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Mode Coupling in the Solar Corona. IV* Magnetohydrodynamic Waves

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Abstract

A general theory for coupling between MHD waves obliquely incident on a stratified medium is developed. Coupling between the Alfvén mode and the magnetoacoustic mode (the fast mode for $v_A > c_s$ and the slow mode for $v_A < c_s$) is affected little by the finiteness of c_s/v_A for $v_A \gg c_s$, and the coupling becomes weaker as c_s/v_A is increased towards unity. Coupling between the fast and slow modes for $v_A \approx c_s$ is discussed qualitatively using solutions of the MHD counterpart of the Booker quartic equation.

1. Introduction

The magnetohydrodynamic (MHD) wave modes are the Alfvén mode, the fast mode and the slow mode. Alfvén waves have a phase speed of the order of the Alfvén speed while the phase speeds of the other two modes are of the order of the maximum and the minimum respectively of the Alfvén speed v_A and the sound speed c_s. Thus the fast mode is magnetoacoustic in character for $v_A > c_s$ and acoustic for $c_s > v_A$, while the slow mode is acoustic for $v_A > c_s$ and magnetoacoustic for $c_s > v_A$. These waves are thought to be important in the transport of energy to the solar corona, and in order to understand the possible forms of energy transport which might be involved it is necessary to understand how the inhomogeneity of the solar atmosphere affects their propagation. Existing discussions of coupling between the various modes have been restricted to consideration of the special case of vertical incidence (Frisch 1964; Poeverlein 1964) or to coupling at a sharp interface (Simon 1958; Stein 1971). Our primary purpose in this paper is to develop the theory of coupling between MHD waves in the general case of oblique incidence.

Frisch (1964) derived two relatively simple results:

(i) Twists ($\phi' \neq 0$) in the magnetic field lines cause coupling between the Alfvén mode and either the fast mode or the slow mode.

(ii) Bends ($\psi' \neq 0$) in the magnetic field lines and other changes (e.g. in the Alfvén speed) cause coupling between the fast and the slow modes.

On the basis of results derived in Part III of this series (Melrose 1977*a*) one can conclude that Frisch's result (i) is strongly dependent on the assumption of vertical incidence. More importantly, coupling between Alfvén and fast mode waves in the limit $c_s/v_A = 0$ was found to be strongest for nearly parallel propagation, and this stronger coupling is the only coupling which is likely to be significant in practice.

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It is shown in Section 4 below that the results obtained in Part III apply when the MHD theory is valid and for finite c_s^2/v_A^2 (with the exception of $c_s^2 \approx v_A^2$).

Coupling between the fast and the slow modes for $v_A^2 \approx c_s^2$ is discussed qualitatively in Section 5. The possible significance of the results obtained in connection with the heating of the solar corona is discussed in Section 6.

2. Formal Development of Theory

It is convenient to generalize the theory developed by Clemmow and Heading (1954) and in Parts I and II of this series (Melrose 1974*a*, 1974*b*) as follows.

Suppose that within the framework of some general theory (MHD theory below) the equations for a small amplitude disturbance which varies harmonically in x, y and t can be written in the form

$$\mathbf{A}\boldsymbol{e}' = -\mathrm{i}(\omega/c)\mathbf{B}\boldsymbol{e} + \mathbf{C}\boldsymbol{e}, \tag{1}$$

where $\boldsymbol{e}$ is an $n\times 1$ matrix and $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $n\times n$ matrices. It is assumed that $\mathbf{B}$ involves no derivatives of the ambient plasma parameters and that each entry in $\mathbf{C}$ is linear in a derivative of the ambient plasma parameters. For example, in the application to magnetoionic waves, $\boldsymbol{e}$ is chosen to consist of E_x, $-E_y$, B_x and B_y, and $\mathbf{C}$ is assumed to be zero. We shall also ignore $\mathbf{C}$ in our detailed discussion, but it is strictly zero neither in the case of the magnetoionic waves nor in the present application to MHD waves.

Proceeding as in Part I, let $\mathbf{R}$ be an $n\times n$ matrix whose columns are right eigenvectors of $\mathbf{A}^{-1}\mathbf{B}$. In fact, the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$, say q_i with $i = 1,\dots,n$, and the eigenvectors may be written down from a knowledge of the wave properties in a homogeneous medium without even knowing the explicit form of $\mathbf{A}^{-1}\mathbf{B}$. The normalization of the eigenvectors is arbitrary and a diagonal matrix $\mathbf{D}$ is introduced to allow the normalization to be changed. By also introducing a new column vector

$$\boldsymbol{g} = \mathbf{D}^{-1}\mathbf{R}^{-1}\boldsymbol{e} \tag{2}$$

and supposing the integrating factor

$$\exp\left(\mathrm{i}(\omega/c)\int \mathrm{d}z\, \mathbf{Q}\right)$$

to be incorporated into $\mathbf{D}$, equation (1) reduces to

$$\boldsymbol{g}' = \mathbf{G}\boldsymbol{g}, \tag{3}$$

with

$$\mathbf{G} = -\mathbf{D}^{-1}\mathbf{R}^{-1}\mathbf{R}'\mathbf{D} - \mathbf{D}^{-1}\mathbf{D}' + \mathbf{D}^{-1}\mathbf{R}^{-1}\mathbf{A}^{-1}\mathbf{C}\mathbf{R}\mathbf{D}. \tag{4}$$

