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Thermal Instability of a Partially Ionized Plasma

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Abstract

A study is made of the thermal hydromagnetic instability of a rotating and finitely conducting composite medium, including frictional effects with neutrals. The prevalent magnetic field is assumed to be uniform and vertical. The effects of the magnetic field and rotation are found to be stabilizing on the thermal instability of such a composite medium. The case of a partially ionized medium is also considered, including the effect of Hall currents in the presence of a uniform vertical magnetic field. For stationary convection, the collisions have no effect while the Hall currents are found to have a destabilizing effect on the thermal instability.

1. Introduction

A detailed account of the thermal instability problem, under varying assumptions of hydrodynamics and hydromagnetics, has been given by Chandrasekhar (1961), while Gupta (1967) has studied the thermal instability of fluids including the effect of Hall currents. The medium has been considered to be fully ionized in both these studies. However, a partially ionized plasma represents a state which often exists in the universe. There are several situations where the interaction between the ionized and neutral gas components becomes important in cosmic physics. Stromgren (1939) has reported that ionized hydrogen is limited to certain rather sharply bounded regions in space surrounding, for example, O-type stars and clusters of such stars, and that the gas outside these regions is essentially non-ionized. Other examples of the existence of such situations are given by Alfvén's (1954) theory on the origin of the planetary system, where a high ionization rate is suggested to appear from collisions between a plasma and a neutral gas cloud and by the absorption of plasma waves due to ion–neutral collisions such as in the solar photosphere and chromosphere and in cool interstellar clouds (Piddington 1954; Lehnert 1959). Lehnert (1972) has found that both ion viscosity and neutral gas friction have a stabilizing influence on cosmical plasma interacting with a neutral gas.

The medium to be considered may therefore be idealized as a composite mixture of a hydromagnetic (ionized) component and a neutral component, the two interacting through mutual collisional (frictional) effects. Such a composite medium cannot be treated as a multicomponent plasma. Hans (1968) and Bhatia (1970) have shown that the collisions have a stabilizing effect on the Rayleigh–Taylor instability. However, for the Kelvin–Helmholtz configuration, Rao and Kalra (1967) and Hans (1968) have found that the collisional effects are in fact destabilizing for a sufficiently large collision frequency. Bhatia and Gupta (1973) have studied the gravitational instability of a finitely conducting hydromagnetic composite plasma, including effects due to finite Larmor radius, Hall currents and collisions with neutrals, and have found that Jeans's criterion remains unchanged in the presence of these effects. Sharma (1976) has studied the thermal hydromagnetic instability of a partially ionized medium.

In the present paper, we examine the thermal instability of a partially ionized medium in the presence of uniform rotation and a uniform magnetic field. In Section 4, we also consider the thermal instability of a partially ionized plasma in the presence of a uniform vertical magnetic field but include the effect of Hall currents.

2. Perturbation Equations

Here we consider an infinite horizontal composite layer of thickness d consisting of a finitely conducting hydromagnetic fluid of density ρ which is permeated with neutrals of density ρ_d, and acted on by a uniform vertical magnetic field $\boldsymbol{H}(0,0,H)$, a uniform rotation $\boldsymbol{\Omega}$ $(0,0,\Omega)$ and a gravity force $\boldsymbol{g}(0,0,-g)$. This layer is heated from below such that a steady adverse temperature gradient β $(=|\mathrm{d}T/\mathrm{d}z|)$ is maintained. We make the assumptions that both the ionized fluid and the neutral gas behave like continuum fluids and that the effects on the neutral component resulting from the presence of the magnetic field, rotation, pressure and gravity are negligible.

Let $\boldsymbol{q}(u,v,w)$, $\boldsymbol{h}(h_x,h_y,h_z)$, θ, δp and $\delta\rho$ denote respectively the perturbations in velocity, magnetic field $\boldsymbol{H}$, temperature T, pressure p and density ρ, with g, ν, κ, η, α, $\boldsymbol{q}_d$ and ν_c being the gravitational acceleration, the kinematic viscosity, the thermal diffusivity, the resistivity, the coefficient of thermal expansion, the velocity of the neutral gas and the mutual collisional frequency between the two components of the composite medium. Then the linearized perturbation equations governing the motion of this medium are

