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Aspects of the Exceptional Group E_8

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Abstract

Methods for calculating branching rules and Kronecker products for the exceptional group E_8 are developed. In particular, tables of branching rules for $E_8 \to SU_9$, $E_8 \to SU_2 \times E_7$ and $E_8 \to SU_3 \times E_6$ are given. The third and fourth symmetrized powers of the adjoint representation of E_8 are also resolved. The relevance of E_8 to unified theories of strong, electromagnetic and weak interactions is briefly considered.

Introduction

The properties of the exceptional groups have recently been a subject of interest to physicists investigating the possible construction of unified gauge theories of strong, weak and electromagnetic interactions (see Gell-Mann *et al.* 1978, and references therein). Ramond (1977) has asked the question 'Is there an exceptional group in your future?', answering in the affirmative. Subsequent developments make the affirmative answer questionable. Nevertheless, it is important that the basic properties of the exceptional groups be known and available.

The basic properties of the exceptional groups have been outlined, and a systematic notation established for describing their irreducible representations (irreps), by Wybourne and Bowick (1977). The calculation of explicit properties such as $3jm$ and $6j$ symbols for E_7 in particular has been considered (Wybourne 1978; Butler *et al.* 1978, 1979). The exceptional group E_8 is truly exceptional in that its defining (or fundamental) and adjoint (or regular) irreps coincide. The high dimension (248) of the adjoint irrep of E_8 further complicates the problem of resolving Kronecker products and branchings. As a consequence there is a paucity of known results for E_8 (McKay *et al.* 1976*b*; Wybourne and Bowick 1977). In this paper we are able to find quite simple and efficient methods for resolving all Kronecker products of E_8 irreps up to the fourth power in the adjoint irrep and thence to resolve the symmetrized third and fourth powers of the adjoint irrep. These calculations were performed leisurely by hand and the most cumbersome case, a Kronecker product of dimension 1 015 808 000, was resolved in 10 minutes while awaiting a coffee break. The bulk of the time was taken up with checking the results. We note that this example is about six orders of magnitude larger than those produced by computer programs which solve the problem by enumeration of weights (cf. Patera and Sankoff 1973; McKay *et al.* 1976*a*).

While for most Lie groups it is possible to deduce branching rules for the various group–subgroup combinations from a knowledge of Kronecker products in the group and subgroup, together with the branching rules for the irreps that arise in the Kronecker square of the fundamental irreps, such a procedure fails for E_8. In this case the Kronecker third powers of the fundamental irrep do not yield sufficient equations to solve for the branchings of three of the five power-3 irreps of E_8. These methods in two cases yield the content of pairs of irreps of E_8, namely the pair $[(43^62),(63^7)]$ and the pair $[(43^62),(3^8)]$. While comparison of the two pairs yields a partial content for each irrep, it is not possible to separate the pairs completely in the traditional manner. Nevertheless, methods are described here which enable us to make a complete separation and thus to determine for the first time the branching rules for all third power irreps of E_8. Inspection of the tables of E_8 Kronecker products shows that the branching rules for all the fourth power irreps of E_8 can be unequivocally resolved and probably even those of fifth power, if not all higher powers of E_8. Particular results are given here for the maximal subgroups SU_9, $SU_2 \times E_7$ and $SU_3 \times E_6$. The relevance (or what currently appears more likely, the irrelevance) of these results to unified gauge theories is briefly considered.

Some Basic Properties of E_8 Irreps

The nontrivial irreps of E_8 may be uniquely labelled by a set of eight integers Λ_i $(i = 1,2,...,8)$ such that (Wybourne and Bowick 1977)

$$\Lambda_i \geqslant \Lambda_{i+1} \geqslant 0, \tag{1}$$

$$2(\Lambda_6+\Lambda_7+\Lambda_8)-(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4+\Lambda_5) = 3n, \tag{2}$$

where n is a non-negative integer. (For an alternative labelling scheme based on the SO_{16} subgroup of E_8 see Qubanchi 1978.) The irreps may be equivalently labelled in Dynkin's notation by the set of non-negative integers $\boldsymbol{a} \equiv a_1 a_2 \ldots a_8$. It is useful to list the irreps in order of increasing power p_Λ (Wybourne 1979) and, for a given power, in order of increasing maximal weights. The power p_Λ of an irrep (Λ) of E_8 is simply the numerical value of Λ_6 (Wybourne 1979). Every irrep of E_8 may be associated with a dimension D_Λ and a second-order Dynkin index $I_\Lambda^{(2)}$ (Dynkin 1952*a*, 1952*b*; Patera *et al.* 1976, 1977). These basic properties are given, for all irreps of E_8 with $p_\Lambda \leqslant 4$, in Table 1. A similar, and more extensive, tabulation including the fourth order Dynkin index $I_\Lambda^{(4)}$ but without ordering with respect to p_Λ has been given by McKay and Patera (1977).

