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Statistics of Fission Widths for Partly Open Fission Channels

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Abstract

The statistical distributions of the fission widths for neutron fission of ^{235}U are examined in the $J^\pi = 3^-$ and 4^- states of the compound nucleus. A χ^2 fit to each set of widths indicates that the $J^\pi = 3^-$ state has two partly open channels. This contention is supported by a further fit to the χ^2 distribution with varying degrees of freedom. Contrary to earlier predictions, it is found that the $J^\pi = 4^-$ state has three partly open channels.

Introduction

In a comprehensive analysis of the then available data, Lynn (1968) concluded that the fission widths of the $J^\pi = 3^-$ states for neutron fission of ^{235}U should exhibit a statistical distribution that is characteristic of a χ_N^2 distribution with N, the number of degrees of freedom, lying between one and two. This he interpreted as being due to the presence of one fully open and one partly open fission channel. In the present paper it is shown that such a distribution is not a χ_N^2 distribution, but an intermediate one, although a least squares fit to the most recent data supports Lynn's qualitative conclusion.

In addition to the above, however, many $J^\pi = 4^-$ levels have since been identified and Lynn's (1968) contention that these should obey a Porter–Thomas (1956) distribution, there being only one partly open channel, is not confirmed by the data. Instead, we find that the distribution corresponding to three partly open fission channels gives by far the best fit to these data.

Two Partly Open Channels

The strong coupling theory of Bohr and Wheeler (1939) provides an estimate of the mean value of the fission width $\langle\Gamma_{fJ}\rangle$, since the value of the fission strength function should be given by

$$\begin{aligned} N_J &= 2\pi \sum_c \Gamma_{fJc}/\langle D\rangle = \sum_c N_{Jc} \\ &= \sum_c \{1+\exp(2\pi(E_c-E)/\hbar\omega_c)\}^{-1}, \end{aligned} \tag{1a}$$

where $\langle D\rangle$ is the average level spacing; E_c the energy of the transition state that provides a channel for fission; $\hbar\omega_c$ the mean width of the transition states, taken to be 0·38 MeV (Lynn 1968); and E the incident neutron energy.

It is usually assumed that the values of E_c are such that one has a number of fully open fission channels above the neutron separation energy. For such fully open channels, the corresponding N_{Jc} is equal to unity. If one channel is partly open, which occurs when $E \sim E_N$, one would expect a distribution which is given by a convolution of two Porter–Thomas distributions of unequal means $\langle\Gamma_1\rangle$ and $\langle\Gamma_2\rangle$, that is,

$$\mathscr{P}_2(\Gamma)\,\mathrm{d}\Gamma = \{2\pi\Gamma(\langle\Gamma_1\rangle\langle\Gamma_2\rangle)^{\frac{1}{2}}\}^{-1}\exp(-\Gamma/2\langle\Gamma_2\rangle)\left(\int_0^1 \frac{\exp(-\alpha z)\,\mathrm{d}z}{\{z(1-z)\}^{\frac{1}{2}}}\right)\mathrm{d}\Gamma\,, \tag{1b}$$

with

$$\alpha = \tfrac{1}{2}\Gamma(\langle\Gamma_2\rangle^{-1}-\langle\Gamma_1\rangle^{-1}).$$

The distribution (1) gives

$$\mathscr{P}_2(x)\,\mathrm{d}x = \{2(\langle\Gamma_1\rangle\langle\Gamma_2\rangle)^{\frac{1}{2}}\}^{-1}\exp\{-\tfrac{1}{4}\Gamma(\langle\Gamma_1\rangle^{-1}+\langle\Gamma_2\rangle^{-1})\}\,\mathrm{I}_0(\tfrac{1}{2}\alpha)\,\mathrm{d}x\,, \tag{2}$$

where the integral representation (Gradshteyn and Ryzhik 1965) for the associated Bessel function has been used, and $x = \Gamma/\langle\Gamma\rangle$ with $\langle\Gamma\rangle = \langle\Gamma_1\rangle+\langle\Gamma_2\rangle$. The Porter–Thomas distribution for each Γ_c is given by

$$\mathscr{P}_1(\Gamma_c) = \{2\pi(\Gamma_c\langle\Gamma_c\rangle)^{\frac{1}{2}}\}^{-1}\exp(-\Gamma_c/2\langle\Gamma_c\rangle)\,. \tag{3}$$

