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Fluctuation-generated Plasma Oscillations

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Abstract

A theory of spontaneous plasma oscillations is developed from basic equations. Longitudinal modes in a one-dimensional system are presented in detail, while possible extensions to three dimensions and the effects of external magnetic fields are indicated. The basic equations lead to a Van der Pol equation. Similarities are noted between plasmas and two-level lasers.

Introduction

We shall consider here a homogeneous plasma with no applied static electromagnetic fields. This may be treated as a black body in thermal equilibrium with its own radiation field. The total energy of the particles may fluctuate about a mean value determined by the temperature and pressure at that time, but for an isolated system the particle energy fluctuations must be equal and opposite to the energy fluctuations of the radiation field.

If the plasma dimensions are much smaller than any collective oscillation wavelength, the fluctuations may be evaluated using the fluctuation-dissipation theorem:

$$\langle j_i\, j_k\rangle_{k,\omega} = \frac{\hbar \mathrm{i}}{\exp(\hbar\omega/k_{\mathrm{B}}T)-1}\{\alpha_{ij}^*(\omega,k)-\alpha_{ji}(\omega,k)\},$$

where $\langle j_i j_k\rangle_{k,\omega}$ is the ensemble-averaged Fourier transform of the current correlation function, expressing the interaction of particle velocities at different positions and times, due to the exchange of thermal photons of momentum $\boldsymbol{k}$ and frequency ω. The α_{ij} are directly related to the dielectric tensor and they determine the effect of the medium on the population of the exchanged thermal photons. The correlations will be important near the zeros of the dielectric tensor (these correspond to poles in the α_{ij}), that is, near the plasma frequency ω_{p}.

Scattered Radiation Spectrum

The correlation between currents at the plasma frequency will affect the scattering of incident radiation on the plasma. The resultant field in the plasma is the sum of the incident, scattered and fluctuating fields. The incident and fluctuating fields are coupled nonlinearly by the Lorentz force so that the scattered field will be proportional to the product of the incident and fluctuating fields plus higher order terms. When the currents are uncorrelated, the scattered wave contains a gaussian spread of frequencies centred on the incident frequency ω_{i}. Given a sufficiently broad-band measuring apparatus, the scattered wave amplitudes will be seen to increase because of the increase in the fluctuating field due to correlations. Higher order nonlinearities may produce further maxima at frequencies $\omega_i \pm n\omega_p$. The scattered radiation spectrum resembles a Raman spectrum with Stokes and anti-Stokes lines.

Non-applicability of Fluctuation-dissipation Theorem

In the preceding sections it has been assumed that:

(1) the collective oscillation wavelength $\lambda_c = (\omega_p/C)^{-1}$ is very much greater than L, the distance between boundaries, and

(2) the radiation field is confined to the plasma.

The second assumption is rarely true because (i) radiation at frequencies above ω_p is weakly screened and can escape, and (ii) radiation near the boundaries can be lost from the system. Thus the plasma may emit its own radiation spectrum which may be coupled to the incident radiation. This so-called collective bremmstrahlung has been widely analysed in the literature.

Interesting phenomena may also occur when the assumption (1) above does not hold. This is the case in experiments carried out in our laboratory. In this situation the fluctuation-dissipation theorem is invalid owing to preferential thermal photon exchange at a series of well-defined frequencies and wave vectors, as analysed below. Physically this is because the plasma becomes temporally inhomogeneous owing to time delays between disturbances and their effects.

Consider a simple one-dimensional plasma of dimension L. If an electrokinetic wave of wave vector k were established, the structure would resonate due to positive feedback at wave vectors $k_n = (n+\frac{1}{2})\pi/L$; this is the Tonks–Dattner resonance condition (Bekefi 1966). Such a system would closely resemble a laser or klystron oscillator, provided end losses and internal dissipation could be overcome. This could be achieved with suitable electromagnetic fields as, for example, in the Impatt diode.

Buildup of Oscillations

The frequency of longitudinal (space-charge) oscillation corresponding to the wave vector k_n may be found from the dispersion relation

$$\varepsilon(k,\omega) = 0.$$

To first order this gives

$$\omega_{k_n}^2 = \omega_p^2 + \tfrac{3}{5}k_n^2 v_0^2 + \hbar k_n^4/4m^2 + \ldots$$

for a quantum plasma. The longitudinal wave does not radiate to first order so that, in considering the interaction of resonant modes, radiative effects can be neglected. Thus the simplified Bohm–Pines Hamiltonian (Pines 1964) will suffice when considering the wave–particle interaction:

$$H = \sum_i p_i^2/2m + \sum_k (2\pi e^2/k^2)(\rho_k^\dagger \rho_k - N) + \sum_{k<k_C} \tfrac{1}{2}\Pi_k^\dagger \Pi_k + \sum_{k<k_C} \tfrac{1}{2}(4\pi e^2/k^2)(\Pi_k^\dagger \rho_k + \Pi_k \rho_k^\dagger),$$

where the p_i are particle momenta, ρ_k and $\rho_k^\dagger$ are Fourier transforms of the space density distribution function and $\rho_k^\dagger = \rho_{-k}$. The distribution function $f(\boldsymbol{v}, \boldsymbol{r}, t)$ is assumed to be separable into $f(\boldsymbol{r}, t)g(\boldsymbol{v}, t)$ so long as inelastic collisional effects are negligible. Π_k and $\Pi_k^\dagger$ are momentum creation and annihilation operators for plasmons which apply for $k < k_C$. This is an effective quantization of the long-range part of the Coulomb potential.