The normalization of $\boldsymbol{g}$ may now be chosen for convenience (by suitably choosing $\mathbf{D}$), and it is convenient to assume that $g_i g_i^*$ (not summed over i) is the energy flux in the z direction for waves in the ith mode. Conservation of total energy requires that the energy flux be independent of z, that is,

$$0 = \sum_{i=1}^{n} \sigma_i \mathrm{d}(g_i g_i^*)/\mathrm{d}z = \sum_{i,j=1}^{n} \sigma_i (g_i^* G_{ij} g_j + g_i G_{ij}^\dagger g_j^*), \tag{5}$$

where σ_i is $+1$ (-1) for upgoing (downgoing) modes and $\mathbf{G}^\dagger$ is the hermitian conjugate of $\mathbf{G}$. However, the g_i are arbitrary and hence equation (5) implies

$$G_{ij} = -\sigma_i \sigma_j G_{ji}^* \quad \text{(no sum)}. \tag{6}$$

In particular, the diagonal components of $\mathbf{G}$ are pure imaginary or zero, and they may be chosen as zero by suitable choice of $\mathbf{D}$.

Let a characteristic length associated with the coupling between modes i and j be defined by

$$L_{ij} = |G_{ij}|^{-1}. \tag{7}$$

The symmetry property (6) implies $|G_{ij}| = |G_{ji}|$, and the coupling ratio Q^{ij} between modes i and j corresponds to

$$Q^{ij} = \{(\omega/2c)\,|q_i - q_j|\,L_{ij}\}^{-1} \quad \text{(no sum)}. \tag{8}$$

Now using equations (4) and (6) one finds

$$|G_{ij}| = |(\mathbf{R}^{-1}\mathbf{R}' - \mathbf{R}^{-1}\mathbf{A}^{-1}\mathbf{C}\mathbf{R})_{ij} \times (\mathbf{R}^{-1}\mathbf{R}' - \mathbf{R}^{-1}\mathbf{A}^{-1}\mathbf{C}\mathbf{R})_{ji}|^{\frac{1}{2}} \quad \text{(no sum)}. \tag{9}$$

The interesting feature of this result is that the expression for $|G_{ij}|$ is independent of $\mathbf{D}$, that is, the normalization chosen for the eigenvectors when constructing $\mathbf{R}$ is immaterial.

Finally, if $\mathbf{C}$ is now neglected, one has

$$L_{ij} = |\Gamma_{ij}\Gamma_{ji}|^{-\frac{1}{2}} \quad \text{(no sum)}, \tag{10}$$

with (cf. Parts I and II)

$$\mathbf{\Gamma} = -\mathbf{R}^{-1}\mathbf{R}'. \tag{11}$$

Although the normalization of the columns of $\mathbf{R}$ may be chosen arbitrarily, to calculate L_{ij} using equation (10), both Γ_{ij} and Γ_{ji} must be calculated because $\mathbf{\Gamma}$ satisfies no symmetry property such as (6).

The neglect of $\mathbf{C}$ may be justified as follows. It is obvious from the form of equation (9) that $\mathbf{C}$ is important only if $|(\mathbf{R}^{-1}\mathbf{A}^{-1}\mathbf{C}\mathbf{R})_{ij}|$ is greater than $|(\mathbf{R}^{-1}\mathbf{R}')_{ij}|$ (or $|(\mathbf{R}^{-1}\mathbf{A}^{-1}\mathbf{C}\mathbf{R})_{ji}|$ is greater than $|(\mathbf{R}^{-1}\mathbf{R}')_{ji}|$). Now the entries in $\mathbf{R}^{-1}\mathbf{A}^{-1}\mathbf{C}\mathbf{R}$ must be of the order of the inverse of the scale length over which the background plasma parameters vary, and entries in $\mathbf{R}^{-1}\mathbf{R}'$ are of order $(L_{ij})^{-1}$. Consequently, the contribution from $\mathbf{C}$ could be important only when the lengths L_{ij} are longer than the characteristic lengths over which the background plasma parameters vary. Besides this probably never being the case, even if it were, the mode coupling would then be relatively weak. In other words, the neglect of $\mathbf{C}$ is justified whenever the lengths L_{ij}, as defined by equation (10), are shorter than the typical lengths over which the plasma parameters vary, and it is only in such cases that mode coupling is likely to be of any significance.

3. Coupling Ratio for MHD Waves

Although it is straightforward to start from the MHD equations and reduce them to a set of equations of the form (1) (this is done in Appendix 1), it is not necessary to do so. All we require to treat the mode coupling in detail is a knowledge of the eigenvalues and eigenvectors of $\mathbf{A}^{-1}\mathbf{B}$, and these may be constructed from a knowledge of the properties of the MHD waves in a homogeneous medium. However, we do need to know what set of field quantities constitute an acceptable set of independent variables. The derivation given in Appendix 1 shows that an acceptable set consists of the three components of the fluid displacement $\boldsymbol{\xi}$ together with $(c/\omega)\boldsymbol{k}.\boldsymbol{\xi}$, and the x and y components of $(c/\omega)\boldsymbol{k}\times(\boldsymbol{\xi}\times\boldsymbol{b})$.