$$\begin{aligned}\rho\,\partial\boldsymbol{q}/\partial t &= -\nabla\delta p+\rho\nu\nabla^2\boldsymbol{q}+\boldsymbol{g}\,\delta\rho\\ &\quad+\rho_d\nu_c(\boldsymbol{q}_d-\boldsymbol{q})+(4\pi)^{-1}(\nabla\times\boldsymbol{h})\times\boldsymbol{H}+2\rho(\boldsymbol{q}\times\boldsymbol{\Omega}),\end{aligned} \tag{1a}$$

$$\partial\boldsymbol{q}_d/\partial t = -\nu_c(\boldsymbol{q}_d-\boldsymbol{q}), \tag{1b}$$

$$\partial\boldsymbol{h}/\partial t = \nabla\times(\boldsymbol{q}\times\boldsymbol{H})+\eta\nabla^2\boldsymbol{h}, \tag{1c}$$

$$\nabla\cdot\boldsymbol{q}=0, \qquad \nabla\cdot\boldsymbol{h}=0, \tag{1d}$$

$$\partial\theta/\partial t = \beta w+\kappa\nabla^2\theta. \tag{1e}$$

3. Dispersion Relation and Discussion

Analysing in terms of normal modes, we seek solutions of the above equations whose dependence on space–time coordinates is of the form

$$f(z)\exp(\mathrm{i}k_x x+\mathrm{i}k_y y+nt), \tag{2}$$

where $f(z)$ is some function of z only, k_x and k_y $(k^2=k_x^2+k_y^2)$ are horizontal wave numbers and n is the frequency of the harmonic disturbance.

Eliminating $\boldsymbol{q}_d$ between equations (1a) and (1b) and using (2), we find that the equations (1) reduce to

$$n'\rho u = -\mathrm{i}k_x\delta p+\rho\nu\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-k^2\right)u+\frac{H}{4\pi}\left(\frac{\mathrm{d}h_x}{\mathrm{d}z}-\mathrm{i}k_x h_z\right)+2\rho\Omega v, \tag{3a}$$

$$n'\rho v = -\mathrm{i}k_y\,\delta p + \rho\nu\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)v + \frac{H}{4\pi}\left(\frac{\mathrm{d}h_y}{\mathrm{d}z} - \mathrm{i}k_y h_z\right) - 2\rho\Omega u\,, \tag{3b}$$

$$n'\rho w = -\frac{\partial(\delta p)}{\partial z} + \rho\nu\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)w + g\alpha\rho\theta\,, \tag{3c}$$

$$\left\{n - \eta\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\right\}h_x = H\frac{\partial u}{\partial z}, \tag{4a}$$

$$\left\{n - \eta\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\right\}h_y = H\frac{\partial v}{\partial z}, \tag{4b}$$

$$\left\{n - \eta\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\right\}h_z = H\frac{\partial w}{\partial z}, \tag{4c}$$

$$\mathrm{i}k_x u + \mathrm{i}k_y v + \frac{\partial w}{\partial z} = 0, \qquad \mathrm{i}k_x h_x + \mathrm{i}k_y h_y + \frac{\partial h_z}{\partial z} = 0, \tag{5}$$

$$\left\{n - \kappa\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\right\}\theta = \beta w\,, \tag{6}$$

where use has also been made of the Boussinesq equation of state

$$\delta\rho = -\alpha\rho\theta \tag{7}$$

and we have written

$$n' = n\left(1 + \frac{\alpha_0 \nu_c}{n + \nu_c}\right), \qquad \alpha_0 = \frac{\rho_d}{\rho}, \qquad \nabla^2 = \frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2. \tag{8}$$

Assume now that the perturbation quantities are of the form

$$[w, \theta, h_z, \xi, \zeta] = [W(z), \Theta(z), K(z), X(z), Z(z)]\exp(\mathrm{i}k_x x + \mathrm{i}k_y y + nt), \tag{9}$$

where ξ and ζ are the z components of the current density and vorticity respectively. Letting $a = kd$, $\sigma = nd^2/\nu$, $p_1 = \nu/\kappa$, $p_2 = \nu/\eta$, $x = x'd$, $y = y'd$, $z = z'd$ and $\mathrm{D} = \mathrm{d}/\mathrm{d}z'$, we find that equations (3)–(6) with the help of the expression (9) give