Treating the Π_k and ρ_k as quantum mechanical operators, the equations of motion in the Heisenberg representation are

$$\dot{\rho}_k = \frac{[\rho_k, H]}{\mathrm{i}\hbar} = \frac{1}{\mathrm{i}\hbar}\left\{(2m)^{-1}\sum_i [\rho_k, p_i^2] - \left(\frac{2\pi e^2}{k^2}\right)\rho_k + \tfrac{1}{2}\left(\frac{4\pi e^2}{k^2}\right)^{\frac{1}{2}}\Pi_k\right\}.$$

The last term in the braces may be neglected in comparison with the other two, the first of which is summed over some 10^{20} particles. Thus

$$\dot{\rho}_k + \left(\frac{2\pi e^2}{k^2}\right)\rho_k = -\mathrm{i}\sum_i \int \mathrm{d}^3\boldsymbol{r}\left(\frac{\boldsymbol{k}\,.\,\boldsymbol{p}_i}{m} + \frac{\hbar k^2}{2m}\right) f(\boldsymbol{r}) \exp(\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{r})$$

$$= -\mathrm{i}\rho_k \sum_i \left(\frac{\boldsymbol{k}\,.\,\boldsymbol{p}_i}{m} + \frac{\hbar k^2}{2m}\right).$$

This equation may be rewritten

$$\dot{\rho}_k + (2\pi e^2/k^2)\rho_k = -\mathrm{i}\rho_k \int \mathrm{d}^3\boldsymbol{v}\, g(\boldsymbol{v})(\boldsymbol{k}\,.\,\boldsymbol{v} + \hbar k^2) + F(t). \tag{1}$$

Here the first term on the right corresponds to a coherent damping of spatial fluctuations by particles moving at the velocity corresponding to that of the fluctuation, i.e. Landau damping (Montgomery and Tidman 1964). The remaining term $F(t)$ may be considered to be due to random velocity fluctuations about the time-averaged mean, and thus represents damping or enhancing space fluctuations. This term is henceforth called 'noise' and is the driving term of oscillations.

The integral in equation (1) can be evaluated for a near-Maxwellian velocity distribution, to give

$$\dot{\rho}_k + (2\pi e^2/k^2)\rho_k = -\mathrm{i}\rho_k(k/m)\langle p\rangle + F(t), \tag{2}$$

where $\langle p\rangle$ is the average linear momentum. The total momentum is a constant G of the motion, so that

$$\langle p\rangle + \hbar k \Pi^\dagger \Pi = G. \tag{3}$$

The time development of Π is given by

$$\dot{\Pi}_k = [\Pi_k, H]/\mathrm{i}\hbar = \tfrac{1}{2}(4\pi e^2/k^2)^{\frac{1}{2}}\rho_k - \Pi_k.$$

Plasmon loss at the boundaries may be included by introducing a quality factor for the boundaries, which are here understood to be inhomogeneities in plasma properties of sufficient magnitude to cause particle reflections and occurring over a spatial distance much smaller than the wavelength of plasma oscillations. Thus introducing the loss term $\kappa\Pi$, with $\kappa \gg 1$, we have

$$\dot{\Pi}_k + \kappa\Pi_k = \tfrac{1}{2}(4\pi e^2/k^2)^{\frac{1}{2}}\rho_k. \tag{4}$$

If we neglect $\dot{\Pi}_k$ in equation (4) in comparison with $\kappa\Pi_k$ and then use this expression together with equations (3) and (2), we obtain

$$\alpha\dot{\Pi}_k + (2\pi e^2/k^2)\alpha\Pi_k = -\mathrm{i}\alpha(k/m)\,\Pi_k(G - \hbar k\Pi_k^\dagger\,\Pi_k) + F(t)\,, \tag{5}$$

where

$$\alpha = 2\kappa(4\pi e^2/k^2)^{-\frac{1}{2}}\,.$$

Equation (5) may be rewritten as

$$\dot{\Pi}_k - \beta(d - \Pi_k^\dagger\,\Pi_k)\Pi_k = F(t)\,, \tag{6}$$

which is a form of the Van der Pol equation (Harnwell 1949).

