J \phi^{\bullet}_J \\ C^{\bullet}_J \phi^{\bullet}_J & D^{\bullet}_J \phi^{\bullet}_J \end{bmatrix}. \tag{22}$$

Hence in general $T\psi$ is a sum of 2^N cross-product vectors of the form (20). This number can be reduced to a sum of two such terms if we choose the parameters p_J (or q_J) such that one of the corner elements in the matrix (22) vanishes for all J's, say,

$$B^{\bullet}_J \phi^{\bullet}_J = 0. \tag{23}$$

For this to hold the p_J's should be such that

$$\det B^{\bullet}_J = 0 \tag{24}$$

and $\phi^{\bullet}_J$ should be an eigenvector $B^{\bullet}_J$ belonging to eigenvalue zero, that is,

$$B^{\bullet}_J \equiv \begin{bmatrix} B^{11}_J & B^{12}_J \\ B^{21}_J & B^{22}_J \end{bmatrix}, \qquad \det B^{\bullet}_J = 0, \tag{25}$$

$$\phi^{\bullet}_J \sim \begin{bmatrix} B^{12}_J \\ -B^{11}_J \end{bmatrix}. \tag{26}$$

From equation (11), the relation (24) is a polynomial in p_J and p_{J+1} from which p_{J+1} can be obtained as a function of p_J. Hence working from a given p_1 all other p_J's can be constructed step by step. At each step a choice of roots is needed. The condition (5) can be used to determine p_1 or, as is more convenient, it may be looked upon as a mild restriction on the values of the parameters $t^a{}_b$. More on this point later.

We have thus shown that independently of q_J it is always possible to find parameters p_J and vectors $\phi^{\bullet}_J$ such that the vector ψ given by equation (20) becomes a sum of two direct-product vectors under the action of the transfer matrix. That is,

$$T\psi = \xi + \eta, \tag{27}$$

with

$$\xi = \xi^{\bullet}_1 \otimes \xi^{\bullet}_2 \otimes \dots \otimes \xi^{\bullet}_N, \tag{28}$$

$$\eta = \eta^{\bullet}_1 \otimes \eta^{\bullet}_2 \otimes \dots \otimes \eta^{\bullet}_N, \tag{29}$$

$$\xi^{\bullet}_J = A^{\bullet}_J \phi^{\bullet}_J, \qquad \eta^{\bullet}_J = D^{\bullet}_J \phi^{\bullet}_J. \tag{30}$$

In fact, with

$$\zeta^{\bullet}_J = C^{\bullet}_J \phi^{\bullet}_J, \tag{31}$$

we have

$$T\psi = \mathrm{tr}\begin{bmatrix} \xi & 0 \\ \zeta & \eta \end{bmatrix}, \tag{32}$$

where

$$\begin{aligned} \zeta = {} & \zeta^{\bullet}_1 \otimes \xi^{\bullet}_2 \otimes \xi^{\bullet}_3 \otimes \dots \otimes \xi^{\bullet}_N \\ & + \eta^{\bullet}_1 \otimes \zeta^{\bullet}_2 \otimes \xi^{\bullet}_3 \otimes \dots \otimes \xi^{\bullet}_N \\ & + \eta^{\bullet}_1 \otimes \eta^{\bullet}_2 \otimes \zeta^{\bullet}_3 \otimes \xi^{\bullet}_4 \otimes \dots \otimes \xi^{\bullet}_N \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \eta^{\bullet}_1 \otimes \eta^{\bullet}_2 \otimes \dots \otimes \eta^{\bullet}_{N-1} \otimes \zeta^{\bullet}_N . \end{aligned} \tag{33}$$

This term, of course, disappears upon taking the trace in (32). Nevertheless it is a useful form for generating other vectors in the next section.

3. Eight-vertex Model Without External Fields

The eight-vertex model without external fields is characterized by the special form (1). If we introduce the parameters

$$a = t_4 + t_3, \quad b = t_4 - t_3, \quad c = t_1 + t_2, \quad d = t_1 - t_2, \tag{34}$$

the quantities occurring in equations (10)–(13) and (15)–(18) are given by

$$t^{\bullet}{}_{11} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad t^{\bullet}{}_{12} = \begin{bmatrix} 0 & d \\ c & 0 \end{bmatrix}, \quad t^{\bullet}{}_{21} = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}, \quad t^{\bullet}{}_{22} = \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}. \tag{35}$$

(a) *Polynomial identities*

The basic polynomial relation for this model is (equations (35) and (11))

$$P_J(p_J, p_{J+1}) \equiv \det B_J^{\bullet}$$
$$= ab(p_J^2+p_{J+1}^2)-cd(1+p_J^2p_{J+1}^2)-p_Jp_{J+1}(a^2+b^2-c^2-d^2) = 0. \quad (36)$$

It gives us the two identities

$$P_J-P_{J+1} = 0, \qquad p_{J+2}P_J-p_JP_{J+1} = 0, \quad (37a, b)$$

and if the successive roots are chosen so that

$$p_J \neq p_{J+2}, \quad (38)$$

these identities respectively imply that

$$(p_J+p_{J+2})(ab-cdp_{J+1}^2)-(a^2+b^2-c^2-d^2)p_{J+1} = 0, \quad (39a)$$

$$(cd-abp_{J+1}^2)+p_Jp_{J+2}(ab-cdp_{J+1}^2) = 0. \quad (39b)$$

If the p_J's are chosen to satisfy the relations (36) and (38) then

$$\phi_J^{\bullet} = n_J\begin{bmatrix} d-cp_Jp_{J+1} \\ -ap_{J+1}+bp_J \end{bmatrix}, \quad (40)$$

where n_J is some normalization.

The vector ψ defined by equations (20) and (40) is such that equations (27)–(30) hold. Furthermore, for this model

$$\xi_J^{\bullet} = A_J^{\bullet}\phi_J^{\bullet} = a_J\phi_{J-1}^{\bullet}, \quad (41)$$

$$\eta_J^{\bullet} = D_J^{\bullet}\phi_J^{\bullet} = d_{J+1}\phi_{J+1}^{\bullet}, \quad (42)$$

with

$$a_J = \frac{r_J\Delta_J}{r_{J+1}\Delta_{J+1}}\frac{n_J}{n_{J-1}}\frac{ad-bcp_J^2}{d-cp_Jp_{J-1}}, \quad (43)$$

$$d_{J+1} = \frac{r_{J+1}}{r_J}\frac{n_J}{n_{J+1}}\frac{bd-acp_{J+1}^2}{d-cp_{J+1}p_{J+2}}. \quad (44)$$

These relations are derived from equations (10), (11) and (13) using (35) and (40). From equations (39) one can verify the identities

$$\frac{ad-bcp_J^2}{d-cp_Jp_{J-1}} = -\frac{(b^2-d^2)p_J-(ab-cdp_J^2)p_{J+1}}{ap_J-bp_{J-1}}, \quad (45)$$

$$\frac{bd-acp_{J+1}^2}{d-cp_{J+1}p_{J+2}} = -\frac{(d^2-a^2)p_{J+1}+(ab-cdp_{J+1}^2)p_J}{ap_{J+2}-bp_{J+1}}. \quad (46)$$

(b) Some eigenvectors of the transfer matrix

Equations (41) and (42) have the important consequence that we can write

$$\xi = (\prod a_J)\psi_{-1}, \qquad \eta = (\prod d_{J+1})\psi_{+1}, \tag{47}$$

where ψ_{-1} is obtained from ψ by replacing $\phi_J^{\bullet}$ by $\phi_{J-1}^{\bullet}$ and ψ_{+1} is obtained from ψ by replacing $\phi_J^{\bullet}$ by $\phi_{J+1}^{\bullet}$.