We note that all the irreps of E_8 are orthogonal and real (Malćev 1944). The group SU_9 occurs as a maximal subgroup of E_8. The characters of SU_9 may be expressed in Schur functions (S-functions) (see Wybourne 1970). Under the reduction $E_8 \to SU_9$ we necessarily have in our notation

$$\Lambda \supset \{\Lambda\}, \tag{3}$$

where we use braces to label the S-functions of SU_9 (Wybourne and Bowick 1977). If $\{\Lambda\}$ is not self-contragredient then $\{\Lambda^*\}$ will also necessarily occur in the right-hand side of the relation (3).

Table 1. Basic properties of E_8 irreps
All irreps of E_8 with $p_A \leqslant 4$ are listed

Irrep (A)	Dynkin label (a)	Power p_A	Dimension D_A	Dynkin index $I_A^{(2)}/8$
(0)	(00000000)	0	1	0
(21^7)	(10000000)	1	248	60
(2^71)	(00000010)	2	3 875	1 500
(3^22^6)	(01000000)	2	30 380	14 700
(42^7)	(20000000)	2	27 000	13 500
(3^8)	(00000001)	3	147 250	85 500
(43^62)	(10000010)	3	779 247	502 740
(4^33^5)	(00100000)	3	2 450 240	1 778 400
(543^6)	(11000000)	3	4 096 000	3 072 000
(63^7)	(30000000)	3	1 763 125	1 365 000
(4^63^2)	(00000100)	4	6 696 000	5 292 000
(4^72)	(00000020)	4	4 881 384	3 936 600
(54^7)	(10000001)	4	26 411 008	22 364 160
(5^24^53)	(01000010)	4	76 271 625	68 890 500
(5^44^4)	(00010000)	4	146 325 270	141 605 100
(64^63)	(20000010)	4	70 680 000	64 980 000
(65^24^5)	(10100000)	4	344 452 500	344 452 500
(6^24^6)	(02000000)	4	203 205 000	206 482 500
(754^6)	(21000000)	4	281 545 875	290 628 000
(84^7)	(40000000)	4	79 143 000	84 249 000

Calculated Results for E_8

Resolution of Kronecker Products

A Kronecker product $(\Lambda)\times(\Lambda')$ will be resolved if all the non-negative integers $k_{\Lambda''}$ are known in

$$(\Lambda)\times(\Lambda') = \sum k_{\Lambda''}\,(\Lambda''), \tag{4}$$

where the summation is over all irreps (Λ''). The resolution may be verified by checking that the identities

$$D_\Lambda \times D_{\Lambda'} = \sum k_{\Lambda''} D_{\Lambda''}, \tag{5}$$

$$D_{\Lambda'} \times I_\Lambda^{(n)} + D_\Lambda \times I_{\Lambda'}^{(n)} = \sum k_{\Lambda''} I_{\Lambda''}^{(n)} \tag{6}$$

are simultaneously satisfied. It is important to require simultaneous satisfaction since either identity by itself can be satisfied by incorrect resolutions. For example, in E_8 the two irreps (6^654) and (765^54) are of the same dimension though they have different values of $I_\Lambda^{(2)}$. In comparatively rare instances two or more irreps may separately coincide in D_Λ and $I_\Lambda^{(2)}$, such as for the $\{31^3\}$ and $\{32^6\}$ irreps of SU_9. In these cases the fourth order Dynkin index $I_\Lambda^{(4)}$ is required. Contragredient partners will necessarily possess common values for the dimension and Dynkin indices. In these cases great caution must be exercised.

It is often possible to resolve Kronecker products by use of equations (5) and (6) and the consequential solution of the resulting linear Diophantine equations, provided the possible values of (Λ'') in equation (4) can be sufficiently restricted. Our resolution of E_8 Kronecker products relies upon producing just such a restricted list of candidates for the (Λ'').

We first note that the range of (Λ'') is restricted by the requirement that (Wybourne 1979)

$$p(\Lambda)+p(\Lambda') \geqslant p(\Lambda'') \geqslant |p(\Lambda)-p(\Lambda')|. \tag{7}$$

The leading term in equation (4) is necessarily

$$(\Lambda'')_{\max} = (\Lambda+\Lambda'), \tag{8}$$

with $k_{\Lambda''_{\max}} = 1$. The identity irrep (0) will occur in equation (4) if and only if $\Lambda \equiv \Lambda^*$ and then only with $k_0 = 1$.