Making use of the integral representation (Gradshteyn and Ryzhik 1965)

$$\int_0^\infty \exp(-ax)\,\mathrm{I}_0(bx)\,x^\mu\,\mathrm{d}x = (a^2-b^2)^{-\frac{1}{2}(\mu+1)}\,\Gamma(\mu+1)\,\mathrm{P}_\mu\big(a/(a^2-b^2)^{\frac{1}{2}}\big)\,, \tag{4}$$

where

$$a = \tfrac{1}{4}(\langle\Gamma_1\rangle^{-1}+\langle\Gamma_2\rangle^{-1})\,,\qquad b = \tfrac{1}{4}(\langle\Gamma_2\rangle^{-1}-\langle\Gamma_1\rangle^{-1}) \tag{5}$$

and $\mathrm{P}_\mu(x)$ is the Legendre function, we find the μth moment of this distribution to be

$$\begin{aligned}\langle\Gamma^\mu\rangle &= 2^\mu(\langle\Gamma_1\rangle\langle\Gamma_2\rangle)^{\frac{1}{2}\mu}\,\Gamma(\mu+1)\,\mathrm{P}_\mu\big(\tfrac{1}{2}(\langle\Gamma_1\rangle\langle\Gamma_2\rangle)^{\frac{1}{2}}(\langle\Gamma_1\rangle^{-1}+\langle\Gamma_2\rangle^{-1})\big)\\ &= 2^\mu(\langle\Gamma_1\rangle\langle\Gamma_2\rangle)^{\frac{1}{2}\mu}\,\Gamma(\mu+1)\,\mathrm{P}_\mu\big(\tfrac{1}{2}\langle\Gamma\rangle/(\langle\Gamma_1\rangle\langle\Gamma_2\rangle)^{\frac{1}{2}}\big)\,.\end{aligned} \tag{6}$$

This raises the question as to whether one can determine $\langle\Gamma_1\rangle$ and $\langle\Gamma_2\rangle$ by calculating two moments of the distribution, e.g.

$$\langle\Gamma\rangle = \langle\Gamma_1\rangle+\langle\Gamma_2\rangle\,,\qquad \langle\Gamma^2\rangle = 3\langle\Gamma\rangle^2-4\langle\Gamma_1\rangle\langle\Gamma_2\rangle\,, \tag{7a}$$

giving

$$\langle\Gamma_1\rangle = \tfrac{1}{2}\{\langle\Gamma\rangle+(\langle\Gamma^2\rangle+2\langle\Gamma\rangle^2)^{\frac{1}{2}}\}\,,\qquad \langle\Gamma_2\rangle = \langle\Gamma\rangle-\langle\Gamma_1\rangle\,. \tag{7b}$$

Therefore, determination of the mean and variance of the distribution determines the partial fission channel widths. This method was tested numerically, but was found to give very inaccurate results unless the population of levels was greater than about 100.

Three Channels

We shall first consider the case where all channels are open. Here the mean fission widths in all channels are equal and the convolution of N Porter–Thomas distributions of the form (3) yields a χ^2 distribution of N degrees of freedom,

$$\mathscr{P}_N(\Gamma)\,\mathrm{d}\Gamma = \frac{1}{\Gamma(\frac{1}{2}N)}\left(\frac{N}{2\langle\Gamma\rangle}\right)^{\frac{1}{2}N}(\Gamma)^{\frac{1}{2}N-1}\exp\left(-\frac{N\Gamma}{2\langle\Gamma\rangle}\right)\mathrm{d}\Gamma\,. \tag{8}$$