The periodicity condition (5) can also be satisfied if there is a sub-periodicity

$$M_{\bullet L+1} = M_{\bullet 1}, \qquad N = \mathscr{I} \times L. \tag{48}$$

where $\mathscr{I}$ is an integer. This type of sequence occurs in the work of Baxter (1973*a*). Assume such a sub-periodicity here, and let ψ_j be the vector obtained from ψ by the replacement $\phi_J^{\bullet} \to \phi_{J+j}^{\bullet}$, in which case

$$\psi_{j+L} = \psi_j. \tag{49}$$

Now, using an Lth root of unity, ω, we can form an eigenvector of the transfer matrix

$$\Psi = \sum_{j=1}^{L} \omega^j \psi_j. \tag{50}$$

We have

$$T\Psi = [\omega(\prod a_J)+\omega^{-1}(\prod d_{J+1})]\Psi. \tag{51}$$

There are an infinite number of such vectors corresponding to the infinite number of values of p_1 with which we can start to form the set of p_J. In addition, different choices of ω can be made. However, not all of these vectors can be linearly independent, and Baxter (1973*a*, Section 7) has shown that the maximum possible number of independent vectors is $2N$. At any rate we have exhibited a class of vectors here which go over into each other under the action of the transfer matrix. These vectors may be said to span a subspace $\mathscr{F}_0$ in the space $\mathscr{F}$ of 2^N component vectors on which the transfer matrix acts. The subspace $\mathscr{F}_0$ is invariant in the sense

$$T\mathscr{F}_0 \subseteq \mathscr{F}_0. \tag{52}$$

(c) Subspaces invariant under action of transfer matrix

We will now show that the space $\mathscr{F}$ introduced above contains subspaces $\mathscr{F}_n$ such that

$$T\mathscr{F}_n \subseteq \mathscr{F}_n, \qquad n = 0,1,2,\ldots,N. \tag{53}$$

The work of the previous subsection is independent of the choice of q_J. We now use the invariance property of the transfer matrix associated with equations (14)–(18). From the relations (35) and (17) we have

$$\det \bar{C}_J^{\bullet} \equiv P_J(q_J, q_{J+1}) = 0, \tag{54}$$

where P_J is defined by equation (36). Using condition (5), the discussion of the previous subsection can be repeated. The set q_J is of the same type as the set p_J

but $q_J \neq p_J$ in order that $M_{\bullet J}$ be nonsingular. In place of $\phi_J^{\bullet}$ we take a new vector $\tau_J^{\bullet}$ which is the eigenvector belonging to the zero eigenvalue of $\bar{C}_J^{\bullet}$. Then

$$\bar{C}_J^{\bullet}\tau_J^{\bullet} = 0, \qquad \tau_J^{\bullet} = n_J'\begin{bmatrix} d - cq_J q_{J+1} \\ -aq_J + bq_{J+1}\end{bmatrix}, \tag{55}$$

$$\bar{A}_J^{\bullet}\tau_J^{\bullet} = a_J'\tau_{J-1}^{\bullet}, \qquad \bar{D}_J^{\bullet}\tau_J^{\bullet} = d_J'\tau_{J+1}^{\bullet}, \tag{56}$$

with

$$a_J' = \frac{r_J' n_J'}{r_{J+1}' n_{J-1}'}\frac{bd - acq_J^2}{d - cq_J q_{J-1}}, \tag{57}$$

$$d_J' = \frac{r_{J+1}' \Delta_J n_J'}{r_J' \Delta_{J+1} n_{J+1}'}\frac{ad - bcq_{J+1}^2}{d - cq_{J+1} q_{J+2}}, \tag{58}$$

where identities similar to (45) and (46) have been used in establishing equations (56)–(58).

With the help of the above equations, following the previous arguments we establish the existence of the subspace $\mathscr{F}_N$ in which the vectors are of the form (subscripts in decreasing order)

$$\psi = \tau_N^{\bullet} \otimes \tau_{N-1}^{\bullet} \otimes \ldots \otimes \tau_2^{\bullet} \otimes \tau_1^{\bullet}. \tag{59}$$

Other invariant subspaces are obtained by considering cross-product vectors which involve both $\phi_J^{\bullet}$ and $\tau_J^{\bullet}$. The construction depends crucially upon the following circumstance: Because of the special nature of the matrices $t^{\bullet}_{\alpha\beta}$ for this model (equations (35)), we have

$$\bar{B}_J^{\bullet} = (\Delta_J/\Delta_{J+1})(B_J^{\bullet} - I^{\bullet}\,\mathrm{tr}_\sigma B_J^{\bullet}), \tag{60}$$

$$C_J^{\bullet} = (\Delta_{J+1}/\Delta_J)(\bar{C}_J^{\bullet} - I^{\bullet}\,\mathrm{tr}_\sigma \bar{C}_J^{\bullet}), \tag{61}$$

where $I^{\bullet} = \sigma^4$ is the unit matrix. Matrices on the right-hand sides annihilate the eigenvectors belonging to the nonzero eigenvalues of $B_J^{\bullet}$ and $\bar{C}_J^{\bullet}$ respectively, and hence when acting on any arbitrary vector they respectively produce multiples of $\phi_J^{\bullet}$ and $\tau_J^{\bullet}$. In particular

$$\bar{B}_J^{\bullet}\tau_J^{\bullet} = b_{J+1}'\phi_J^{\bullet}, \tag{62}$$

$$b_{J+1}' = \frac{r_{J+1} r_J n_J'}{\Delta_{J+1} n_J}\frac{(ap_J - bp_{J+1})(d - cq_J q_{J+1}) - (aq_J - bq_{J+1})(d - cp_J p_{J+1})}{d - cp_J p_{J+1}} \tag{63}$$

and

$$\zeta_J^{\bullet} = C_J^{\bullet}\phi_J^{\bullet} = c_J\tau_J^{\bullet}, \tag{64}$$

$$c_J = -\frac{r_{J+1}' r_J' n_J}{\Delta_J n_J'}\frac{(aq_{J+1} - bq_J)(d - cp_J p_{J+1}) - (ap_{J+1} - bp_J)(d - cq_J q_{J+1})}{d - cq_J q_{J+1}}. \tag{65}$$

Collecting the various results,

$$\hat{t}^{\bullet}_{J}\,\phi^{\bullet}_{J} = \begin{bmatrix} a_J\,\phi^{\bullet}_{J-1} & 0 \\ c_J\,\tau^{\bullet}_J & d_{J+1}\,\phi^{\bullet}_{J+1} \end{bmatrix}, \tag{66}$$

$$\hat{t}^{\bullet}_{J}\,\tau^{\bullet}_{J} = \begin{bmatrix} a'_J\,\tau^{\bullet}_{J-1} & b'_{J+1}\,\phi^{\bullet}_J \\ 0 & d'_{J+1}\,\tau^{\bullet}_{J+1} \end{bmatrix}. \tag{67}$$

Now consider the action of the transfer matrix on one of the vectors ζ_j which occur on the right-hand side of equation (33). From the relations (41), (42) and (64) the typical form is

$$\zeta_j = \phi^{\bullet}_2 \otimes \phi^{\bullet}_3 \otimes \ldots \otimes \phi^{\bullet}_j \otimes \tau^{\bullet}_j \otimes \phi^{\bullet}_j \otimes \ldots \otimes \phi^{\bullet}_{N-1}\,. \tag{68}$$

In the expression for $T\zeta_j$ we can insert the matrices $M_{\bullet j}$ appropriately so that $\phi^{\bullet}_l \to \hat{t}^{\bullet}\,\phi^{\bullet}_l$ and $\tau^{\bullet}_j \to \hat{t}^{\bullet}\,\tau^{\bullet}_j$ provided the $M_{\bullet J}$ now satisfy the condition

$$M_{\bullet N} = M_{\bullet 2}\,. \tag{69}$$

Writing out the trace with matrices (66) and (67) it can be verified that the vectors appearing in the diagonal elements are linear combinations of vectors ζ_j. Thus

$$T\zeta_j = \sum_k a_{jk}\,\zeta_k\,, \tag{70}$$

where a_{jk} are certain coefficients involving scalar quantities. Thus we have a subspace $\mathscr{F}_1$ that is invariant under the action of the transfer matrices. It is characterized by the form of vectors (68) in which only one $\tau^{\bullet}_j$ occurs in the cross product, preceded and followed by a $\phi^{\bullet}_j$, and by the condition (69). Interchanging the roles of $\phi^{\bullet}_j$ and $\tau^{\bullet}_j$, taking into account the descending sequence (59) of $\tau^{\bullet}_j$, we find the subspace $\mathscr{F}_{N-1}$.