$$\left(\mathrm{D}^2 - a^2\right)\left\{\mathrm{D}^2 - a^2 - \sigma\left(1 + \frac{\alpha_0 \nu_c d^2/\nu}{\sigma + \nu_c d^2/\nu}\right)\right\}W + \frac{Hd}{4\pi\rho\nu}\left(\mathrm{D}^2 - a^2\right)\mathrm{D}K$$
$$- \left(\frac{g\alpha d^2}{\nu}\right)a^2\Theta - \left(\frac{2\Omega d^3}{\nu}\right)\mathrm{D}Z = 0, \tag{10a}$$

$$\left\{\mathrm{D}^2 - a^2 - \sigma\left(1 + \frac{\alpha_0 \nu_c d^2/\nu}{\sigma + \nu_c d^2/\nu}\right)\right\}Z = \left(\frac{Hd}{4\pi\rho\nu}\right)\mathrm{D}X + \left(\frac{2\Omega d}{\nu}\right)\mathrm{D}W, \tag{10b}$$

$$(\mathrm{D}^2 - a^2 - p_2\sigma)K = -(Hd/\eta)\mathrm{D}W, \tag{11a}$$

$$(\mathrm{D}^2 - a^2 - p_2\sigma)X = -(Hd/\eta)\mathrm{D}Z, \tag{11b}$$

$$(\mathrm{D}^2 - a^2 - p_1\sigma)\Theta = -(\beta d^2/\kappa)W. \tag{11c}$$

Eliminating K, X, Z and Θ between equations (10) and (11), we finally get

$$\left[\left(D^2-a^2-p_2\sigma\right)\left\{D^2-a^2-\sigma\left(1+\frac{\alpha_0\nu_c d^2/\nu}{\sigma+\nu_c d^2/\nu}\right)\right\}+QD^2\right]$$

$$\times\left[\left(D^2-a^2\right)\left(D^2-a^2-p_2\sigma\right)\left(D^2-a^2-p_1\sigma\right)\left\{D^2-a^2-\sigma\left(1+\frac{\alpha_0\nu_c d^2/\nu}{\sigma+\nu_c d^2/\nu}\right)\right\}\right.$$

$$\left.-Q(D^2-a^2)(D^2-a^2-p_1\sigma)D^2+Ra^2(D^2-a^2-p_2\sigma)\right]W$$

$$=T(D^2-a^2-p_2\sigma)^2(D^2-a^2-p_1\sigma)D^2W, \qquad (12)$$

where $R=g\alpha\beta d^4/\nu\kappa$ is the Rayleigh number, $Q=H^2d^2/4\pi\rho\nu\eta$ is the Chandrasekhar number and $T_A=4\Omega^2d^4/\nu^2$ is the Taylor number.

Consider the case in which both the boundaries are free and the medium adjoining the fluid is nonconducting. The boundary conditions appropriate to the problem are (Chandrasekhar 1961)

$$\left.\begin{array}{lll} W=D^2W=\Theta=0, & & DZ=0, \\ X=0 & \text{and} & \boldsymbol{h}\ \text{continuous} \end{array}\right\} \quad \text{at} \quad z'=0 \text{ and } 1. \qquad (13)$$

With these boundary conditions it can be shown that all the even-order derivatives of W must vanish for $z'=0$ and 1, and hence the proper solution of W characterizing the lowest mode is

$$W=W_0\sin\pi z, \qquad (14)$$

where W_0 is a constant. Substituting the solution (14) in equation (12), we obtain the dispersion relation

$$R_1=\frac{(1+x)(1+x+p_1\sigma/\pi^2)\{(1+x+p_2\sigma/\pi^2)(1+x+A\sigma/\pi^2)+Q_1\}}{x(1+x+p_2\sigma/\pi^2)}$$

$$+\frac{T_1(1+x+p_2\sigma/\pi^2)(1+x+p_1\sigma/\pi^2)}{x\{(1+x+p_2\sigma/\pi^2)(1+x+A\sigma/\pi^2)+Q_1\}}, \qquad (15)$$

where we have written for convenience

$$A=1+\frac{\alpha_0\nu_c d^2/\nu}{\sigma+\nu_c d^2/\nu}, \qquad (16)$$

and $x=a^2/\pi^2$, $R_1=R/\pi^4$, $Q_1=Q/\pi^2$ and $T_1=T_A/\pi^4$.

