Statistics of Fission Widths for Partly Open Fission Channels

J. L. Cook and E. K. Rose
AAEC Research Establishment, Private Mail Bag, Sutherland, N.S.W. 2232.

Abstract

The statistical distributions of the fission widths for neutron fission of $^{235}$U are examined in the $J^* = 3^-$ and $4^-$ states of the compound nucleus. A $\chi^2$ fit to each set of widths indicates that the $J^* = 3^-$ state has two partly open channels. This contention is supported by a further fit to the $\chi^2$ distribution with varying degrees of freedom. Contrary to earlier predictions, it is found that the $J^* = 4^-$ state has three partly open channels.

Introduction

In a comprehensive analysis of the then available data, Lynn (1968) concluded that the fission widths of the $J^* = 3^-$ states for neutron fission of $^{235}$U should exhibit a statistical distribution that is characteristic of a $\chi^2$ distribution with $N$, the number of degrees of freedom, lying between one and two. This he interpreted as being due to the presence of one fully open and one partly open fission channel. In the present paper it is shown that such a distribution is not a $\chi^2$ distribution, but an intermediate one, although a least squares fit to the most recent data supports Lynn’s qualitative conclusion.

In addition to the above, however, many $J^* = 4^-$ levels have since been identified and Lynn’s (1968) contention that these should obey a Porter–Thomas (1956) distribution, there being only one partly open channel, is not confirmed by the data. Instead, we find that the distribution corresponding to three partly open fission channels gives by far the best fit to these data.

Two Partly Open Channels

The strong coupling theory of Bohr and Wheeler (1939) provides an estimate of the mean value of the fission width $\langle \Gamma_{\ell} \rangle$, since the value of the fission strength function should be given by

$$N_{\ell} = 2\pi \sum_{c} \Gamma_{\ell c} \langle D \rangle = \sum_{c} N_{\ell c}$$

$$= \sum_{c} \{1 + \exp(2\pi (E_c - E)/h\omega_c)\}^{-1}, \quad (1a)$$

where $\langle D \rangle$ is the average level spacing; $E_c$ the energy of the transition state that provides a channel for fission; $h\omega_c$ the mean width of the transition states, taken to be 0.38 MeV (Lynn 1968); and $E$ the incident neutron energy.
It is usually assumed that the values of $E_c$ are such that one has a number of fully open fission channels above the neutron separation energy. For such fully open channels, the corresponding $N_f$, is equal to unity. If one channel is partly open, which occurs when $E \sim E_{N_i}$, one would expect a distribution which is given by a convolution of two Porter–Thomas distributions of unequal means $\langle \Gamma_1 \rangle$ and $\langle \Gamma_2 \rangle$, that is,

$$\mathcal{P}_2(\Gamma) d\Gamma = \left\{2\pi\Gamma(\langle \Gamma_1 \rangle, \langle \Gamma_2 \rangle) \right\}^{-1} \exp(-\Gamma/2\langle \Gamma_2 \rangle) \left( \int_0^1 \exp(-\alpha z) \frac{dz}{z(1-z)^{1/4}} \right) d\Gamma,$$  

(1b)

with

$$\alpha = \frac{1}{2} \Gamma(\langle \Gamma_2 \rangle^{-1} - \langle \Gamma_1 \rangle^{-1}).$$

The distribution (1) gives

$$\mathcal{P}_2(x) dx = \left\{2\langle \Gamma_1 \rangle \langle \Gamma_2 \rangle \right\}^{-1} \exp\left\{-\frac{1}{4} \Gamma(\langle \Gamma_1 \rangle^{-1} + \langle \Gamma_2 \rangle^{-1})\right\} I_0(\frac{1}{2}x) dx,$$  