In general the vectors in subspace $\mathscr{F}_n$ are cross products of n $\tau^{\bullet}_j$ vectors and $N-n$ $\phi^{\bullet}_j$ vectors arranged according to the rule that a $\tau^{\bullet}_j$ can be followed by a $\tau^{\bullet}_{j-1}$ or a $\phi^{\bullet}_j$ in the cross product, while a $\phi^{\bullet}_j$ can be followed by a $\tau^{\bullet}_j$ or a $\phi^{\bullet}_{j+1}$. If the $M_{\bullet j}$'s are chosen so that

$$M_{\bullet 1} = M_{\bullet N-2n+1} \tag{71}$$

or if

$$(p_1, q_1) = (p_{L+1}, q_{L+1})\,, \qquad \text{with} \quad N-2n = \mathscr{I}\times L\,, \tag{72}$$

$\mathscr{I}$ being an integer, it can be verified that under the action of the transfer matrix such a vector goes into a linear combination of vectors of the same type. Hence equation (53) and the statement at the beginning of this subsection are established.

(d) Reduction to a generalized ice-type model

The invariant subspaces of the previous subsection were obtained by Baxter (1973*b*) as families of vectors, by first reducing the problem to an Ising-like form and then noting that it satisfied an ice-type condition. The equations were interpreted in terms of some new vertex diagrams where the ice-type condition is expressed by saying that at each vertex 'there are two arrows pointing into the vertex and two arrows pointing out ... Thus the number n of down arrows in a row of vertical bonds is the same for each row of the lattice.' The number n can be identified with the subscript n for the subspace $\mathscr{F}_n$. We shall not go into the diagrammatic interpretation here.

The main point of Baxter's reduction is that one converts the eigenvalue problem of a $2^N \times 2^N$ transfer matrix to sub-problems in each invariant subspace, where they can be further looked upon as (integral–summation) equations for functions defined on integers. This is a direct consequence of applying adequate notation to express the results of the previous subsection. The notation was introduced by Baxter (1973*b*).

Define a two-index symbol $\Phi^{\bullet}_{l,l'}$ for $l' = l \pm 1$ by

$$\phi^{\bullet}_l = \Phi^{\bullet}_{l,l+1}, \qquad \tau^{\bullet}_l = \Phi^{\bullet}_{l+1,l}. \tag{73}$$

The cross-product vector ψ belonging to any of the subspaces of the previous subsection is fully specified then by integers l_i ($i = 1, 2, \ldots, N+1$) such that

$$l_{i+1} = l_i \pm 1. \tag{74}$$

We have

$$\psi \equiv \psi(l_1, l_2, \ldots, l_{N+1}) = \Phi^{\bullet}_{l_1 l_2} \otimes \Phi^{\bullet}_{l_2 l_3} \otimes \ldots \otimes \Phi^{\bullet}_{l_N l_{N+1}}, \tag{75}$$

$$T\psi(l_1, l_2, \ldots, l_{N+1}) = \sum_m \prod_{J=1}^{N} W(m_J\, m_{J+1} \,|\, l_J\, l_{J+1})\, \psi(m_1, m_2, \ldots, m_{N+1}), \tag{76}$$

where the summation in (76) is over sets of integers m such that

$$m_J = l_J \pm 1. \tag{77}$$

Each pair (l_1, l_2) gives rise to a pair (m_1, m_2) with $m_1 = l_1 \pm 1$, $m_2 = l_2 \pm 1$. The function W is defined for the eight possible values of its arguments by equations (66) and (67) (four values each):

$$W(l-1, l \,|\, l, l+1) = a_l, \qquad W(l-1, l+2 \,|\, l, l+1) = 0, \tag{78a, b}$$

$$W(l+1, l \,|\, l, l+1) = c_l, \qquad W(l, l+1 \,|\, l-1, l) = d_l, \tag{78c, d}$$

$$W(l, l-1 \,|\, l+1, l) = a'_l, \qquad W(l-1, l \,|\, l, l-1) = c'_l, \tag{78e, f}$$

$$W(l+2, l-1 \,|\, l+1, l) = 0, \qquad W(l+1, l \,|\, l, l-1) = d'_l. \tag{78g, h}$$

The vanishing weights (78b) and (78g) ensure that the vector remains in the same subspace. We can make use of this to specify the vector ψ by giving a smaller set of integers. Let x_i ($i = 1, 2, \ldots, n$) be the positions (integers) where $\tau^{\bullet}_i$ occurs in the cross product. We have $1 \leqslant x_1 < x_2 < \ldots < x_n \leqslant N$. Given the initial value of $l\,(l_1)$, all other l_i values in equation (75) can be calculated using the x_i and the definitions (73). We can therefore use l and the set X of the values x_i to specify the vector ψ belonging to the subspace $\mathscr{F}_n$. Equation (76) now takes the form

$$T\psi(l, X) = \sum_{m,Y} \prod_{J=1}^{N} W_J(mY \,|\, lX)\, \psi(mY), \tag{79}$$

with

$$W_J(mY\,|\,lX) \equiv W(m_J m_{J+1}\,|\,l_J l_{J+1}), \tag{80}$$

where l_J and m_J are calculated from l and X and m and Y respectively.

In the subspace $\mathscr{F}_n$ we take the linear combinations

$$\Psi = \sum_{l=1}^{L} \sum_{X} f(l, X)\,\psi(l, X), \tag{81}$$

where L is determined by the periodicity condition $N-2n = \mathscr{I}\times L$ of equation (72). Note that the eigenvalue equation for the transfer matrix

$$T\Psi = \Lambda\Psi \tag{82}$$

is satisfied in the subspace $\mathscr{F}_n$ if the coefficients $f(l, X)$ satisfy the equation

$$\Lambda f(l, X) = \sum_{m,Y}\left(\prod_{J=1}^{N} W_J(mY\,|\,lX)\right) f(m, Y). \tag{83}$$

This may be looked upon as a transformation of the eigenvalue equation from the space $\mathscr{F}$ of cross-product vectors to the space of functions on the integers $1 \leqslant l \leqslant L$ and x_i $(1 \leqslant x_1 < x_2 < \ldots < x_n)$. Equation (83) can be further put in the canonical form of ice-type models (Lieb and Wu 1972, Section IV) by noting that on the right-hand side $m = l \pm 1$ and

$$\Lambda f(l, X) = \sum_{Y} D_{\mathrm{L}}(l, X, Y) f(l+1, Y) + \sum_{Y} D_{\mathrm{R}}(l, X, Y) f(l-1, Y), \tag{84}$$

with

$$D_L(l, X, Y) = \prod_{J=1}^{N} W_J(l+1\; Y\,|\,lX), \tag{85a}$$

$$D_{\mathrm{R}}(l, X, Y) = \prod_{J=1}^{N} W_J(l-1\; Y\,|\,lX). \tag{85b}$$

In equation (84) the sum over y_i is restricted to the ranges

$$1 \leqslant y_1 \leqslant x_1,\; x_1 \leqslant y_2 \leqslant x_2,\; \ldots,\; x_{n-1} \leqslant y_n \leqslant x_n$$

in the first term and to the ranges

$$x_1 \leqslant y_1 \leqslant x_2,\; x_2 \leqslant y_2 \leqslant x_3,\; \ldots,\; x_n \leqslant y_n \leqslant N$$

in the second. By introducing $x_0 = 0$, $x_{n+1} = N+1$ we can describe these intervals respectively in the form:

$$x_{j-1} \leqslant y_j \leqslant x_j; \quad x_j \leqslant y_j \leqslant x_{j+1}; \quad \text{with} \quad x_j < x_{j+1}$$

and no two y's equal. The products of N factors in D_{L} and D_{R} can then be expressed as products of n factors defined in these intervals. The new functions can be labelled by values of l_J at the Jth position $(J = 1, 2, \ldots, N+1)$ within the interval which contains y_i and the variables which define the interval. Thus

$$D_{\mathrm{L}}(l, X, Y) = \prod_{j=1}^{n} U(l+1-2j \mid x_{j-1}, y_j, x_j), \tag{86a}$$

$$D_{\mathrm{R}}(l, X, Y) = \prod_{j=1}^{n} U(l-1-2j \mid x_j, y_j, x_{j+1}), \tag{86b}$$

where within the respective intervals $l_j = l \pm 1 - 2j + J$. The precise form of U can be simplified by appropriate choice of the two normalizations n_J and n'_J and the two ratios r_J and r'_J, occurring in the expressions for weights (43), (44), (57), (58), (63), (65) and (78).