The wave properties for the MHD waves in a homogeneous medium with the displacement current neglected may be written down in the following form (Melrose 1975). Firstly, the refractive indices for the Alfvén (A), fast (+) and slow (−) modes are given by

$$\mu^{\mathrm{A}} = c/v_{\mathrm{A}}|\cos\theta|, \qquad \mu^{\pm} = c/v_{\phi\pm}, \tag{12a, b}$$

with

$$v_{\phi\pm}^2 = \tfrac{1}{2}(c_{\mathrm{s}}^2+v_{\mathrm{A}}^2)\pm\tfrac{1}{2}\{(c_{\mathrm{s}}^2+v_{\mathrm{A}}^2)^2-4c_{\mathrm{s}}^2v_{\mathrm{A}}^2\cos^2\theta\}^{\frac{1}{2}}. \tag{13}$$

The normalized (to unity) fluid displacements are

$$\boldsymbol{\xi}^{\mathrm{A}} = \boldsymbol{a}, \qquad \boldsymbol{\xi}^{\pm} = \cos(\Psi^{\pm}-\theta)\boldsymbol{\kappa}+\sin(\Psi^{\pm}-\theta)\boldsymbol{\tau}, \tag{14a, b}$$

with

$$\tan\Psi^{\pm} = \frac{c_{\mathrm{s}}^2\sin\theta\cos\theta}{v_{\phi\pm}^2-v_{\mathrm{A}}^2-c_{\mathrm{s}}^2\sin^2\theta} = \frac{v_{\phi\pm}^2-c_{\mathrm{s}}^2\cos^2\theta}{c_{\mathrm{s}}^2\sin\theta\cos\theta}. \tag{15}$$

The triad of unit vectors $\boldsymbol{\kappa}$, $\boldsymbol{\tau}$ and $\boldsymbol{a}$ was defined by equations (11a, b, c) of Part II:

$$\boldsymbol{\kappa} = (r,0,q)/(r^2+q^2)^{\frac{1}{2}}, \tag{16a}$$

$$\boldsymbol{\tau} = \big(-\alpha q, -(r^2+q^2)^{\frac{1}{2}}\beta, \alpha r\big)/(r^2+q^2)^{\frac{1}{2}}(\alpha^2+\beta^2)^{\frac{1}{2}}, \tag{16b}$$

$$\boldsymbol{a} = \big(\beta q, -(r^2+q^2)^{\frac{1}{2}}\alpha, -\beta r\big)/(r^2+q^2)^{\frac{1}{2}}(\alpha^2+\beta^2)^{\frac{1}{2}}, \tag{16c}$$

where two minor omissions are corrected. The quantities α and β are as defined by equations (12) of Part II.

The eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ may be found by substituting

$$\mu = (r^2+q^2)^{\frac{1}{2}}, \qquad \mu\cos\theta = q\cos\psi+r\sin\psi\cos\phi \tag{17a, b}$$

in equations (12a) and (12b). One obtains a quadratic equation

$$(q\cos\psi+r\sin\psi\cos\phi)^2 = c^2/v_{\mathrm{A}}^2, \tag{18}$$

whose two solutions correspond to upgoing and downgoing Alfvén modes, and a quartic equation

$$Aq^4+Bq^3+Cq^2+Dq+E = 0, \tag{19}$$

with

$$A = \cos^2\psi, \tag{20a}$$

$$B = 2r\cos\psi\sin\psi\cos\phi, \tag{20b}$$

$$C = r^2(\cos^2\psi+\sin^2\psi\cos^2\phi)-(c^2/v_{\mathrm{A}}^2+c^2/c_{\mathrm{s}}^2), \tag{20c}$$

$$D = 2r^3 \cos\psi \sin\psi \cos\phi\,, \tag{20d}$$

$$E = r^4 \sin^2\psi \cos^2\phi - r^2(c^2/v_A^2 + c^2/c_s^2) + (c^2/c_s^2)(c^2/v_A^2)\,. \tag{20e}$$

In the cases of interest the four solutions of equation (19) may be interpreted as upgoing fast and slow modes and downgoing fast and slow modes. (No attempt has been made to prove that two of the solutions of equation (19) always correspond to the plus sign in equation (13), with the other two corresponding to the minus sign, but it does seem that this is the case.)

The six columns of **R** may now be written down as follows. Each column may be chosen to correspond to entries ξ_x, ξ_y, ξ_z, $(c/\omega)\boldsymbol{k}\,.\,\boldsymbol{\xi}$, $(c/\omega)[\boldsymbol{k}\times(\boldsymbol{\xi}\times\boldsymbol{b})]_x$ and $(c/\omega)[\boldsymbol{k}\times(\boldsymbol{\xi}\times\boldsymbol{b})]_y$. Also, because the normalization of each column is arbitrary, one may multiply each column by any suitable factor to simplify it. Two columns correspond to the Alfvén modes and are given by substituting a solution of equation (18) in

$$\boldsymbol{e} = \begin{bmatrix} q\beta \\ -\mu\alpha \\ -\beta r \\ 0 \\ \beta q \mu \cos\theta \\ -\alpha\mu^2\cos\theta \end{bmatrix}, \tag{21a}$$

where equation (14a) has been used. In equation (21a) μ and $\cos\theta$ are to be constructed by inserting the relevant solution of equation (18) in (17a, b). The other four columns are constructed by inserting each of the four solutions of equation (19) in

$$\boldsymbol{e} = \begin{bmatrix} -\alpha q \sin(\Psi-\theta) + r\sin\theta\cos(\Psi-\theta) \\ -\mu\beta\sin(\Psi-\theta) \\ q\sin\theta\cos(\Psi-\theta) + \alpha r\sin(\Psi-\theta) \\ \mu^2\sin\theta\cos(\Psi-\theta) \\ -\alpha\mu q\sin\Psi \\ -\beta\mu^2\sin\Psi \end{bmatrix}, \tag{21b}$$

where equation (14b) has been used. Again μ and $\cos\theta$ are to be constructed using equations (17a, b) and Ψ is to be constructed by inserting the resulting expression for v_ϕ^2 in place of $v_{\phi\pm}^2$ in equation (15). Having thus constructed the q's and **R**, the remainder of the calculation of the coupling coefficients Γ_{ij} and of the coupling ratios Q^{ij} is straightforward (but very tedious in general).