(2)

where the integral representation (Gradshteyn and Ryzhik 1965) for the associated Bessel function has been used, and $x = \Gamma/\langle \Gamma \rangle$ with $\langle \Gamma \rangle = \langle \Gamma_1 \rangle + \langle \Gamma_2 \rangle$. The Porter–Thomas distribution for each $\Gamma_c$ is given by

$$\mathcal{P}_1(\Gamma_c) = \left\{2\pi(\langle \Gamma_c \rangle \right\}^{-1} \exp(-\Gamma_c/2\langle \Gamma_c \rangle).$$  

(3)

Making use of the integral representation (Gradshteyn and Ryzhik 1965)

$$\int_0^\infty \exp(-ax) I_0(bx) x^\mu dx = (a^2 - b^2)^{-(\mu+1)/2} \Gamma(\mu+1) P_\mu(a/(a^2 - b^2)^{1/2}),$$  

(4)

where

$$a = \frac{1}{2}(\langle \Gamma_1 \rangle^{-1} + \langle \Gamma_2 \rangle^{-1}), \quad b = \frac{1}{2}(\langle \Gamma_2 \rangle^{-1} - \langle \Gamma_1 \rangle^{-1})$$

(5)

and $P_\mu(x)$ is the Legendre function, we find the $\mu$th moment of this distribution to be

$$\langle \Gamma^n \rangle = 2^n(\langle \Gamma_1 \rangle \langle \Gamma_2 \rangle)^{\mu} \Gamma(\mu+1) P_\mu\left(\frac{\langle \Gamma_1 \rangle \langle \Gamma_2 \rangle}{\langle \Gamma_1 \rangle/\langle \Gamma_2 \rangle}\right) \langle \Gamma_1 \rangle^{-1} \langle \Gamma_2 \rangle^{-1}.$$  

(6)

This raises the question as to whether one can determine $\langle \Gamma_1 \rangle$ and $\langle \Gamma_2 \rangle$ by calculating two moments of the distribution, e.g.

$$\langle \Gamma \rangle = \langle \Gamma_1 \rangle + \langle \Gamma_2 \rangle,$$

$$\langle \Gamma^2 \rangle = 3\langle \Gamma^2 \rangle - 4\langle \Gamma_1 \rangle \langle \Gamma_2 \rangle,$$  

(7a)

giving

$$\langle \Gamma_1 \rangle = \frac{1}{3} \{\langle \Gamma \rangle + (\langle \Gamma^2 \rangle + 2\langle \Gamma_1 \rangle \langle \Gamma_2 \rangle)^{1/2}\}, \quad \langle \Gamma_2 \rangle = \langle \Gamma \rangle - \langle \Gamma_1 \rangle.$$  

(7b)

Therefore, determination of the mean and variance of the distribution determines the partial fission channel widths. This method was tested numerically, but was found to give very inaccurate results unless the population of levels was greater than about 100.

**Three Channels**

We shall first consider the case where all channels are open. Here the mean fission widths in all channels are equal and the convolution of $N$ Porter–Thomas distributions
of the form (3) yields a χ² distribution of N degrees of freedom,

$$P_N(\Gamma) d\Gamma = \frac{1}{\Gamma(\frac{1}{2}N)} \left(\frac{N}{2\langle\Gamma\rangle}\right)^{\frac{1}{2}N} (\Gamma)^{\frac{1}{2}N-1} \exp\left(-\frac{N\Gamma}{2\langle\Gamma\rangle}\right) d\Gamma.$$  (8)

If, however, we have one further partly open channel, the convolution yields

$$P_{N+1}(\Gamma) d\Gamma = \frac{1}{(2\langle\Gamma_{N+1}\rangle)^{\frac{1}{2}}} \frac{1}{\Gamma(\frac{1}{2}(N+1))} \left(\frac{N}{2\langle\Gamma\rangle}\right)^{\frac{1}{2}N} (\Gamma)^{\frac{1}{2}(N-1)} \exp\left(-\frac{\Gamma}{2\langle\Gamma_{N+1}\rangle}\right)$$

$$\times \, _1F_1\left(\frac{1}{2}N, \frac{1}{2}(N+1), a\Gamma\right) d\Gamma,$$  (9)

where

$$a = \frac{1}{2}(\langle\Gamma_{N+1}\rangle^{-1} - N\langle\Gamma\rangle^{-1})$$

and \(_1F_1(\alpha, \beta, z)\) is the confluent hypergeometric function as defined by Gradshteyn and Ryzhik (1965).