This reduction of the problem to an ice-type one suggests that a generalized Bethe ansatz for $f(l, X)$ may solve equation (84). Baxter (1973*c*) has shown that this is indeed the case, and the final equations obtained for the eigenvalues are the same as were obtained by Baxter (1972*a*) from a functional matrix equation.

4. Functional Matrix Equation

The functional matrix equation was derived by Baxter (1972*a*) as a generalization of some ice-model results. Parameterization of vertex weights in terms of elliptic functions provides the basic motivation for such a generalization. It is possible that the method works because of the basic ice-type structure of the problem. It is not clear what types of transfer matrices satisfy such functional equations and what types of functional matrix equations are helpful in solving the eigenvalue problem.

In a limited way the problem may be approached as follows. We have studied the action of the transfer matrix on certain vectors and seen that a demand for simplification expressed by equation (27) has led to several useful results, including the construction of invariant subspaces. We now propose to study the action of the transfer matrix on other matrices and ask for similar simplification but with the added requirement that these matrices commute with the transfer matrix. The commutation requirement is a natural one: we want the matrices to have common eigenvectors with the transfer matrix. This proves to be sufficient for deriving the functional matrix equation for the eight-vertex model. For the more general problem we only have certain basic equations whose solution may help in deciding whether such functional equations can be derived in other cases.

(*a*) *Action of transfer matrix on matrices*

The matrices we consider are defined by analogy with the expression (4) for the transfer matrix itself. Let $q^{\bullet}$ be objects defined by

$$q^{\bullet} = q^{a}\sigma^{a} \equiv q^{\alpha\beta}{}_{mn}, \tag{87}$$

where q^a are $L \times L$ matrices, with $\alpha, \beta = 1, 2$ and $m, n = 1, 2, ..., L$. Then the matrices Q on which the action of the transfer matrices is to be considered are

$$Q = \mathrm{tr}_L\Big(q^{\bullet} \overset{\sigma}{\otimes} q^{\bullet} \overset{\sigma}{\otimes} \dots \overset{\sigma}{\otimes} q^{\bullet}\Big), \qquad (N \text{ terms}). \tag{88}$$

Using the relation

$$(\mathrm{tr}_m A)(\mathrm{tr}_n B) = \mathrm{tr}_{mn}(A \otimes B), \tag{89}$$

we find

$$TQ = \mathrm{tr}_{2L}\left((t^{\bullet}_{\cdot} q^{\bullet}) \overset{\sigma}{\otimes} (t^{\bullet}_{\cdot} q^{\bullet}) \overset{\sigma}{\otimes} \dots \overset{\sigma}{\otimes} (t^{\bullet}_{\cdot} q^{\bullet})\right), \tag{90}$$

with

$$(t^{\bullet}_{\cdot} q^{\bullet}) = t^{\alpha\beta}{}_{\lambda\mu} q^{\beta\gamma}{}_{mn} \equiv \begin{bmatrix} t^{\bullet}{}_{11} q^{\bullet} & t^{\bullet}{}_{12} q^{\bullet} \\ t^{\bullet}{}_{21} q^{\bullet} & t^{\bullet}{}_{22} q^{\bullet} \end{bmatrix}. \tag{91}$$

Equation (91) represents an ordinary product in σ-type indices and a cross product between the $L \times L$ matrices of q^a and 2×2 matrices ρ_a. The trace in equation (90) is over the resulting $2L \times 2L$ matrix. This trace remains unchanged under a variety of operations (cf. Section 2*b*), the simplest of which is a similarity transformation of all elements $(t^{\bullet}_{\cdot} q^{\bullet})$ in (90) by a single matrix P. If $P^{-1}(t^{\bullet}_{\cdot} q^{\bullet})P$ is partitioned as suggested by equation (91) and P is chosen so that an $L \times L$ corner block vanishes identically, then the trace in equation (90) decomposes into a sum of two traces over L-dimensional matrices. It is sufficient to consider the form

$$P = \begin{bmatrix} I_L & p \\ \mathcal{O}_L & I_L \end{bmatrix}, \tag{92}$$

where p is an arbitrary matrix, I_L the unit matrix and $\mathcal{O}_L$ the null matrix. From equations (91) and (92)

$$P^{-1}(t^{\bullet}_{\cdot} q^{\bullet})P = \begin{bmatrix} q^{\bullet}_1 & q^{\bullet}_3 \\ q^{\bullet}_4 & q^{\bullet}_2 \end{bmatrix}. \tag{93}$$

We need only the blocks

$$q^{\bullet}_1 = t^{\bullet}{}_{11} q^{\bullet} - p t^{\bullet}{}_{21} q^{\bullet}, \qquad q^{\bullet}_2 = t^{\bullet}{}_{21} q^{\bullet} p + t^{\bullet}{}_{22} q^{\bullet}, \tag{94}$$

$$q^{\bullet}_3 = t^{\bullet}{}_{11} q^{\bullet} p + t^{\bullet}{}_{12} q^{\bullet} - t^{\bullet}{}_{21} p q^{\bullet} p - t^{\bullet}{}_{22} p q^{\bullet} = \mathcal{O}_L. \tag{95}$$

In these equations the products between $t^{\bullet}$ and $q^{\bullet}$ are in 2×2 matrix indices and those between p and $q^{\bullet}$ are between $L \times L$ matrices. The order of factors is important. Both p and $q^{\bullet}$ are to be chosen to satisfy (95). Since this equation is linear and homogeneous in $q^{\bullet}$, one may choose p to simplify it and make it solvable. The 2×2 indices decouple from the $L \times L$ indices if p is diagonal,

$$p = p_m \delta_{mn}, \qquad m, n = 1, 2, \dots, L. \tag{96}$$

Equation (95) now becomes

$$q^{\bullet}_3 \equiv \mathcal{B}^{\bullet}(mn) q^{\bullet}(mn) = 0, \tag{97}$$

with

$$\mathcal{B}^{\bullet}(mn) = (p_n t^{\bullet}{}_{11} + t^{\bullet}{}_{12} - p_m p_n t^{\bullet}{}_{21} - p_m t^{\bullet}{}_{22}). \tag{98}$$

It is to be satisfied for all pairs (m, n). The trivial solution $q^{\bullet}(mn) = 0$ can be used where convenient. Nontrivial solutions can occur for those (m, n) for which p_m and

p_n satisfy the polynomial relation

$$\det[\mathscr{B}^{\bullet}(mn)] \equiv P(mn) = 0. \tag{99}$$

In that case equation (97) determines the ratio of elements in the two columns of the 2×2 matrix $q^{\bullet}(mn)$ and one can write

$$q^{\bullet}(mn) = \begin{bmatrix} \mathscr{B}^{12}(mn)\tau_+ & \mathscr{B}^{12}(mn)\tau_- \\ -\mathscr{B}^{11}(mn)\tau_+ & -\mathscr{B}^{11}(mn)\tau_- \end{bmatrix}, \tag{100}$$

where the arbitrary factors $\tau_\pm$ can also depend on t and m, n. This relation holds only for some limited number of pairs (m, n).