If, however, we have one further partly open channel, the convolution yields

$$\mathscr{P}_{N+1}(\Gamma)\,\mathrm{d}\Gamma = \frac{1}{(2\langle\Gamma_{N+1}\rangle)^{\frac{1}{2}}}\frac{1}{\Gamma(\frac{1}{2}N+\frac{1}{2})}\left(\frac{N}{2\langle\Gamma\rangle}\right)^{\frac{1}{2}N}(\Gamma)^{\frac{1}{2}(N-1)}\exp\left(-\frac{\Gamma}{2\langle\Gamma_{N+1}\rangle}\right) \times {}_1\mathrm{F}_1\big(\tfrac{1}{2}N,\tfrac{1}{2}(N+1),a\Gamma\big)\,\mathrm{d}\Gamma\,, \tag{9}$$

where

$$a = \tfrac{1}{2}(\langle\Gamma_{N+1}\rangle^{-1} - N\langle\Gamma\rangle^{-1})$$

and ${}_1\mathrm{F}_1(\alpha,\beta,z)$ is the confluent hypergeometric function as defined by Gradshteyn and Ryzhik (1965).

In particular, for $N = 2$ we obtain

$$\mathscr{P}_3(\Gamma)\,\mathrm{d}\Gamma = \langle\Gamma_0\rangle^{-1}(1-2\langle\Gamma_3\rangle/\langle\Gamma_0\rangle)^{-\frac{1}{2}}\exp(-\Gamma/\langle\Gamma_0\rangle)\,\mathrm{erf}(a^{\frac{1}{2}}\Gamma^{\frac{1}{2}})\,\mathrm{d}\Gamma\,, \tag{10}$$

where

$$\mathrm{erf}(z) = 2\pi^{-\frac{1}{2}}\int_0^z \exp(-t^2)\,\mathrm{d}t$$

and

$$\langle\Gamma_0\rangle = \langle\Gamma_1\rangle + \langle\Gamma_2\rangle\,, \qquad \langle\Gamma_1\rangle = \langle\Gamma_2\rangle\,.$$

The nth moment of this distribution is given by

$$\int_0^\infty \Gamma^n\mathscr{P}_3(\Gamma)\,\mathrm{d}\Gamma = \frac{(-1)^n}{\langle\Gamma_0\rangle(2\langle\Gamma_3\rangle)^{\frac{1}{2}}}\left[\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left(\frac{1}{y(y+a)^{\frac{1}{2}}}\right)\right]_{y=1/\langle\Gamma_0\rangle}, \tag{11}$$

which yields

$$\langle\Gamma\rangle = 2\langle\Gamma_1\rangle + \langle\Gamma_3\rangle\,, \tag{12a}$$

$$\langle\Gamma^2\rangle = 2\langle\Gamma\rangle^2 - 2\langle\Gamma\rangle\langle\Gamma_3\rangle + 3\langle\Gamma_3\rangle^2\,, \tag{12b}$$

$$\langle\Gamma_3\rangle = \tfrac{1}{3}\langle\Gamma\rangle + \tfrac{1}{3}(3\langle\Gamma^2\rangle - 5\langle\Gamma\rangle^2)^{\frac{1}{2}}\,. \tag{12c}$$

These relations enable $\langle\Gamma_1\rangle$, $\langle\Gamma_2\rangle$ and $\langle\Gamma_3\rangle$ to be calculated from the first and second moments.

Should $\langle\Gamma_1\rangle$, $\langle\Gamma_2\rangle$ and $\langle\Gamma_3\rangle$ be different values, indicating that three fission channels were partly open to different degrees, the convolution of three Porter–Thomas distributions yields

$$\mathscr{P}_3(\Gamma) = (\langle\Gamma\rangle/2\pi)^{\frac{1}{2}}(\langle\Gamma_1\rangle\langle\Gamma_2\rangle\langle\Gamma_3\rangle)^{-\frac{1}{2}}\exp(-\alpha x)\,x^{\frac{1}{2}}\int_0^1 \exp(-\gamma x z^2)\,\mathrm{I}_0\big(\beta x(1-z^2)\big)\,\mathrm{d}z\,, \tag{13}$$

where

$$x = \Gamma/\langle\Gamma\rangle\,, \qquad \alpha = \langle\Gamma\rangle/2\langle\Gamma_3\rangle\,, \qquad \beta = \tfrac{1}{4}\langle\Gamma\rangle(\langle\Gamma_2\rangle^{-1} - \langle\Gamma_1\rangle^{-1})\,,$$

$$\gamma = \tfrac{1}{2}\langle\Gamma\rangle\{\langle\Gamma_3\rangle^{-1} - \tfrac{1}{2}(\langle\Gamma_1\rangle^{-1} + \langle\Gamma_2\rangle^{-1})\} \quad \text{and} \quad \langle\Gamma_1\rangle > \langle\Gamma_2\rangle > \langle\Gamma_3\rangle\,.$$