In particular, for \(N = 2\) we obtain

$$P_3(\Gamma) d\Gamma = \langle\Gamma_0\rangle^{-1} (1 - 2\langle\Gamma_3\rangle/\langle\Gamma_0\rangle)^{-\frac{1}{2}} \exp(-\Gamma/\langle\Gamma_0\rangle) \text{erf}(a^\frac{1}{2}\Gamma) d\Gamma,$$  (10)

where

$$\text{erf}(z) = 2\pi^{-\frac{1}{2}} \int_0^z \exp(-t^2) dt,$$

and

$$\langle\Gamma_0\rangle = \langle\Gamma_1\rangle + \langle\Gamma_2\rangle, \quad \langle\Gamma_1\rangle = \langle\Gamma_2\rangle.$$

The \(n\)th moment of this distribution is given by

$$\int_0^\infty \Gamma^n P_3(\Gamma) d\Gamma = \frac{(-1)^n}{\langle\Gamma_0\rangle(2\langle\Gamma_3\rangle)^{\frac{1}{2}}} \left[ \int_0^\infty \frac{1}{y(y+a)^{\frac{1}{2}}} \right] y = \frac{1}{\langle\Gamma_0\rangle},$$  (11)

which yields

$$\langle\Gamma\rangle = 2\langle\Gamma_1\rangle + \langle\Gamma_3\rangle,$$  (12a)

$$\langle\Gamma^2\rangle = 2\langle\Gamma^2\rangle - 2\langle\Gamma\rangle\langle\Gamma_3\rangle + 3\langle\Gamma_3\rangle^2,$$  (12b)

$$\langle\Gamma_3\rangle = \frac{1}{3}\langle\Gamma\rangle + \frac{1}{3}(3\langle\Gamma^2\rangle - 5\langle\Gamma_3\rangle^2)^{\frac{1}{2}}.$$  (12c)

These relations enable \(\langle\Gamma_1\rangle, \langle\Gamma_2\rangle\) and \(\langle\Gamma_3\rangle\) to be calculated from the first and second moments.

Should \(\langle\Gamma_1\rangle, \langle\Gamma_2\rangle\) and \(\langle\Gamma_3\rangle\) be different values, indicating that three fission channels were partly open to different degrees, the convolution of three Porter–Thomas distributions yields

$$P_3(\Gamma) = \langle\Gamma\rangle/2\pi^{\frac{3}{2}} (\langle\Gamma_1\rangle\langle\Gamma_2\rangle\langle\Gamma_3\rangle)^{-\frac{1}{2}} \exp(-ax) x^\frac{1}{2} \int_0^1 \exp(-yxz^2) I_0(\beta x(1-z^2)) dz,$$  (13)

where

$$x = \frac{\Gamma}{\langle\Gamma\rangle}, \quad a = \frac{\langle\Gamma\rangle}{2\langle\Gamma_3\rangle}, \quad \beta = \frac{1}{2}(\langle\Gamma\rangle\langle\Gamma_2\rangle^{-1} - \langle\Gamma_1\rangle^{-1})$$

and \(\langle\Gamma_1\rangle > \langle\Gamma_2\rangle > \langle\Gamma_3\rangle\).
The distribution (13) tends to a \( \chi^2 \) distribution in the limit \( \langle \Gamma_1 \rangle = \langle \Gamma_2 \rangle = \langle \Gamma_3 \rangle \) and a \( \chi^2 \) distribution in the limit \( \langle \Gamma_3 \rangle = 0 \). To calculate the three average widths from the moments, we would require three moments. For example, the equations analogous to (7a) are