We have thus shown that it is always possible to construct a matrix Q from equations (88), (99) and (100) such that

$$TQ = Q_1 + Q_2, \tag{101}$$

where the matrices Q_i are of the form (88) with $q^{\bullet}$ replaced by $q_i^{\bullet}$, and from equations (94) and (96)

$$q_1^{\bullet}(mn) = (t^{\bullet}_{11} - p_m t^{\bullet}_{21})q^{\bullet}(mn), \tag{102a}$$

$$q_2^{\bullet}(mn) = (p_n t^{\bullet}_{21} + t^{\bullet}_{22})q^{\bullet}(mn). \tag{102b}$$

The product $\bar{Q}T$ can be considered in a similar way. Instead of $t^{\bullet}_{\cdot} q^{\bullet}$ we now have $\bar{q}^{\bullet} t^{\bullet}_{\cdot}$. Equation (97) is replaced by

$$\bar{q}^{\bullet}(mn)\mathscr{B}^{\bullet}(mn) = 0, \tag{103}$$

where $\mathscr{B}^{\bullet}(mn)$ is still given by (98) so that the solvability condition is again (99). Equation (103) determines the ratio of elements in rows of $\bar{q}^{\bullet}$, which is

$$\bar{q}^{\bullet} = \begin{bmatrix} \mathscr{B}^{21}\tau'_+ & -\mathscr{B}^{11}\tau'_+ \\ \mathscr{B}^{21}\tau'_- & -\mathscr{B}^{11}\tau'_- \end{bmatrix}. \tag{104}$$

Accordingly, it is always possible to construct a matrix $\bar{Q}$ from equations (88), (99) and (104) such that

$$\bar{Q}T = \bar{Q}_1 + \bar{Q}_2, \tag{105}$$

where the $\bar{Q}_i$ are obtained from equation (88) by replacing $q^{\bullet}$ by $\bar{q}_i^{\bullet}$, and

$$\bar{q}_1^{\bullet}(mn) = \bar{q}^{\bullet}(mn)(t^{\bullet}_{11} - p_m t^{\bullet}_{21}), \tag{106a}$$

$$\bar{q}_2^{\bullet}(mn) = \bar{q}^{\bullet}(mn)(p_n t^{\bullet}_{21} + t^{\bullet}_{22}). \tag{106b}$$

This behaviour is analogous to the action on vectors (Section 2). If we take $n = m \pm 1$, the matrix $\mathscr{B}^{\bullet}(mn)$ reduces to the matrices $B^{\bullet}$ and $\bar{B}^{\bullet}$ of equations (11) and (16). For the eight-vertex model the same polynomial relation is obtained in both cases. This circumstance leads to the disposition of nonzero elements in the $q^{\bullet}$ matrices found in Baxter's work (1972*a*).

(*b*) *Transformation of parameters*

The four quantities $q_i^{\bullet}$ and $\bar{q}_i^{\bullet}$ have been obtained as particular functions of the parameters t ($\equiv t^a{}_b$) of the transfer matrix. Are there certain other values t_i and $\bar{t}_i$ of these parameters in terms of which Q_i and $\bar{Q}_i$ become the same functions of the new parameters as Q and $\bar{Q}$ are of t? Consider one of these cases. If there exists an $L\times L$ matrix $\mathscr{X}_1$ and a scalar function ϕ_1 such that

$$q_1^{\bullet}(t) = \phi_1(t)\mathscr{X}_1 q^{\bullet}(t_1)\mathscr{X}_1^{-1}, \tag{107}$$

then

$$Q_1 = \{\phi_1(t)\}^N Q(t_1). \tag{108}$$

Choosing $\mathscr{X}_1$ to be a diagonal matrix, for each nontrivial pair m, n we have two homogeneous equations for two unknowns $\mathscr{X}_{1m}$ and $\mathscr{X}_{1n}$. The solvability condition is independent of $\phi_1(t)$:

$$R_1(mn) \equiv q_1^{11}(t\,|\,mn)q^{21}(t_1\,|\,mn) - q_1^{21}(t\,|\,mn)q^{11}(t_1\,|\,mn) = 0. \tag{109}$$

Considered as a polynomial in p_m and p_n, this equation should be satisfied by the same set of values of p_m and p_n which satisfy the basic polynomial (99). Hence $R_1(mn)$ can differ by at most a factor of the form $(\alpha+\beta p_m)$ from $P(mn)$, that is,

$$R_1(mn)+(\alpha+\beta p_m)P(mn) = 0. \tag{110}$$

Since we require this to be identically satisfied, we can equate coefficients of different powers of p_m and p_n simultaneously to zero, obtaining equations for linearly occurring unknowns t_1 ($\equiv t_{1b}^{\,a}$), α and β. Depending on the nature of the system $t^a{}_b$, different types of linear systems will be obtained. For the eight-vertex model the system is in fact overdetermined rather than underdetermined, but easily solvable.

Similarly, for the remaining three cases we have

$$R_2(mn) = q_2^{11}(t\,|\,mn)q^{21}(t_2\,|\,mn) - q_2^{21}(t\,|\,mn)q^{11}(t_2\,|\,mn), \tag{111a}$$

$$R_2(mn)+(\alpha+\beta p_n)P(mn) = 0, \tag{111b}$$

$$\bar{R}_i(mn) = \bar{q}_i^{11}(t\,|\,mn)\bar{q}^{21}(\bar{t}_i\,|\,mn) - \bar{q}_i^{21}(t\,|\,mn)\bar{q}^{11}(\bar{t}_i\,|\,mn), \tag{111c}$$

$$\bar{R}_1(mn)+(\alpha+\beta p_m)P(mn) = 0, \tag{111d}$$

$$\bar{R}_2(mn)+(\alpha+\beta p_n)P(mn) = 0. \tag{111e}$$

Note that the arbitrary factors $\tau_\pm$ and $\tau'_\pm$ do not affect these equations. Once the t's are determined by these relations, equation (107) can then be used to find ϕ by considering the elements $n = m = 1$. Here we also have to restrict the factors τ, to be consistent with the structure assumed for $q^{\bullet}$. Finally, it has to be verified that the new t's indeed lead to the same p_m and p_n, if used in the basic equation (99).

It is not clear that the linear systems for the new t's described above will be solvable in every case. Even if they were it may not be very helpful since by themselves they are not sufficient to establish the functional equations.

(c) *Eight-vertex model without fields*

(i) *Solutions*

In this subsection we first give the solutions of the systems described above for the special case of the eight-vertex model without fields. The properties of Q matrices which lead to the functional equation depend upon the choice of (m,n) values for which the $q^{\bullet}$ are taken to be nonvanishing. Following Baxter (1972*a*) we choose

$$m = n = 1, \qquad m = n \pm 1, \qquad \text{and} \qquad m = n = L. \tag{112}$$

The first and last elements are the same—a requirement of periodicity. Alternative choices may be possible but this has not been explored. The possibility of taking nonvanishing elements for both $m = n+1$ and $m = n-1$ is a consequence of the symmetry of the basic polynomial in m and n. It seems that the elements corresponding to one or the other possibility could be set equal to zero without altering the results. Recalling now that

$$\mathscr{B}^{\bullet}(mn) = \begin{bmatrix} ap_m - bp_n & d - cp_m p_n \\ c - dp_m p_n & bp_m - ap_n \end{bmatrix}, \tag{113}$$

with the substitutions (35) in equation (98), we have

$$P(mn) = p_m^2 + p_n^2 - \mu p_m p_n - \nu(1 + p_m^2 p_n^2) = 0, \tag{114}$$

where

$$\mu = (a^2+b^2-c^2-d^2)/ab, \qquad \nu = cd/ab. \tag{115}$$

For $m = n = 1$ we have a quartic

$$p_1^4 - \{(2-\mu)/\nu\}p_1^2 + 1 = 0. \tag{116}$$

This equation can always be solved to yield an initial value of p_1 which can then be substituted into (114) to successively solve for all other p_m. Because of the symmetry between m and n, $P(m, m+1) = 0$ implies $P(m+1, m) = 0$ and nonzero values of $q^{\bullet}$ are obtained from equation (100) for all (m,n) pairs (112).