The distribution (13) tends to a χ_3^2 distribution in the limit $\langle\Gamma_1\rangle = \langle\Gamma_2\rangle = \langle\Gamma_3\rangle$ and a χ_2^2 distribution in the limit $\langle\Gamma_3\rangle = 0$. To calculate the three average widths from the moments, we would require three moments. For example, the equations analogous to (7a) are

$$\langle\Gamma\rangle = \langle\Gamma_1\rangle + \langle\Gamma_2\rangle + \langle\Gamma_3\rangle, \tag{14a}$$

$$\langle\Gamma^2\rangle = 3\langle\Gamma\rangle^2 - 4(\langle\Gamma_1\rangle\langle\Gamma_2\rangle + \langle\Gamma_1\rangle\langle\Gamma_3\rangle + \langle\Gamma_2\rangle\langle\Gamma_3\rangle), \tag{14b}$$

$$\begin{aligned}\langle\Gamma^3\rangle = 15\langle\Gamma\rangle^3 - 6(&\langle\Gamma_1\rangle^2\langle\Gamma_2\rangle + \langle\Gamma_1\rangle\langle\Gamma_2\rangle^2 + \langle\Gamma_2\rangle^2\langle\Gamma_3\rangle + \langle\Gamma_2\rangle\langle\Gamma_3\rangle^2 \\ &+ \langle\Gamma_1\rangle^2\langle\Gamma_3\rangle + \langle\Gamma_1\rangle\langle\Gamma_3\rangle^2).\end{aligned} \tag{14c}$$

These can be solved by numerical direct search methods, but once again a large population of levels is required to give reliable results.

Table 1. **^{235}U resonance parameters**

E (eV)	Γ_f (eV)	E (eV)	Γ_f (eV)	E (eV)	Γ_f (eV)	E (eV)	Γ_f (eV)
$J^\pi = 3^-$		$J^\pi = 3^-$		$J^\pi = 4^-$		$J^\pi = 4^-$	
0·285	0·100	28·720	0·070	1·135	0·115	23·420	0·007
2·028	0·010	30·590	0·100	2·840	0·100	26·490	0·125
3·145	0·116	34·850	0·090	3·613	0·050	27·820	0·068
5·446	0·024	41·880	0·020	4·845	0·005	29·650	0·030
5·830	0·075	48·300	0·054	6·390	0·010	30·860	0·020
6·210	0·185	48·800	0·026	7·080	0·027	32·070	0·050
9·740	0·085			8·780	0·085	33·530	0·023
10·800	0·700			9·280	0·109	34·390	0·038
12·390	0·027			10·180	0·046	35·200	0·120
13·280	0·040			11·660	0·005	38·360	0·065
14·020	0·460			12·850	0·055	39·410	0·052
14·510	0·013			13·690	0·070	40·520	0·160
18·050	0·110			15·400	0·047	43·940	0·060
20·130	0·074			16·080	0·017	44·600	0·180
23·630	0·100			16·680	0·078	45·790	0·023
24·250	0·028			18·960	0·040	47·980	0·024
24·500	0·043			19·300	0·060	49·430	0·020
25·590	0·200			20·620	0·045	51·270	0·140
27·160	0·050			21·070	0·024	55·080	0·058
28·360	0·116			22·940	0·046	56·500	0·092