\[
\langle \Gamma \rangle = \langle \Gamma_1 \rangle + \langle \Gamma_2 \rangle + \langle \Gamma_3 \rangle, \\
\langle \Gamma^2 \rangle = 3\langle \Gamma \rangle^2 - 4(\langle \Gamma_1 \rangle \langle \Gamma_2 \rangle + \langle \Gamma_1 \rangle \langle \Gamma_3 \rangle + \langle \Gamma_2 \rangle \langle \Gamma_3 \rangle), \\
\langle \Gamma^3 \rangle = 15\langle \Gamma \rangle^3 - 6(\langle \Gamma_1 \rangle^2 \langle \Gamma_2 \rangle + \langle \Gamma_1 \rangle \langle \Gamma_2 \rangle \langle \Gamma_3 \rangle + \langle \Gamma_2 \rangle^2 \langle \Gamma_3 \rangle + \langle \Gamma_2 \rangle \langle \Gamma_3 \rangle^2 + \langle \Gamma_1 \rangle \langle \Gamma_3 \rangle^2 + \langle \Gamma_1 \rangle^2 \langle \Gamma_3 \rangle^2). 
\]

These can be solved by numerical direct search methods, but once again a large population of levels is required to give reliable results.

<table>
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<tr>
<th>( E ) (eV)</th>
<th>( \Gamma_i ) (eV)</th>
<th>( E ) (eV)</th>
<th>( \Gamma_i ) (eV)</th>
<th>( E ) (eV)</th>
<th>( \Gamma_i ) (eV)</th>
<th>( E ) (eV)</th>
<th>( \Gamma_i ) (eV)</th>
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<td>J(^\pi) = 3(^-)</td>
<td>J(^\pi) = 4(^-)</td>
<td>J(^\pi) = 4(^-)</td>
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<td>55.080</td>
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<td>28.360</td>
<td>0.116</td>
<td>22.940</td>
<td>0.046</td>
<td>56.500</td>
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**Table 1.** \(^{235}\text{U} \) resonance parameters

Analysis of Experimental Data

The resonance parameters quoted by Mughabghab and Garber (1973) for \(^{235}\text{U} \), with \( J^\pi \) assignments taken from their recommendations and those of de Saussure and Perez (1969), are shown in Table 1. The range of \( x = \Gamma_i/\langle \Gamma \rangle \) was divided into four intervals: \( 0 \leq x \leq 0.5, 0.5 < x \leq 1, 1 < x \leq 2 \) and \( 2 < x \leq \infty \). The integrals over these intervals were calculated and a \( \chi^2 \) fit was carried out for the \( J^\pi = 3^- \) states to the two-channel distribution \( \mathcal{P}_2 \) (equation 2) and the three-channel distribution \( \mathcal{P}_3 \) (equation 13). It was found that the minimum \( \chi^2 \) of 0.22 occurred in the two-channel case with

\[
\langle \Gamma_1(J^\pi=3^-) \rangle = 0.084 \pm 0.04 \text{ eV}, \quad \langle \Gamma_2(J^\pi=3^-) \rangle = 0.028 \pm 0.02 \text{ eV}. \tag{15}
\]
A plot of $\chi^2$ versus $\langle \Gamma_2 \rangle$ with $\langle \Gamma \rangle$ fixed at 0.112 eV is shown in Fig. 1a. Normally it is assumed that if channels are only partly open then $N$ in equation (8) becomes a non-integer $v$. A fit to a $\chi_v^2$ distribution was also carried out and a minimum $\chi^2$ of 0.24 was found at $v = 2.1 \pm 0.3$. The plot of $\chi^2$ for this case is shown in Fig. 1b. Thus there are probably two partly open channels, in agreement with the results of Lynn (1968).