Since $P(mn)$ is unchanged for $p_m \to -p_m$ and $p_n \to -p_n$, we have the possibility of taking different signs for $\mathscr{B}^{11}$ in equations (100) and (104), and we shall make use of this in subsection (ii) below. It is of importance when we consider the commutation properties of Q matrices but not for parameter transformations.

From (35)

$$t^{\bullet}_{11} - p_m t^{\bullet}_{21} = \begin{bmatrix} a & -cp_m \\ -dp_m & b \end{bmatrix} \tag{117}$$

and, using this in equation (109),

$$\begin{aligned} R_1(mn) \equiv\ & p_m(-d_1\, ad - d_1(d^2-b^2)) + p_n(a_1\, ad - d_1\, ab) \\ & + p_m^2 p_n(-a_1\, cb + d_1\, cd + c_1(d^2-b^2)) + p_n^2 p_m(c_1\, ab) \\ & + p_m^3(b_1\, cb) + p_m^3 p_n^2(-c_1\, cd) = 0. \end{aligned} \tag{118}$$

From equations (118) and (114), we see that $\alpha = 0$ in (110), and equating coefficients of powers to zero we obtain six equations, two of which are identical. Taking $\beta = -c_1 ab$, they yield the system

$$\begin{bmatrix} d & 0 & 0 & -b \\ 0 & c & -a & 0 \\ 0 & ad & -cd & d^2-b^2 \\ -cb & 0 & a^2-c^2 & cd \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} = 0. \tag{119}$$

The determinant vanishes identically and the solution is immediate.

The working for other transformations is similar and need not be given here. Although all four systems are different, the remarkable result is that

$$t_1 = \bar{t}_2 = (a_1, b_1, c_1, d_1), \qquad t_2 = \bar{t}_1 = (a_2, b_2, c_2, d_2), \tag{120}$$

$$\frac{a_1}{d_1} = \frac{b}{d}, \qquad \frac{b_1}{d_1} = \frac{a}{d}\frac{b^2-d^2}{a^2-c^2}, \quad \frac{c_1}{d_1} = \frac{c}{d}\frac{b^2-d^2}{a^2-c^2}, \tag{121a}$$

$$\frac{a_2}{d_2} = \frac{b}{d}\frac{a^2-d^2}{b^2-c^2}, \quad \frac{b_2}{d_2} = \frac{a}{d}, \qquad \frac{c_2}{d_2} = \frac{c}{d}\frac{a^2-d^2}{b^2-c^2}. \tag{121b}$$

One can verify that $\mu_1 = \mu_2 = \mu$ and $\nu_1 = \nu_2 = \nu$, where μ_i and ν_i are defined in analogy with equations (115). Thus the new parameters in turn lead to the same polynomial (114).

To determine ϕ_1 we take the ratios (121a) as given. Since χ_1 is taken to be diagonal, for $m = n = 1$ equation (107) gives

$$q_1^{\bullet}(t \mid 11) = \phi_1(t) q^{\bullet}(t_1 \mid 11), \tag{122}$$

or using equations (100), (113), (102a) and (117),

$$\phi_1(t)\,\theta_1 = \frac{ad - cbp_1^2}{d_1 - c_1 p_1^2} = \frac{d^2 - b^2 + ab - cdp_1^2}{a_1 - b_1}, \tag{123}$$

with

$$\theta_1 = \tau_+(t_1 \mid 11)/\tau_-(t \mid 11) = \tau_-(t_1 \mid 11)/\tau_+(t \mid 11). \tag{124}$$

Similarly,

$$\phi_2(t)\,\theta_2 = \frac{bd - cap_1^2}{d_2 - c_2 p_1^2} = \frac{d^2 - a^2 + ab - cdp_1^2}{b_2 - a_2}, \tag{125}$$

with

$$\theta_2 = \tau_+(t_2 \mid 11)/\tau_+(t \mid 11) = \tau_-(t_2 \mid 11)/\tau_-(t \mid 11), \tag{126}$$

and

$$\bar{\phi}_1(t)\,\bar{\theta}_1 = \frac{ca - bdp_1^2}{c_2 - d_2 p^2} = \frac{c^2 - b^2 + ab - cdp_1^2}{a_2 - b_2}, \tag{127}$$

$$\bar{\phi}_2(t)\,\bar{\theta}_2 = \frac{cb - adp_1^2}{c_1 - d_1 p_1^2} = \frac{c^2 - a^2 + ab - cdp_1^2}{b_1 - a_1}, \tag{128}$$

with

$$\bar{\theta}_i = \tau'_+(\bar{t}_i \mid 11)/\tau'_+(t \mid 11) = \tau'_-(\bar{t}_i \mid 11)/\tau'_-(t \mid 11). \tag{129}$$

The identity of alternative forms in equations for the ϕ's may be established using the relations (116) and (121). One can choose the τ's in such a way that

$$\phi_1(t) = \bar{\phi}_2(t) \qquad \text{and} \qquad \phi_2(t) = \bar{\phi}_1(t). \tag{130}$$

It should be noted that the choice of the τ's for other values of m, n is still arbitrary.

Now writing Q_{R} for Q and Q_{L} for $\bar{Q}$, we have established that

$$T(t) Q_{\mathrm{R}}(t) = \Phi_1(t)\, Q_{\mathrm{R}}(t_1) + \Phi_2(t)\, Q_{\mathrm{R}}(t_2), \tag{131a}$$

$$Q_{\mathrm{L}}(t)\, T(t) = \Phi_1(t)\, Q_{\mathrm{L}}(\bar{t}_1) + \Phi_2(t)\, Q_{\mathrm{L}}(\bar{t}_2), \tag{131b}$$

with

$$\Phi_i(t) = \{\phi_i(t)\}^N. \tag{132}$$

The last relation follows from equation (108).

(ii) *Commuting Q Matrices and Functional Equation*

For the rest of this subsection the symbol Q is used to denote a different matrix which satisfies the relations:

$$[T(t), Q(t)] = 0, \tag{133a}$$

$$T(t)\, Q(t) = \Phi_1(t)\, Q(t_1) + \Phi_2(t)\, Q(t_2), \tag{133b}$$

$$[Q(t), Q(s)] = 0. \tag{133c}$$

Equation (133c) needs to hold only when the parameters t and s are related in some way.

Let F and G be matrices such that

$$Q = Q_{\mathrm{R}} F = G\, Q_{\mathrm{L}}. \tag{134}$$

Then equations (131a, b) imply equations (133a, b). The relation (133c) is satisfied if

$$Q_{\mathrm{L}}(t)\, Q_{\mathrm{R}}(s) = Q_{\mathrm{L}}(s)\, Q_{\mathrm{R}}(t). \tag{135}$$

Both sides of (135) can be expressed as traces of L^2 dimensional matrices and are equal if there exists an $L^2 \times L^2$ matrix $\mathscr{Y}$ such that

$$\mathscr{Y}\left(q^{\bullet}_{\mathrm{L}}(t) \overset{L}{\otimes} q^{\bullet}_{\mathrm{R}}(s)\right) = \left(q^{\bullet}_{\mathrm{L}}(s) \overset{L}{\otimes} q^{\bullet}_{\mathrm{R}}(t)\right)\mathscr{Y}. \tag{136}$$

Here $q^{\bullet}_{\mathrm{R}}$ is the same as $q^{\bullet}$ given by equation (100) and $q^{\bullet}_{\mathrm{L}}$ is $\bar{q}^{\bullet}$ given by equation (104) with p_m, p_n replaced by $-p_m, -p_n$, as explained below equation (116) in the previous subsection.