Analysis of Experimental Data

The resonance parameters quoted by Mughabghab and Garber (1973) for ^{235}U, with J^π assignments taken from their recommendations and those of de Saussure and Perez (1969), are shown in Table 1. The range of $x = \Gamma/\langle\Gamma\rangle$ was divided into four intervals: $0 \leqslant x \leqslant 0{\cdot}5, 0{\cdot}5 \leqslant x \leqslant 1, 1 \leqslant x \leqslant 2$ and $2 \leqslant x \leqslant \infty$. The integrals over these intervals were calculated and a χ^2 fit was carried out for the $J^\pi = 3^-$ states to the two-channel distribution $\mathscr{P}_2$ (equation 2) and the three-channel distribution $\mathscr{P}_3$ (equation 13). It was found that the minimum χ^2 of 0·22 occurred in the two-channel case with

$$\langle\Gamma_1(J^\pi{=}3^-)\rangle = 0{\cdot}084 \pm 0{\cdot}04 \text{ eV}, \qquad \langle\Gamma_2(J^\pi{=}3^-)\rangle = 0{\cdot}028 \pm 0{\cdot}02 \text{ eV}. \tag{15}$$

A plot of χ^2 versus $\langle \Gamma_2 \rangle$ with $\langle \Gamma \rangle$ fixed at 0·112 eV is shown in Fig. 1*a*. Normally it is assumed that if channels are only partly open then N in equation (8) becomes a non-integer ν. A fit to a χ_ν^2 distribution was also carried out and a minimum χ^2 of 0·24 was found at $\nu = 2 \cdot 1 \pm 0 \cdot 3$. The plot of χ^2 for this case is shown in Fig. 1*b*. Thus there are probably two partly open channels, in agreement with the results of Lynn (1968).