For stationary convection, $\sigma=0$ and equation (15) reduces to

$$R_1=\frac{1+x}{x}\left((1+x)^2+Q_1+\frac{T_1(1+x)}{(1+x)^2+Q_1}\right). \qquad (17)$$

The collision frequency thus vanishes with σ and the collisions, naturally, have no effect on the thermal instability for stationary convection.

For overstable convection, remembering that σ can be complex, the real part σ_r must be zero with the imaginary part $\sigma_i \neq 0$. Putting

$$\sigma/\pi^2 = i\sigma_1, \tag{18}$$

we thus have σ_1 real. Equation (15) then becomes

$$R_1 = \frac{(1+x)(1+x+ip_1\sigma_1)\{(1+x+ip_2\sigma_1)(1+x+iA\sigma_1)+Q_1\}}{x(1+x+ip_2\sigma_1)} + \frac{T_1(1+x+ip_2\sigma_1)(1+x+ip_1\sigma_1)}{x\{(1+x+ip_2\sigma_1)(1+x+iA\sigma_1)+Q_1\}}. \tag{19}$$

In order to study the effect of rotation on the thermal instability of a composite hydrodynamic ($Q_1 \to 0$) medium, we examine the form of dR_1/dT_1. For $Q_1 = 0$, from equation (19) it follows that

$$\frac{dR_1}{dT_1} = \frac{(1+x)^2 + p_1\sigma_1^2 A + i\sigma_1(1+x)(p_1 - A)}{x\{(1+x)^2 + \sigma_1^2 A^2\}}. \tag{20}$$

For the imaginary part of dR_1/dT_1 to vanish we then have $p_1 = A$, where A is as defined by equation (16) above. Substituting this value of p_1 in equation (20) we obtain

$$dR_1/dT_1 = x^{-1}, \tag{21}$$

which is always positive. Thus with increase in the Taylor number, the Rayleigh number also increases thereby showing that the rotation has a stabilizing effect on the thermal instability.

To investigate the effect of the magnetic field on the thermal instability of a composite non-rotating medium, we consider the form of dR_1/dQ_1. For $T_1 = 0$ equation (19) yields

$$\frac{dR_1}{dQ_1} = \frac{1+x}{x}\frac{1+x+ip_1\sigma_1}{1+x+ip_2\sigma_1} = \frac{1+x}{x}\frac{(1+x)^2 + p_1p_2\sigma_1^2 + i\sigma_1(1+x)(p_1 - p_2)}{(1+x)^2 + p_2^2\sigma_1^2}. \tag{22}$$

For dR_1/dQ_1 to be real we then have $p_1 = p_2$, and substituting this in equation (22) we obtain

$$dR_1/dQ_1 = (1+x)/x, \tag{23}$$

which again is always positive. The magnetic field thus also has a stabilizing effect on the thermal instability.

4. Effects of Hall Currents

We again consider the infinite horizontal composite layer described in Section 2, except that rotation is no longer present. With N and e denoting respectively the electron number density and the electronic charge, the linearized perturbation equations governing the motion of the medium now are

$$\rho\, \partial \boldsymbol{q}/\partial t = -\nabla \delta p + \rho \nu \nabla^2 \boldsymbol{q} + \boldsymbol{g}\, \delta\rho + (4\pi)^{-1}(\nabla \times \boldsymbol{h}) \times \boldsymbol{H} + \rho_{\mathrm{d}} \nu_{\mathrm{c}}(\boldsymbol{q}_{\mathrm{d}} - \boldsymbol{q}), \quad (24a)$$

$$\partial \boldsymbol{h}/\partial t = (\boldsymbol{H}.\nabla)\boldsymbol{q} + \eta \nabla^2 \boldsymbol{h} - (4\pi Ne)^{-1} \nabla \times \{(\nabla \times \boldsymbol{h}) \times \boldsymbol{H}\}, \quad (24b)$$

together with equations (1b), (1d) and (1e). This new set of equations, within the Boussinesq approximation and using perturbation quantities of the form (9), gives