For a later purpose it is relevant to note that the choice of variables made in equations (21a, b) is not always the most convenient. In place of ξ_x, ξ_y, and ξ_z it is sometimes more convenient to choose E_x, E_y and $\boldsymbol{b}\,.\,\boldsymbol{\xi}$, where $\boldsymbol{E}$ is proportional to $\boldsymbol{\xi}\times\boldsymbol{b}$. This alternative choice corresponds to replacing some rows in **R**, as it is constructed above, by linear combinations of these rows. In particular, if we choose E_x, E_y, B_z, B_y, $\boldsymbol{b}\,.\,\boldsymbol{\xi}$ and $(c/\omega)\boldsymbol{k}\,.\,\boldsymbol{\xi}$ as the six variables, then equations (21a, b) are replaced by respectively

$$e = \begin{bmatrix} r\sin^2\theta - \alpha q\cos\theta \\ -\mu\alpha\beta\cos\theta \\ \mu\beta q\cos\theta \\ -\mu^2\alpha\cos\theta \\ 0 \\ 0 \end{bmatrix} \tag{22a}$$

and

$$e = \begin{bmatrix} \beta q \\ -\alpha\mu \\ \alpha\mu q \\ \mu^2\beta \\ -\cot\Psi \\ -\mu\cos(\Psi-\theta)/\sin\Psi \end{bmatrix}. \tag{22b}$$

4. Alfvén and Fast Mode Waves

(a) Conditions $v_A \gg c_s$

It is straightforward to show that the refractive indices for the fast mode and the Alfvén mode are equal only for $\sin\theta = 0$ for $v_A \gg c_s$. This implies that the coupling point between these two modes is for parallel propagation, as pointed out in Part III, but only for $v_A \gg c_s$. For $v_A < c_s$, $\sin\theta = 0$ is the coupling point between the Alfvén and slow modes.

Firstly, let us show that the MHD theory reproduces the results of cold plasma theory in the regime where the two theories overlap. Specifically, we wish to show that the theory developed in this paper reproduces the results of Part III in detail in the limit $c_s \to 0$. (We exclude the range of very small θ where the waves become circularly polarized in the cold plasma theory.)

In the limit $v_A/c_s \to \infty$ the properties of the fast mode reduce to those of the magnetoacoustic mode of cold plasma theory. Suppose the matrix **R** is constructed using equations (22a, b) with the first four columns chosen to correspond to A+, F+, A−, F− with the final two columns being S+ and S− (F and S denote fast and slow respectively). By inspection the upper left 4×4 submatrix is equivalent to the matrix **R** written down in equation (9) of Part III. This equivalence (the normalization is unimportant) is the basis of the proof of the equivalence of the two theories. A proof requires that the determinant of **R** be proportional to the determinant of the leading 4×4 submatrix in the limit $c_s \to 0$.

Consider the fifth and sixth columns of **R**. Now r must be less than about c/v_A (because otherwise Alfvén and fast mode waves would be evanescent) and hence one has $q^2 \gg r^2$ and $q^2 \approx \mu^2 \approx c^2/c_s^2\cos^2\psi$ for the slow mode. Also equation (15) implies that Ψ is of order c_s^2/v_A^2 for the slow mode, and $\Psi-\frac{1}{2}\pi$ is of order c_s^2/v_A^2 for the fast mode. The ordering of the entries in the fifth and sixth columns is therefore c/c_s, c/c_s, $(c/c_s)^2$, $(c/c_s)^2$, v_A^2/c_s^2 and $(c/c_s)v_A^2/c_s^2$. The fifth row has entries of order

0, 0, c_s^2/v_A^2, c_s^2/v_A^2, v_A^2/c_s^2 and v_A^2/c_s^2, which may be replaced by $0, 0, 0, 0, ., .$ (where the dot stands for any finite entry) when expanding in c_s^2/v_A^2. The sixth row has entries of order $0, 0$, c/v_A, c/v_A, $(c/c_s)v_A^2/c_s^2$ and $(c/c_s)v_A^2/c_s^2$, whose fifth and sixth entries are of order $(v_A/c_s)^3$ larger than the third and fourth. Finally, when one notes that the largest terms in the leading 4×4 submatrix are of order c^2/v_A^2, and when one develops the determinant across the sixth row, one finds that the leading terms are proportional to the determinant of the leading 4×4 submatrix and the corrections are of order c_s/v_A smaller. Thus to the lowest order in c_s/v_A the present theory reproduces the results of Part III.

(*b*) *Conditions* $v_A \gtrsim c_s$

Coupling between Alfvén and fast mode waves is necessarily strong for nearly parallel propagation for $v_A \gtrsim c_s$. Although nearly parallel propagation has been treated in the limit $c_s/v_A = 0$ in Part III, it is relevant to generalize the result obtained to the case where c_s/v_A is finite and of order unity.

Firstly, let us prove that there is a coupling point for Alfvén and fast mode waves for parallel propagation. For the (upgoing) Alfvén mode it is elementary to show that parallel propagation requires

$$\phi = 0\,, \qquad r = (c/v_A)\sin\psi\,, \tag{23a, b}$$

which corresponds to

$$q = (c/v_A)\cos\psi\,. \tag{24}$$

Substituting equations (23) in the quartic equation (19) with the expressions (20), one finds that the resulting equation can be factorized as

$$\left(q-\frac{c\cos\psi}{v_A}\right)\left\{-\frac{c^2}{c_s^2}\left(q+\frac{c\cos\psi}{v_A}\right)\right.$$
$$\left.+\cos\psi\left(q^2+\frac{c^2\sin^2\psi}{v_A^2}\right)\left(q\cos\psi+\frac{c(1+\sin^2\psi)}{v_A}\right)\right\} = 0\,. \tag{25}$$

The solution $q = (c/v_A)\cos\psi$ corresponds to the upgoing fast mode for $v_A > c_s$ (and to the upgoing slow mode for $v_A < c_s$). Thus the stated coupling point does exist.