Analogy with Laser

Equation (6) is analogous to the time-development equation for the radiation field in a two-level laser (Risken 1965; Risken and Vollmer 1967), but the coupling constants are quite different. In the plasma the coupling comes from the Fourier transform for a Coulomb potential, while in the laser the coupling comes from an electromagnetic dipole moment. The analogy is, however, quite physical and we can consider the plasma as a two-level laser. The ground state of an interacting Fermi gas is a mixture of combinations of a filled core and a set of low lying quasi-particle excitations. In the case under consideration, the boundaries produce an effective force driving the system towards an eigenstate which is reached by a process described by equation (6). This state is not stable for two reasons: (i) It has a finite lifetime associated with the imaginary part of its self energy, which is of order k^{-2}; thus quantum damping should increase as the mode number increases. (ii) On a more classical level, the oscillations will be damped by inelastic collisions. Here kinetic energy is interchanged with potential energy and the former is velocity dependent only to first order while the latter is position dependent. Such an interchange could mean that our separation of the distribution function into $f(\boldsymbol{r}, t)g(\boldsymbol{v}, t)$ would not be valid. In effect, then, the present analysis is only justified on a time scale that is short compared with the inelastic collision time. Thus the waves should grow in the Van der Pol manner over a time scale equal to the shorter of the two damping times and decay thereafter. If the oscillation period is much shorter than either of these decay times a relatively long-lived oscillation will be maintained. If the converse is true, short erratic bursts of oscillation are to be expected.

The driving term $F(t)$ is a stochastic variable describing the inherent fluctuations of the system. Physically this can be visualized as arising from two sources:

(1) Particles moving in the correct way at the correct time and place produce positive feedback. Such particles may participate directly in the growth of the wave.

(2) Fast particles in the tail of the velocity distribution, moving faster than the wave velocity in the medium, may leave a core of longitudinal Čerenkov radiation which is fed back in the correct phase. The wave is established in the wake of such particles and they need not participate in the wave directly.

In a qualitative way the noise creates a driving force which moves the system from its incoherent ground state towards the coherent oscillatory state. The transition is effected by the emission of plasmons which are later reabsorbed. If external energy can be provided to inhibit the reabsorption, the system may reach threshold and lase.

Extensions of the Theory

In three dimensions the same principles as outlined above should hold, but the $\boldsymbol{k}$ vector will have transverse components. For highly asymmetrical systems the mode patterns should resemble those of a laser of equivalent geometry. For a quantum plasma the dominant mode and direction may be determined by finding the lowest possible $\boldsymbol{k}$ vector.

It has been assumed here that the plasma oscillations do not produce a radiation field or interact with the radiation field of the plasma. The latter is essentially incoherent. We have neglected the effect of the magnetic field, and the nonlinear Lorentz force of this field, and the effect of boundary radiation including surface waves. When these effects are included it is found that, to second order, the radiation accompanying the wave satisfies an inhomogeneous Klein–Gordon equation, which has been treated by Montgomery and Tidman (1964). Thus the effect of longitudinal oscillations is not confined to scattering experiments but also causes weak emission at the resonant frequencies. This radiation field should be included in the equations of motion, but the main effect will be to produce further damping.

In an external magnetic field the normal collective oscillatory modes are helicons. Electrons then have a regular circular drift on top of their random motion. Also the Fermi surface for a quantum plasma becomes distorted and split along the field direction, which in turn alters the effective masses. In this case, radiative effects come in at zero order since helicons have transverse fields. There is in addition a synchrotron spectrum radiated by accelerated electrons along the direction of the tangent to the electron trajectory. The spectrum is peaked around integral multiples of the Doppler-shifted cyclotron frequency. If the amplitude of any of these radiations is appreciable at frequencies above the plasma frequency, the system is capable of radiating directly from local fluctuations. This is a complex problem which is currently being examined.

Conclusions and Some Typical Values

The fluctuation theory presented here is a simple low order theory. Plasma behaviour is usually much more complex and higher order wave–particle interactions can be expected to produce significant effects. The many and diverse cooperative phenomena observed in solid and gaseous plasmas exemplify this.

Some typical numerical values for spontaneous emission due to resonances in a solid state plasma would be:

$$L = 2\times 10^{-3}\ \mathrm{m}, \qquad k_n = (n+\tfrac{1}{2})\,.\,\tfrac{1}{2}\pi \times 10^3\ \mathrm{m}^{-1}, \qquad k_0 \approx 10^3\ \mathrm{m}^{-1};$$

$$\omega_k^2 = \omega_\mathrm{p}^2 + \tfrac{1}{3}v_0^2 k_0^2 = \omega_\mathrm{p}^2 + \tfrac{1}{3}(k_\mathrm{B}T/2m)\times 10^6 \approx \omega_\mathrm{p}^2 + 10^{16}, \quad \omega_\mathrm{p}^2 \approx 10^{24}\ \mathrm{rad}^2\,\mathrm{s}^{-2},$$

and thus

$$\omega_k \approx \omega_\mathrm{p} \approx 10^{12}\ \mathrm{rad\,s}^{-1}; \qquad \nu_\mathrm{coll} \approx 10^{11}\ \mathrm{s}^{-1}.$$

Here we could expect reasonably long lived bursts of radiation at ω_p since the inelastic collision rate should be much less than $10^{11}\ \mathrm{s}^{-1}$. However, the modes would be closely spaced giving a large line width. A purer sample with, say, $\omega_\mathrm{p}^2 = 10^{16}$ and a smaller dimension L of 10^{-4} m would give much more widely separated resonant modes, all of which could be observable.

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