$$\left(\mathrm{D}^2 - a^2\right)\left\{\mathrm{D}^2 - a^2 - \sigma\left(1 + \frac{\alpha_0 \nu_{\mathrm{c}}}{n + \nu_{\mathrm{c}}}\right)\right\} W + \frac{Hd}{4\pi\rho\nu}\left(\mathrm{D}^2 - a^2\right)\mathrm{D}K = \frac{g\alpha d^2}{\nu} a^2 \Theta, \quad (25a)$$

$$\left\{\mathrm{D}^2 - a^2 - \sigma\left(1 + \frac{\alpha_0 \nu_{\mathrm{c}}}{n + \nu_{\mathrm{c}}}\right)\right\} Z = -\frac{Hd}{4\pi\rho\nu}\mathrm{D}X, \quad (25b)$$

$$(\mathrm{D}^2 - a^2 - p_2\sigma)K = -(Hd/\eta)\mathrm{D}W + (Hd/4\pi Ne)\mathrm{D}X, \quad (25c)$$

$$(\mathrm{D}^2 - a^2 - p_2\sigma)X + (H/4\pi Ned)(\mathrm{D}^2 - a^2)\mathrm{D}K = -(Hd/\eta)\mathrm{D}Z, \quad (25d)$$

$$(\mathrm{D}^2 - a^2 - p_1\sigma)\Theta = -(\beta d^2/\kappa)W. \quad (25e)$$

Eliminating K, X, Z and Θ between equations (25a)–(25e), we then obtain

$$\begin{aligned}&\{(\mathrm{D}^2 - a^2)(\mathrm{D}^2 - a^2 - p_1\sigma)(\mathrm{D}^2 - a^2 - A\sigma) + Ra^2\}\\ &\times\{(\mathrm{D}^2 - a^2 - p_2\sigma)^2(\mathrm{D}^2 - a^2 - A\sigma) - Q\mathrm{D}^2(\mathrm{D}^2 - a^2 - p_2\sigma)\\ &\quad - M(\mathrm{D}^2 - a^2)(\mathrm{D}^2 - a^2 - A\sigma)\mathrm{D}^2\}W\\ &= Q(\mathrm{D}^2 - a^2)(\mathrm{D}^2 - a^2 - p_1\sigma)\mathrm{D}^2\{(\mathrm{D}^2 - a^2 - p_2\sigma)(\mathrm{D}^2 - a^2 - A\sigma) - Q\mathrm{D}^2\}W, \quad (26)\end{aligned}$$

where A is as defined by equation (16) above and $M = (H/4\pi Ne)^2$ denotes a non-dimensional number accounting for Hall currents.

We also assume here that the partially ionized layer is confined between two free boundaries and the adjoining medium is electrically nonconducting. Using the appropriate boundary conditions (13) and the proper solution (14), we then obtain the dispersion relation

$$\begin{aligned}R_1 &= x^{-1}(1+x)(1+x+\mathrm{i}p_1\sigma_1)(1+x+\mathrm{i}A\sigma_1)\\ &+ \frac{Q_1(1+x)(1+x+\mathrm{i}p_1\sigma_1)\{(1+x+\mathrm{i}p_2\sigma_1)(1+x+\mathrm{i}A\sigma_1)+Q_1\}}{x\{(1+x+\mathrm{i}p_2\sigma_1)^2(1+x+\mathrm{i}A\sigma_1)+Q_1(1+x+\mathrm{i}p_2\sigma_1)+M(1+x)(1+x+\mathrm{i}A\sigma_1)\}}.\end{aligned} \quad (27)$$

In the limit of vanishing Hall currents ($M \to 0$), equation (27) reduces to the result (2.25) obtained by Sharma (1976). For the case of stationary convection ($\sigma = 0$), equation (27) becomes

$$R_1 = (1+x)[\{(1+x)^2 + Q_1\}^2 + M(1+x)^3]/x\{(1+x)^2 + Q_1 + M(1+x)\}. \quad (28)$$

The collision frequency thus vanishes with σ and the collisions have no effect on the thermal instability of a partially ionized Hall plasma.

To investigate the effect of the Hall parameter M on R_1, we examine the form of dR_1/dM. From equation (28) it follows that

$$dR_1/dM = -Q_1(1+x)^2\{(1+x)^2+Q_1\}/x\{(1+x)^2+Q_1+M(1+x)\}^2, \qquad (29)$$

which is always negative, thus showing the destabilizing effect of Hall currents.

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