In the neighbourhood of the coupling it is relatively simple to calculate the properties of the upgoing Alfvén and fast modes, i.e. the two modes of interest. However, the properties of the other four modes are required, in principle at least, to treat the coupling, and, although the properties of the downgoing Alfvén mode are relatively simple, the properties of the other three modes are very cumbersome. In particular, one must solve the cubic equation included in (25) for the q's for the three modes and substitute the solutions in equations (17a, b) and in the equation for Ψ (cf. equation 15) to find the relevant quantities to insert in equations (21b) or (22b). Fortunately, we find below that we do not require explicit expressions for the wave properties of the four modes which are not involved in the coupling.

Let the matrix $\mathbf{R}$ be constructed from equations (21a, b) with the first and second columns corresponding to the upgoing Alfvén and fast modes, the third column to the downgoing Alfvén mode and the remaining three columns to the other three modes. It is necessary to retain finite β in the first two columns but not in the remaining four. The properties of the upgoing Alfvén mode follow directly from equation (21a). For the upgoing fast mode, one requires the value of Ψ in equation (21b), and the relation (15) with $\theta = 0$ implies $\Psi = \frac{1}{2}\pi$ for the fast mode. Thus we find

$$\mathbf{R} = \begin{bmatrix} q\beta & -q\alpha & 0 & R_{14} & R_{15} & R_{16} \\ -\mu\alpha & -\mu\beta & R_{23} & 0 & 0 & 0 \\ -r\beta & r\alpha & 0 & R_{34} & R_{35} & R_{36} \\ 0 & 0 & 0 & R_{44} & R_{45} & R_{46} \\ q\mu\beta & -q\mu\alpha & 0 & R_{54} & R_{55} & R_{56} \\ -\mu^2\alpha & -\mu^2\beta & R_{63} & 0 & 0 & 0 \end{bmatrix}. \tag{26}$$

In this result r and q are given by equations (23b) and (24) respectively, θ is zero for both modes and one has $\mu = c/v_{\mathrm{A}}$.

To calculate the coupling coefficients Γ_{12} and Γ_{21} using equation (11), it is necessary to construct the first two rows of the matrix $\mathbf{R}^{-1}$ which is the inverse of (26). To this end, let D_i be the determinant of the 3×3 matrix obtained from

$$\begin{bmatrix} R_{14} & R_{15} & R_{16} \\ R_{34} & R_{35} & R_{36} \\ R_{44} & R_{45} & R_{46} \\ R_{54} & R_{55} & R_{56} \end{bmatrix}$$

by deleting the ith row ($i = 1, 3, 4$ or 5). One finds that the determinant of $\mathbf{R}$ reduces to

$$\det\mathbf{R} = \mu\sin^2\theta(-\mu R_{23}+R_{63})(qD_1+rD_3-q\mu D_5). \tag{27}$$

Hence, from the cofactors of the nonzero elements in the first two columns of the matrix (26), one finds

$$R_{11}^{-1} = \frac{\beta D_1}{\sin^2\theta(qD_1+rD_3-q\mu D_5)}, \qquad R_{21}^{-1} = \frac{-\alpha D_1}{\sin^2\theta(qD_1+rD_3-q\mu D_5)}, \tag{28a}$$

$$R_{12}^{-1} = \frac{-\alpha R_{63}}{\mu\sin^2\theta(-\mu R_{23}+R_{63})}, \qquad R_{22}^{-1} = \frac{-\beta R_{63}}{\mu\sin^2\theta(-\mu R_{23}+R_{63})}, \tag{28b}$$

$$R_{13}^{-1} = \frac{-\beta D_3}{\sin^2\theta(qD_1+rD_3-q\mu D_5)}, \qquad R_{23}^{-1} = \frac{\alpha D_3}{\sin^2\theta(qD_1+rD_3-q\mu D_5)}, \tag{28c}$$

$$R_{15}^{-1} = \frac{-\beta D_5}{\sin^2\theta(qD_1+rD_3-q\mu D_5)}, \qquad R_{25}^{-1} = \frac{\alpha D_5}{\sin^2\theta(qD_1+rD_3-q\mu D_5)}, \tag{28d}$$

$$R_{16}^{-1} = \frac{\alpha R_{23}}{\mu\sin^2\theta(-\mu R_{23}+R_{63})}, \qquad R_{26}^{-1} = \frac{\beta R_{23}}{\mu\sin^2\theta(-\mu R_{23}+R_{63})}. \tag{28e}$$

Only two steps remain in the derivation of the coupling. The first is the calculation of the coupling coefficients. As in Part III, the only derivatives which need be retained in the first two columns of the matrix (26) are those of α and β, and then using equation (11) one obtains

$$\Gamma_{12} = (\alpha'\beta - \beta'\alpha)/\sin^2\theta = -\Gamma_{21}. \tag{29}$$