The simplest possibility is to take $\mathscr{Y}$ to be diagonal, i.e.

$$\mathscr{Y}_{mm',pp'} = \delta_{mp}\, \delta_{m'p'}\, y_{mm'}. \tag{137}$$

We can also assume that τ_+ (τ'_+) is proportional to τ_- (τ'_-) for all mn, and recall that nonvanishing $q^{\bullet}$ matrices occur only for certain mn values, given by (112).

Then from equations (100) and (104) we find that (136) is equivalent to the equations

$$G(11;11) = 1 = G(LL;LL), \tag{138a}$$

$$y_{1m} = y_{1\,m-1}\,G(1m;1\,m-1), \tag{138b}$$

$$y_{m1} = y_{m-1\,1}\,G(m1;m-1\,1), \tag{138c}$$

$$y_{m,m'} = y_{m-1,m'-1}\,G(mm';m-1,m'-1), \qquad m,m' \geqslant 2, \tag{138d}$$

$$G(mm',nn') = H(mm',nn'\,|\,ts)/H(mm',nn'\,|\,st), \tag{138e}$$

$$H(mm',nn'\,|\,ts) = \{\mathscr{B}^{21}(t\,|\,mn)\,\mathscr{B}^{12}(s\,|\,m'n') - \mathscr{B}^{11}(t\,|\,mn)\,\mathscr{B}^{11}(s\,|\,m'n')\}$$

$$\times\,\tau'_{+}(t\,|\,mn)\,\tau_{+}(s\,|\,m'n'). \tag{138f}$$

We note that

$$G(11;11) \sim \frac{\tau'_{+}(t\,|\,11)\,\tau_{+}(s\,|\,11)}{\tau'_{+}(s\,|\,11)\,\tau_{+}(t\,|\,11)}. \tag{139}$$

When $s = t_1 = \bar{t}_2$ or $s = t_2 = \bar{t}$, the same ratio of τ's can be found from equations (123)–(129), and the first part of (138a) is seen to hold as an identity.

One sets $y_{11} = 1$ and obtains all other elements $y_{mm'}$ by iteration from equations (138 b–f) and (139). The elements in which either m or m' or both are equal to L can be constructed in more than one way and consistency is to be achieved by exploiting the freedom in the choice of τ and τ'. This has not been carried out in detail, but in any case the point has been reached where periodicity conditions, e.g. the last part of equation (138a), have to be studied and recourse to elliptic functions is needed. With this reservation, we have shown that the three equations (133) can be satisfied: that is, one can construct a matrix $Q(t)$ which commutes with the transfer matrix $T(t)$ and the matrices $Q(t_1)$ and $Q(t_2)$ and satisfies the functional equation (133b).

It will be noted that in the present derivation we treat the τ's as functions of parameters, which are then chosen to satisfy the required equations. In Baxter's (1972*a*) work they are taken as constants. Presumably, the use of elliptic functions and the normalization of various matrix elements in his derivation implicitly satisfy the necessary requirements.

5. Eight-vertex Model With External Field

In the presence of an external field the parameter array $t^{a}{}_{b}$ is nondiagonal. It is simpler, however, to give the matrices $t^{\bullet}{}_{\alpha\beta}$ which can be directly interpreted in terms of vertex diagrams (see Baxter 1972*a*, where the matrices are designated $\mathbf{R}(\alpha,\beta)$):

$$t^{\bullet}{}_{11} = \begin{bmatrix} aVH & 0 \\ 0 & bV/H \end{bmatrix}, \qquad t^{\bullet}{}_{12} = \begin{bmatrix} 0 & d \\ c & 0 \end{bmatrix}, \tag{140a}$$

$$t^{\bullet}{}_{21} = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}, \qquad t^{\bullet}{}_{22} = \begin{bmatrix} bH/V & 0 \\ 0 & a/VH \end{bmatrix}, \tag{140b}$$

where V and H represent the vertical and horizontal field effects; for the field-free case $V = H = 1$.

Forming the matrices of equations (10)–(12), we note that

$$\det B_J^{\bullet} = ab(V^{-2}p_J^2 + V^2 p_{J+1}^2) - (a^2 + b^2 - c^2 - d^2)p_J p_{J+1} - cd(1 + p_J^2 p_{J+1}^2). \quad (141)$$

In the absence of vertical fields ($V = 1$) this is the same polynomial as that for the field-free case. That is, the presence of a horizontal field ($H \neq 1$) does not affect the basic polynomial, but it does affect the matrices and the vectors $\phi^{\bullet}$ and $\tau^{\bullet}$. Because of this difference it is no longer possible to obtain the equations (66) and (67) which lead to invariant subspaces in Section 3. No straightforward generalization appears to be possible. Problems in the functional-equation approach are different, but these have not been explored so far.

6. Conclusions

When the matrix Q can be constructed in diagonal form the functional equation (133b) gives the eigenvalues of the transfer matrix. On the other hand, the equations of the generalized ice-model form can be solved to yield the same equations for the eigenvalue. Baxter (1973*c*) has shown this equivalence in detail. Both approaches are made possible and shown to be equivalent because of a series of 'mathematical flukes'. In the present paper we have arrived at several of these flukes in a different way and have shown that some of them, at least, no longer hold for the more general problem of the last section.

It seems quite natural to seek subspaces invariant under the action of the transfer matrix, but the manner in which they were obtained is too finely balanced (cf. equations (66) and (67)) and should not be expected to work for other cases. The formulae of Section 2*b* do not exhaust the possibilities offered by the invariance of the trace. The device of making corner elements zero is too restrictive. Similar remarks apply to the Q-matrix approach. At every step one opts for the simplest alternatives which turn out to be adequate for the particular problem. Generalizations have a tendency to become very rapidly intractable (e.g. try taking nondiagonal matrices for p in equation (92), $\mathscr{X}$ in (107) or $\mathscr{Y}$ in (136)). In a basic sense, constructing invariant subspaces is equivalent to finding matrices that commute with T, but the two approaches described above are not identical in detail in spite of many similarities. Attention should be directed to a better exploitation of the freedom offered by the invariance of the trace.

In conclusion we note three problems which perhaps do not require very radically new methods: (i) to find the dimension of the subspace $\mathscr{F}_n$ and to show that the totality of $\mathscr{F}_n$ spans the whole of $\mathscr{F}$, (ii) to prove by algebraic means that a generalized Bethe ansatz solves equation (84), and (iii) to investigate solutions of the linear systems arising from (110) and (111) for other models to see if commuting Q matrices can be set up by this method.

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Appendix

(*a*) *Parameterization in Terms of Elliptic Functions and Uniformization Theory of Algebraic Functions*

The parameterization of vertex weights was first introduced by Baxter (1972*a*) from considerations of commuting transfer matrices and was later derived from the basic polynomial relation (Baxter 1973*a*). The latter derivation is related to the theory of uniformization of algebraic functions (see e.g. Sansone and Gerretsen 1969; Siegel 1969). This connection is pointed out briefly here since it is part of a much more powerful theory applicable to any polynomial in two variables.