A fit of the $J^* = 4^-$ levels to the distribution (13) gave a smaller $\chi^2$ value of 0.14, with

$$\langle \Gamma_1(J^* = 4^-) \rangle = 0.035 \pm 0.002 \text{ eV},$$  \hspace{1cm} (16a)

$$\langle \Gamma_2(J^* = 4^-) \rangle = 0.017 \pm 0.001 \text{ eV},$$  \hspace{1cm} (16b)

$$\langle \Gamma_3(J^* = 4^-) \rangle = 0.0077 \pm 0.0007 \text{ eV},$$  \hspace{1cm} (16c)

whereas the fit to the case given by the distribution (10), with $\langle \Gamma_1 \rangle = \langle \Gamma_2 \rangle$, gave a much larger $\chi^2$ minimum of 0.23. The plot of $\chi^2$ as a function of $\langle \Gamma_2 \rangle$ is shown in

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**Fig. 1.** Plots of $\chi^2$ versus:

(a) $\langle \Gamma_2 \rangle$, with $\langle \Gamma \rangle$ fixed at 0.112 eV, for a fit of the $J^* = 3^-$ states to a two-channel distribution;

(b) $v$, for fits of the $J^* = 3^-$ and $4^-$ states to $\chi^2$ distributions;

(c) $\langle \Gamma_2 \rangle$, with $\langle \Gamma \rangle$ fixed at 0.06 eV, for a fit of the $J^* = 4^-$ states to a three-channel distribution.
Fig. 1c. These values indicate a confidence level of 97\% to 99\% from the tables of Abramowitz and Stegun (1965). The fit to a $\chi^2$ distribution for this case gave a value of $\chi^2 = 0.14$ for $v = 2.7 \pm 0.3$, indicating the presence of about three partly open channels (Fig. 1b).

<table>
<thead>
<tr>
<th>$J^\pi$</th>
<th>$v_{\text{eff}}$</th>
<th>$\langle \Gamma_1 \rangle$ (eV)</th>
<th>$\langle \Gamma_2 \rangle$ (eV)</th>
<th>$\langle \Gamma_3 \rangle$ (eV)</th>
<th>$\langle D \rangle$ (eV)</th>
<th>$N_1$</th>
<th>$N_2$</th>
<th>$N_3$</th>
<th>$E_1$ (keV)</th>
<th>$E_2$ (keV)</th>
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<td>3$^-$</td>
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<td>0.084</td>
<td>0.028</td>
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<td>0.0</td>
<td>-43.5</td>
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<tr>
<td>4$^-$</td>
<td>3.9</td>
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<td>0.017</td>
<td>0.0077</td>
<td>0.795</td>
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The effective value of $v$ from equation (8) is given by

$$v_{\text{eff}}(J^\pi) = 2\langle \Gamma \rangle^2/(\langle \Gamma^2 \rangle - \langle \Gamma \rangle^2).$$

The results for both channels are shown in Table 2. The energies for the transition states were calculated from equation (1a). It can be seen from values of the fission strength functions $N_{Jc}$ that all channels are only partly open and to different degrees, the most open being a channel in the $J^\pi = 3^-$ state.

We conclude that the present work provides evidence for the validity of the theory of partly open fission channels ($\mathcal{P}_2, \mathcal{P}_3$). The fits of the theoretical distributions are summarized in Table 3.

<table>
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<tr>
<th>$J^\pi$</th>
<th>Range of $x = \Gamma/\langle \Gamma \rangle$</th>
<th>$\langle \mathcal{P}_2 - \mathcal{P}_3 \rangle$ distribution</th>
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<td>0.43</td>
</tr>
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<td>0.5 to 1.0</td>
<td>0.35</td>
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<tr>
<td></td>
<td>1.0 to 2.0</td>
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<td>2.0 to $\infty$</td>
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<tr>
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<td>1.0 to 2.0</td>
<td>0.28</td>
<td>0.24</td>
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<td>2.0 to $\infty$</td>
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References


Manuscript received 21 March 1975