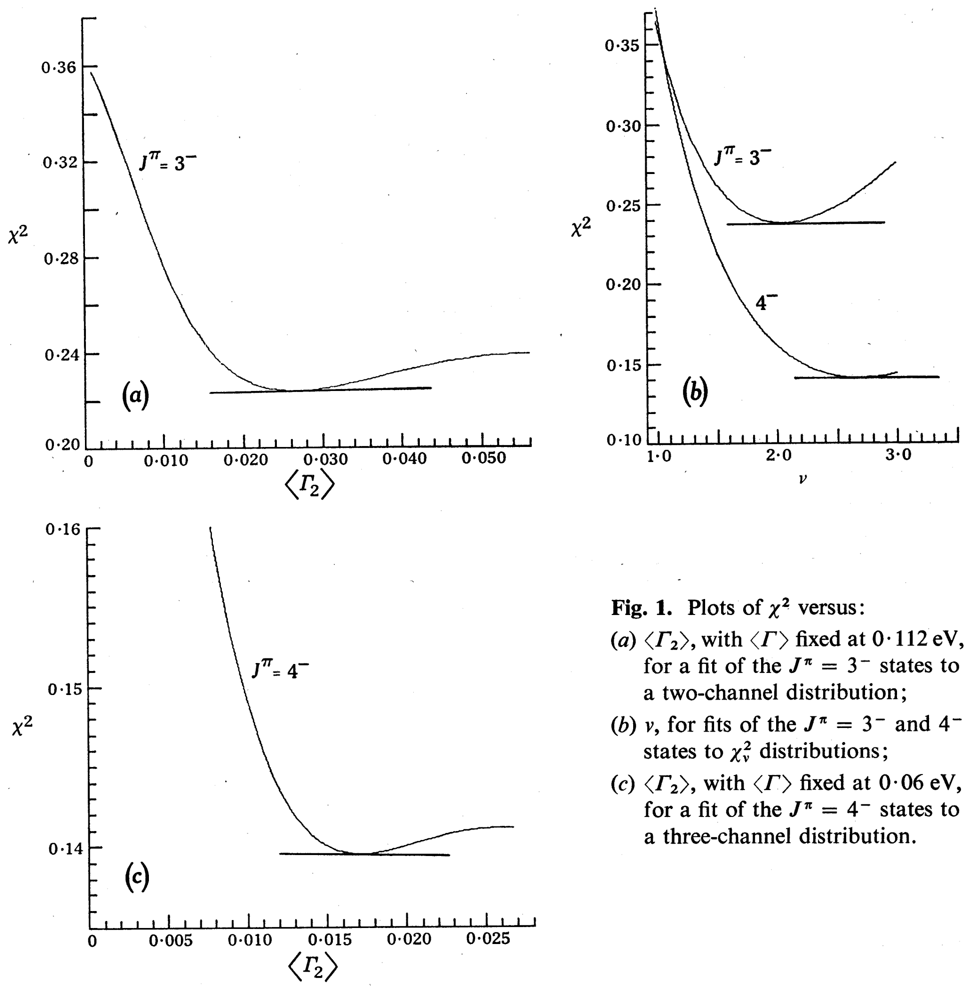

Fig. 1. Plots of χ^2 versus:
(*a*) $\langle \Gamma_2 \rangle$, with $\langle \Gamma \rangle$ fixed at 0·112 eV, for a fit of the $J^\pi = 3^-$ states to a two-channel distribution;
(*b*) ν, for fits of the $J^\pi = 3^-$ and 4^- states to χ_ν^2 distributions;
(*c*) $\langle \Gamma_2 \rangle$, with $\langle \Gamma \rangle$ fixed at 0·06 eV, for a fit of the $J^\pi = 4^-$ states to a three-channel distribution.

A fit of the $J^\pi = 4^-$ levels to the distribution (13) gave a smaller χ^2 value of 0·14, with

$$\langle \Gamma_1(J^\pi = 4^-) \rangle = 0 \cdot 035 \pm 0 \cdot 002 \text{ eV}, \tag{16a}$$

$$\langle \Gamma_2(J^\pi = 4^-) \rangle = 0 \cdot 017 \pm 0 \cdot 001 \text{ eV}, \tag{16b}$$

$$\langle \Gamma_3(J^\pi = 4^-) \rangle = 0 \cdot 0077 \pm 0 \cdot 0007 \text{ eV}, \tag{16c}$$

whereas the fit to the case given by the distribution (10), with $\langle \Gamma_1 \rangle = \langle \Gamma_2 \rangle$, gave a much larger χ^2 minimum of 0·23. The plot of χ^2 as a function of $\langle \Gamma_2 \rangle$ is shown in

Fig. 1*c*. These values indicate a confidence level of 97% to 99% from the tables of Abramowitz and Stegun (1965). The fit to a χ_ν^2 distribution for this case gave a value of $\chi^2 = 0\cdot14$ for $\nu = 2\cdot7\pm0\cdot3$, indicating the presence of about three partly open channels (Fig. 1*b*).

Table 2. Channel parameters for ^{235}U fission widths

J^π	ν_{eff}	$\langle\Gamma_1\rangle$ (eV)	$\langle\Gamma_2\rangle$ (eV)	$\langle\Gamma_3\rangle$ (eV)	$\langle D\rangle$ (eV)	N_1	N_2	N_3	E_1 (keV)	E_2 (keV)	E_3 (keV)
3^-	1·2	0·084	0·028	0·0	0·795	0·672	0·213	0·0	−43·5	78·9	>200
4^-	3·9	0·035	0·017	0·0077	0·795	0·279	0·132	0·0609	57·6	114	166

The effective value of ν from equation (8) is given by

$$\nu_{\text{eff}}(J^\pi) = 2\langle\Gamma\rangle^2/(\langle\Gamma^2\rangle-\langle\Gamma\rangle^2). \tag{17}$$

The results for both channels are shown in Table 2. The energies for the transition states were calculated from equation (1a). It can be seen from values of the fission strength functions N_{Jc} that all channels are only partly open and to different degrees, the most open being a channel in the $J^\pi = 3^-$ state.

We conclude that the present work provides evidence for the validity of the theory of partly open fission channels ($\mathscr{P}_2, \mathscr{P}_3$). The fits of the theoretical distributions are summarized in Table 3.

Table 3. Fits of theoretical distributions to experimental data

J^π	Range of $x = \Gamma/\langle\Gamma\rangle$	Experiment	$\mathscr{P}_2$–$\mathscr{P}_3$ distribution	χ_ν^2 distribution
3^-	0·0 to 0·5	0·50	0·43	0·39 ($\nu = 2\cdot1$)
	0·5 to 1·0	0·35	0·23	0·24
	1·0 to 2·0	0·15	0·20	0·24
	2·0 to ∞	0·0	0·14	0·13
4^-	0·0 to 0·5	0·43	0·36	0·34 ($\nu = 2\cdot7$)
	0·5 to 1·0	0·30	0·28	0·27
	1·0 to 2·0	0·28	0·24	0·27
	2·0 to ∞	0·0	0·12	0·12

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