We write the polynomial (36) in terms of new variables $w = p_J$ and $z = p_{J+1}$ in the form

$$P(w,z) = w^2+z^2-\mu wz-\nu(1+w^2z^2)\,, \tag{A1}$$

$$\mu = (a^2+b^2-c^2-d^2)/ab\,, \qquad \nu = cd/ab\,. \tag{A2}$$

These equations determine w as a function of z. The theory of uniformization enables us to find a single complex variable u and two functions of this variable $w = w(u)$ and $z = z(u)$ such that (A1) is satisfied for all u. The polynomial determines these functions. The crucial quantity is the genus p of the polynomial, and it is found as follows. We note that the number n of roots of (A1) is two, say w_1 and w_2, given by

$$w = [\mu z \pm \{\mu^2z^2-4(1-\nu z^2)(z^2-\nu)\}^{\frac{1}{2}}]/2(1-\nu z^2)\,. \tag{A3}$$

The discriminant

$$(w_1-w_2)^2 = \nu(z^4-\rho z^2+1)/(1-\nu z^2)^2\,, \tag{A4}$$

with

$$\rho = (4+4\nu^2-\mu^2)/4\nu\,, \tag{A5}$$

vanishes at four values of z given by

$$z^2 = \tfrac{1}{2}\{\rho\pm(\rho^2-4)^{\frac{1}{2}}\}\,. \tag{A6}$$

Each of these is a ramification point of order 1 for the algebraic function w defined here. The point at infinity is an ordinary point. Hence the ramification number $m = 4$ and the genus p is given by (Sansone and Gerretsen 1969, Section 12.5.5)

$$p = \tfrac{1}{2}m-n+1 = 1\,. \tag{A7}$$

Polynomials of genus 1 can be uniformized by elliptic functions (see e.g. Sansone and Gerretsen 1969; Siegel 1969). From the symmetry of the polynomial the same elliptic function is indicated for w and z, and hence equation (A3) should appear as the addition theorem for the required elliptic functions. We shorten the work by noting that the algebraic function

$$\eta^2 = z^4 - \rho z^2 + 1 = (1-\zeta^2)(1-k^2\zeta^2). \tag{A8}$$

with ζ and k given by

$$z = k^{\frac{1}{2}}\zeta, \qquad k^2 + 1 - \rho k = 0, \tag{A9}$$

is known to be uniformized by (e.g. Sansone and Gerretsen, Section 12.6.3, p. 315)

$$\zeta = \operatorname{sn} u, \qquad \eta = \operatorname{cn} u \operatorname{dn} u. \tag{A10}$$

Substitution in equation (A1) gives

$$w = [\mu k^{\frac{1}{2}} \operatorname{sn} u \pm 2v^{\frac{1}{2}} \operatorname{cn} u \operatorname{dn} u]/2(1 - kv \operatorname{sn}^2 u). \tag{A11}$$

This suggests

$$\mu = 2\operatorname{cn}(2\eta)\operatorname{dn}(2\eta), \qquad v = k\operatorname{sn}^2(2\eta) \tag{A12}$$

and

$$w = k^{\frac{1}{2}} \operatorname{sn}(u \pm 2\eta). \tag{A13}$$

To get a parameterization of weights, we introduce a new variable ξ so that the two equations (A2) give (Baxter 1973*a*)

$$c = av^{\frac{1}{2}}\xi^{-1}, \qquad d = bv^{\frac{1}{2}}\xi \tag{A14}$$

and

$$\mu ab = a^2 + b^2 - a^2 v\xi^{-2} - b^2 v\xi^2$$

or

$$(a/b)^2(\xi^2 - v) - (a/b)\mu\xi^2 + \xi^2 - v\xi^4 = 0. \tag{A15}$$

The same uniformization argument can be repeated for equation (A15) with the result

$$a/b = \operatorname{sn}(y)/\operatorname{sn}(y \pm 2\eta), \qquad \xi = k^{\frac{1}{2}}\operatorname{sn}(y), \tag{A16}$$

where the expressions (A12) have been used. The new variable y can be further replaced by $v+\eta$ to give the parameterization obtained by Baxter,

$$a : b : c : d = \operatorname{sn}(v+\eta) : \operatorname{sn}(v-\eta) : \operatorname{sn}(2\eta) : k\operatorname{sn}(v+\eta)\operatorname{sn}(v-\eta)\operatorname{sn}(2\eta). \tag{A17}$$

Using this parameterization in equations (121a, b), one verifies

$$a_1 : b_1 : c_1 : d_1 = \operatorname{sn}(v+3\eta) : \operatorname{sn}(v+\eta) : \operatorname{sn}(2\eta) : k\operatorname{sn}(v+3\eta)\operatorname{sn}(v+\eta)\operatorname{sn}(2\eta), \tag{A18a}$$

$$a_2 : b_2 : c_2 : d_2 = \operatorname{sn}(v-\eta) : \operatorname{sn}(v-3\eta) : \operatorname{sn}(2\eta) : k\operatorname{sn}(v-\eta)\operatorname{sn}(v-3\eta)\operatorname{sn}(2\eta). \tag{A18b}$$

These equations show that the transformations derived here agree with those of Baxter (1972*a*).

(b) Polynomial Relations satisfied by Ratios of Elements in $\phi_J^{\bullet}$ and $\tau_J^{\bullet}$

The equation $B_J^{\bullet}\phi_J^{\bullet} = 0$ only determines the ratio x_J of elements in $\phi_J^{\bullet}$ and gives two expressions for it:

$$x_J \equiv \phi_{1J}/\phi_{2J} = -B_J^{12}/B_J^{11} = -B_J^{22}/B_J^{21},$$

that is (cf. equation (40)),

$$x_J = -\frac{d-cp_J p_{J+1}}{ap_{J+1}-bp_J} = -\frac{bp_{J+1}-ap_J}{c-dp_J p_{J+1}}. \tag{A19}$$

One can eliminate p_{J+1} from these equations and obtain the relation

$$x_J^2+p_J^2-\mu_1 x_J p_J - \nu_1(1+x_J^2 p_J^2) = 0, \tag{A20}$$

with

$$\mu_1 = (a^2-b^2+c^2-d^2)/ac = 2\,\mathrm{cn}(v-\eta)\,\mathrm{dn}(v-\eta), \tag{A21a}$$

$$\nu_1 = db/ac = k\,\mathrm{sn}^2(v-\eta). \tag{A21b}$$

This shows that x_J also is an sn function and its argument is obtained from that of p_J by adding $\pm(v-\eta)$.

Alternatively, one can eliminate p_J from equations (A19) and obtain

$$x_J^2+p_{J+1}^2-\mu_2 x_J p_{J+1} - \nu_2(1+x_J^2 p_{J+1}^2) = 0, \tag{A22}$$

with

$$\mu_2 = (a^2-b^2-c^2+d^2)/cb = 2\,\mathrm{cn}(v+\eta)\,\mathrm{dn}(v+\eta), \tag{A23a}$$

$$\nu_2 = ad/cb = k\,\mathrm{sn}^2(v+\eta). \tag{A23b}$$

This shows that x_J is obtained from p_{J+1} by altering the argument by $\pm(v+\eta)$, as it should, since we saw in (*a*) of this appendix that p_{J+1} is obtained from p_J by changing the argument by $\pm 2\eta$.

If in equation (A20) we change J to $J+1$ and eliminate p_{J+1} between the resulting equation and (A22), the polynomial relation obtained between x_J and x_{J+1} is of fourth degree in each and factors into two polynomials. We have

$$PP' = 0, \tag{A24}$$

with

$$P \equiv x_J^2+x_{J+1}^2-\mu x_J x_{J+1} - \nu(1+x_J^2 x_{J+1}^2), \tag{A25a}$$

$$P' \equiv x_J^2+x_{J+1}^2-\mu' x_J x_{J+1} - \nu'(1+x_J^2 x_{J+1}^2), \tag{A25b}$$

where μ and ν are given by equations (A2) and (A12) and

$$\mu' = \{\alpha^2(c^2+d^2)-\beta^2(a^2+b^2)\}/ab\beta^2$$

$$= 2\,\mathrm{cn}(v-\eta)\,\mathrm{dn}(v-\eta), \tag{A26a}$$

$$\nu' = \nu\alpha^2/\beta^2 = k\,\mathrm{sn}^2(2v), \tag{A26b}$$

with

$$\alpha = a^2-b^2, \qquad \beta = c^2-d^2.$$

Thus the roots of equation (A24) corresponding to $P = 0$ imply that x_{J+1} is obtained from x_J by changing the argument by $\pm 2\eta$, whereas the roots corresponding to $P' = 0$ imply that the relevant change is $\pm 2v$, corresponding to a reversal of the roles of v and η in the two polynomials.

Similar remarks to the above apply to the ratio of elements of $\tau_J^{\bullet}$. Equations (C.8) and (C.10) of Baxter (1973*b*) correspond to taking the increment $+2\eta$ in all cases.

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