The other step is the calculation of the difference in the q's. From equations (12), (13) and (24) we find

$$q_1 - q_2 \approx \frac{\mu_1^2 - \mu_2^2}{q_1 + q_2} = \frac{v_{\mathrm{A}}}{2c\cos\psi}\left(\frac{c^2}{v_{\mathrm{A}}^2\cos^2\theta} - \frac{c^2}{v_{\phi+}^2}\right)$$

$$\approx \frac{c}{2v_{\mathrm{A}}\cos\psi}\tan^2\theta \qquad \text{for} \quad v_{\mathrm{A}}^2 \gg c_{\mathrm{s}}^2\,, \tag{30a}$$

$$\approx \frac{c}{2v_{\mathrm{A}}\cos\psi}\left(\sin^2\theta + \sin\theta\{(1+t^2)^{\frac{1}{2}} - t\}\right) \qquad \text{for} \quad v_{\mathrm{A}}^2 \approx c_{\mathrm{s}}^2,\ \sin\theta \ll 1\,, \tag{30b}$$

with

$$t \equiv \frac{x}{\sin\theta}, \qquad x \equiv \frac{v_{\mathrm{A}}^2 - c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2 + c_{\mathrm{s}}^2}. \tag{31a, b}$$

(Specifically, the approximate form for $v_{\mathrm{A}}^2 \approx c_{\mathrm{s}}^2$ applies for $x \ll \frac{1}{2}$.) For $v_{\mathrm{A}}^2 \gg c_{\mathrm{s}}^2$ the coupling coefficient reduces to

$$Q = \frac{4v_{\mathrm{A}}}{\omega}\frac{\sin\psi\,|\cos\psi|}{\theta^4}\,|(\psi' - \rho')\phi - (\psi - \rho)\phi'|\,, \tag{32}$$

which is identical with the corresponding result from cold plasma theory (cf. equation (40) of Part III). For $t\sin\theta \ll 1$, that is, for

$$x \ll 1\,, \tag{33}$$

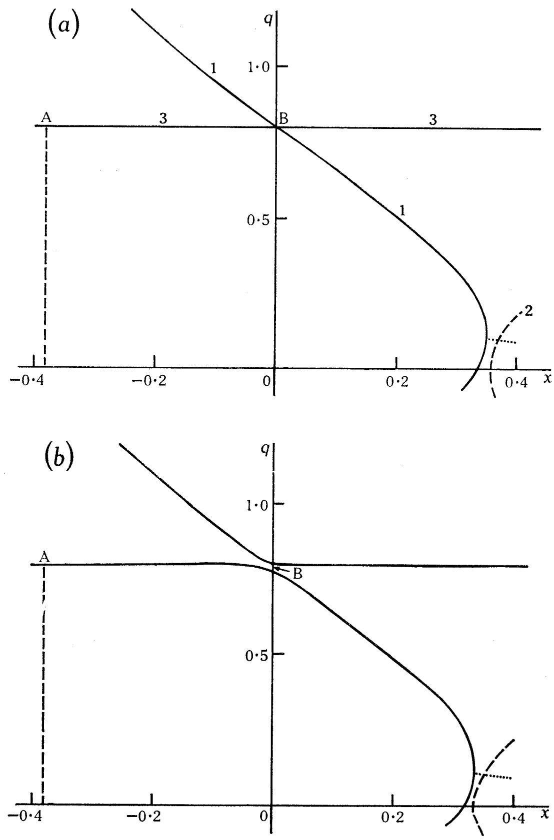
the coupling coefficient becomes, in place of equation (32),

$$Q = \frac{4v_{\mathrm{A}}}{\omega}\frac{\sin\psi\,|\cos\psi|}{\theta^3}\frac{|(\psi' - \rho')\phi - (\psi - \rho)\phi'|}{(t^2+1)^{\frac{1}{2}} - t}. \tag{34}$$

Thus whereas Q varies with θ roughly as θ^{-3} for $v_{\mathrm{A}}^2 \gg c_{\mathrm{s}}^2$, for $v_{\mathrm{A}}^2 \approx c_{\mathrm{s}}^2$ the coupling is less efficient with Q varying as θ^{-2} for $t \lesssim \theta$.

5. Fast and Slow Modes

It follows from equation (25) that for $c_{\mathrm{s}} = v_{\mathrm{A}}$ and $\sin\theta = 0$ there is a 'double' coupling point. At this point the upgoing Alfvén, fast and slow modes all have the same q value, specifically $q = (c/v_{\mathrm{A}})\cos\psi$. It would be of interest to consider the relative efficiency of the couplings between all three modes in the neighbourhood of this double coupling point. However, the algebra involved becomes excessively cumbersome and we do not attempt an analytic treatment here. (An analytic treatment is given in Part V (Melrose 1977*b*, present issue pp. 661–9) using a simplified mode-coupling theory.) In this section we discuss qualitatively the coupling between fast and slow modes for $\sin\theta \approx 0$ and small x as defined by equation (31b).



Figs 1*a*–1*d*. Roots of the quartic equation (19) plotted as functions of q (in units of c/c_s) versus $x \equiv (v_A^2 - c_s^2)/(v_A^2 + c_s^2)$ for fixed c_s. The solid curves denote real solutions while the dashed and dotted curves respectively denote the real and imaginary parts of complex roots. The graphs are for the particular parameter values:

(*a*) $r = (c/c_s)\sin\psi$, $\cos\psi = 0\cdot8$, $\phi = 0$;
(*b*) $r = (c/c_s)(\sin\psi + 0\cdot01)$, $\cos\psi = 0\cdot8$, $\phi = 0$;
(*c*) $r = (c/c_s)\sin\psi$, $\cos\psi = 0\cdot2$, $\phi = 0$;
(*d*) $r = (c/c_s)(\sin\psi + 0\cdot003)$, $\cos\psi = 0\cdot2$, $\phi = 0$.

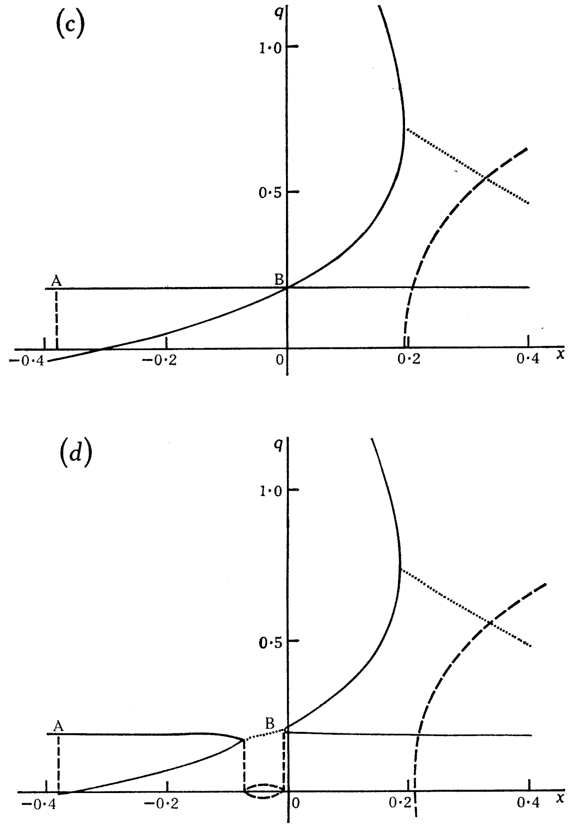
The discussion is based on graphs of solutions of the quartic equation (19). Note that the Alfvén mode is not considered at all here.

A preliminary point is that in a slowly varying medium approximate solutions of equation (1) for $\boldsymbol{e}$ are given by the WKB formula (e.g. Budden 1961, Ch. 18)

$$\boldsymbol{e} = A_i \boldsymbol{e}^{[i]} \exp\left(\mathrm{i}(\omega/c) \int \mathrm{d}z\, q_i\right), \qquad i = 1, ..., 6, \tag{35}$$

where $\boldsymbol{e}^{[i]}$ is the eigenvector of $\mathbf{A}^{-1}\mathbf{B}$ corresponding to the eigenvalue q_i. The A_i in equation (35) are functions of z and describe the amplitude of the disturbance. A solution of the form (35) is implied by the expression 'waves in a natural mode' of the inhomogeneous medium.

Fig. 1*a* displays the roots of the quartic equation (19) for $c_s = \text{const.}$, $r = (c/c_s)\sin\psi$, $\cos\psi = 0\cdot8$ and $\phi = 0$. The branch labelled 3 may be shown to be



Figs 1*c* and 1*d*

the upgoing fast mode for $c_s > v_A$ $(x < 0)$ and the upgoing slow mode for $v_A > c_s$ $(x > 0)$. The 'double' coupling point is at $x = 0$, $\phi = 0$ and $r = (c/c_s)\sin\psi$ $(= (c/v_A)\sin\psi)$, and in this case the curves pass through this point. Suppose that at point A in Fig. 1*a* the waves are directed upward in the fast mode. As v_A/c_s increases the waves remain in branch 3, which implies that they change from the fast mode to the slow mode at point B. From the viewpoint of mode-coupling theory, one would say that the coupling ratio at the coupling point is infinite and that a complete change of mode occurs. (This conclusion is subject to the important proviso that coupling to the Alfvén mode is neglected.)

In Fig. 1*b* the solutions are displayed for $r = (c/c_s)(\sin\psi + 0\cdot01)$, $\cos\psi = 0\cdot8$ and $\phi = 0$. In this case the solutions pass close to the coupling point at B but not directly through it. Although there is not a single branch like that labelled 3 in Fig. 1*a*, one might still expect fast-mode waves at A to be partially converted into slow-mode waves at B. Qualitatively, the inhomogeneity in the plasma causes an intrinsic uncertainty in wavenumber, say an uncertainty Δq in q, and if this uncertainty exceeds the difference in the q values between two modes then the two modes cannot be regarded as distinct. In fact the coupling ratio Q is defined as effectively in the ratio of Δq to the difference in the two q values. Hence for $Q \gg 1$, in the neighbourhood of point B the fast-mode waves would change into slow-mode waves as implied by Fig. 1*a*, while for $Q \ll 1$ the waves remain in the fast mode and follow the continuous curve on which it is located at point A.

In Fig. 1*c* the roots are displayed for $r = (c/c_s)\sin\psi$, $\cos\psi = 0\cdot 2$ and $\phi = 0$. This illustrates the manner in which the curves in Fig. 1*a* deform as the direction of the magnetic field is changed subject to the condition that the curves pass through the coupling point. In Fig. 1*d* the parameters are $r = (c/c_s)(\sin\psi + 0\cdot 003)$, $\cos\psi = 0\cdot 2$ and $\phi = 0$. As in the small change from Figs 1*a* to 1*b* the curves pass in the neighbourhood of the coupling point but not through it. Unlike Fig. 1*b*, fast-mode waves would be reflected before reaching point B. In this case conversion to the slow mode requires tunnelling through a region (near the point B) where the waves are evanescent. The tunnelling is effective in converting fast-mode waves at $x < 0$ into slow-mode waves at $x > 0$ only if the distance over which the waves are evanescent times the imaginary part of q in the region of evanescence is much less than unity.

The curve corresponding to the upgoing fast mode for $c_s =$ const. and v_A/c_s increasing extends only to a finite value of x (for $r \neq 0$). The maximum value of x achieved is at the reflection point. In Fig. 1*b* the reflection point occurs for $x > 0$ and it can be ignored in discussing the coupling. In Fig. 1*d* there are two reflection points and no coupling point. What occurs is that as $\cos\psi$ decreases from $0\cdot 8$ to $0\cdot 2$, corresponding to the transition from Figs 1*b* to 1*d*, the coupling point and the reflection point for the fast mode move closer together, coincide and then separate as two reflection points, one for the fast mode and the other for the slow mode.

6. Discussion

From a formal viewpoint we have seen that it is straightforward to treat the general case of coupling between MHD waves obliquely incident on a stratified medium using the method of Clemmow and Heading (1954). We have used this method to show that fast Alfvén-mode coupling for $v_A \gg c_s$ is equivalent to magneto-acoustic Alfvén-mode coupling in cold plasma theory (Part III), as one would expect. More generally, it is clear that existing treatments of the coupling between MHD waves (Frisch 1964; Poeverlein 1964) apply only to very special cases, and would lead to misleading results if generalized. The most important coupling occurs for nearly parallel propagation, which case could not be considered properly by the earlier authors due to their assumption of 'vertical incidence'.

From a practical viewpoint, application of Clemmow and Heading's (1954) method to obliquely incident MHD waves leads to an excessively cumbersome theory. The treatment of coupling between all three modes near the double coupling point at $\sin\theta = 0$ and $c_s = v_A$ has not been pursued here simply because of the lengthy algebra involved. However, this special case is of interest in practice. For example, in the theory of the heating of the solar corona discussed in Part III, an upward flux in the fast mode must pass through a region with $c_s = v_A$ to reach the corona (or the flux be converted into the Alfvén mode before reaching this region). One would like to be able to estimate the relative efficiencies of conversion of an initial fast-mode flux for $c_s > v_A$ into fluxes in all three modes for $v_A > c_s$. In Section 5 we discussed the fast–slow mode coupling from a qualitative viewpoint, but we did not consider this coupling quantitatively nor did we consider the simultaneous coupling between all three modes. A quantitative treatment is given in the following paper (Part V, Melrose 1977*b*) where a simplified version of mode–coupling theory is developed.

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Appendix

The MHD equations (with the displacement current neglected) are

$$\eta\,\mathrm{d}\boldsymbol{v}/\mathrm{d}t = -\operatorname{grad} p + c^{-1}\boldsymbol{j}\times\boldsymbol{B}, \qquad \partial\eta/\partial t + \operatorname{div}(\eta\boldsymbol{v}) = 0, \tag{A1,2}$$

$$\operatorname{curl}\boldsymbol{E} = -c^{-1}\,\partial\boldsymbol{B}/\partial t, \qquad \operatorname{curl}\boldsymbol{B} = 4\pi c^{-1}\boldsymbol{j}, \tag{A3,4}$$

$$\boldsymbol{E} = -c^{-1}\boldsymbol{v}\times\boldsymbol{B}, \qquad \mathrm{d}(\eta p^{-\gamma})/\mathrm{d}t = 0, \tag{A5,6}$$

where η is the fluid density, $\boldsymbol{v}$ is the fluid velocity, p is the pressure, γ is the adiabatic index and the other symbols are as in Parts I and II. Let each quantity be separated into a part which describes the background value (subscript 0) and a part associated with the wave motion (subscript 1), and let $\boldsymbol{v}_0$ and $\boldsymbol{E}_0$ be zero. Now linearize the equations in the small quantities, Fourier transform with respect to x, y and t, choosing the axes such that k_y vanishes, and replace $\boldsymbol{v}_1$ by $-\mathrm{i}\omega\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is the fluid displacement. On eliminating $\boldsymbol{j}_0$, $\boldsymbol{j}_1$, $\boldsymbol{E}_1$ and p_1 we then find

$$-\omega^2\eta_0\,\xi_x = -c_s^2 k_x\eta_1 + (4\pi)^{-1}\{B_{1z}B'_{0x} + B_{0z}B'_{1x} - \mathrm{i}k_x(B_{0y}B_{1y} + B_{0z}B_{1z})\}, \tag{A7}$$

$$-\omega^2\eta_0\,\xi_y = (4\pi)^{-1}\{B_{1z}B'_{0y} + B_{0z}B'_{1y} + \mathrm{i}k_x B_{0x}B_{1y}\}, \tag{A8}$$

$$\begin{aligned}\omega^2\eta_0\,\xi_z = -(c_s^2\eta_1)' + (4\pi)^{-1}\{&-B_{0x}B'_{1x} - B'_{0x}B_{1x} - B_{0y}B'_{1y}\\ &- B'_{0y}B_{1y} + \mathrm{i}k_x B_{0x}B_{1z}\},\end{aligned} \tag{A9}$$

$$\eta_1 + \eta'_0\,\xi_z + \mathrm{i}k_x\eta_0\,\xi_x + \eta_0\,\xi'_z = 0, \tag{A10}$$

$$B_{1x} = \xi'_x B_{0z} - \xi'_z B_{0x} - \xi_z B'_{0x}, \tag{A11}$$

$$B_{1y} = \mathrm{i}k_x B_{0x}(\xi_y - \xi_x) + \xi'_y B_{0z} - \xi'_z B_{0y} - \xi_z B'_{0y}, \tag{A12}$$

$$B_{1z} = \mathrm{i}k_x(B_{0x}\xi_z - \xi_x B_{0z}). \tag{A13}$$

The final equation (A13) involves no derivatives and may be used to eliminate B_{1z}.

The resulting six equations may be written in the form (1) of Section 2 with the components of $\boldsymbol{e}$ consisting of ξ_x, ξ_y, ξ_z, η_1, B_{1x} and B_{1y}. Furthermore any one of the six variables may be multiplied by a function of the background variables (the resulting derivatives of which are incorporated into $\mathbf{C}$), and any other six linear combinations of these six variables may be chosen in their stead.

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