Table 2. **Kronecker products for E_8**

Product	Evaluation
$(21^7)\times(21^7)$	$\{(42^7)+(2^71)+(0)\}+[(3^22^6)+(21^7)]$
$(2^71)\times(21^7)$	$(43^62)+(3^8)+(3^22^6)+(2^71)+(21^7)$
$(2^71)\times(2^71)$	$\{(4^72)+(4^33^5)+(42^7)+(3^8)+(2^71)+(0)\}+[(4^63^2)+(43^62)+(3^22^6)+(21^7)]$
$(3^22^6)\times(21^7)$	$(543^6)+(4^33^5)+(43^62)+(42^7)+(3^8)+(3^22^6)+(2^71)+(21^7)$
$(3^22^6)\times(2^71)$	$(5^24^53)+(54^7)+(543^6)+(4^63^2)+(4^33^5)+2(43^62)+(42^7)+(3^8)+2(3^22^6)+(2^71)+(21^7)$
$(3^22^6)\times(3^22^6)$	$\{(6^24^6)+(64^63)+(5^44^4)+(543^6)+(4^72)+2(4^33^5)+(43^62)+2(42^7)+(3^8)+2(2^71)+(0)\}+[(65^24^5)+(63^7)+(5^24^53)+(54^7)+(543^6)+(4^63^2)+2(43^62)+(3^8)+2(3^22^6)+(21^7)]$
$(42^7)\times(21^7)$	$(63^7)+(543^6)+(43^62)+(42^7)+(3^22^6)+(21^7)$
$(42^7)\times(2^71)$	$(64^63)+(54^7)+(543^6)+(4^33^5)+(43^62)+(42^7)+(3^8)+(3^22^6)+(2^71)$
$(42^7)\times(3^22^6)$	$(754^6)+(65^24^5)+(64^63)+(63^7)+(5^24^53)+(54^7)+2(543^6)+(4^63^2)+(4^33^5)+2(43^62)+(42^7)+(3^8)+2(3^22^6)+(2^71)+(21^7)$
$(42^7)\times(42^7)$	$\{(84^7)+(6^24^6)+(64^63)+(543^6)+(4^33^5)+(4^72)+2(42^7)+(2^71)+(0)\}+[(754^6)+(63^7)+(5^24^53)+(543^6)+(43^62)+(3^22^6)+(21^7)]$
$(3^8)\times(21^7)$	$(54^7)+(4^63^2)+(43^62)+(3^8)+(3^22^6)+(2^71)$
$(43^62)\times(21^7)$	$(64^63)+(5^24^53)+(54^7)+(543^6)+(4^72)+2(43^62)+(4^33^5)+(4^63^2)+(42^7)+(3^8)+(3^22^6)+(2^71)$
$(4^33^5)\times(21^7)$	$(65^24^5)+(5^44^4)+(5^24^53)+(54^7)+(543^6)+(4^63^2)+(4^33^5)+(43^62)+(3^8)+(3^22^6)$
$(543^6)\times(21^7)$	$(754^6)+(6^24^6)+(65^24^5)+(64^63)+(63^7)+(5^24^53)+(54^7)+2(543^6)+(4^33^5)+(43^62)+(42^7)+(3^22^6)$
$(63^7)\times(21^7)$	$(84^7)+(754^6)+(64^63)+(63^7)+(543^6)+(42^7)$
$(3^8)\times(2^71)$	$(5^74)+(5^44^4)+(5^24^53)+(54^7)+(4^72)+(4^63^2)+(543^6)+(4^33^5)+2(43^62)+(3^8)+(3^22^6)+(42^7)+(2^71)+(21^7)$
$(4^72)\times(21^7)$	$(65^63)+(5^74)+(5^24^53)+(4^72)+(4^63^2)+(43^62)$
$(4^63^2)\times(21^7)$	$(65^54^2)+(5^74)+(5^44^4)+(54^7)+(4^72)+(4^63^2)+(4^33^5)+(43^62)+(3^8)$
$(54^7)\times(21^7)$	$(75^7)+(6^25^6)+(65^54^2)+(64^63)+(65^24^5)+(5^74)+(5^44^4)+(5^24^53)+2(54^7)+(543^6)+(4^63^2)+(4^33^5)+(43^62)+(3^8)$

The above conditions severely restrict the range of (Λ'') in the Kronecker product though not always sufficiently to avoid Diophantine equations with several redundant terms. At this juncture we note that the method used by Wybourne and Bowick (1977) unequivocally resolves the E_8 products if the branching rules for the $E_8 \to SU_9$ reductions are known. One simply reduces Λ and Λ' to linear combinations of S-functions appropriate to SU_9, forms their products and then uses the relation (3) and the branching rules to invert the S-functions back into E_8 irreps. Clearly, considerable labour is involved if all the S-function multiplications are carried out. However, many S-function products cannot yield partitions that satisfy the relation

(3) and thus in obtaining possible candidates for (Λ'') we need only consider those products that can yield S-functions whose defining partitions satisfy the conditions (1) and (2). Indeed in many cases the selection of the (Λ'') can be deduced by simply combining the S-function products $\{\Lambda\}\{\Lambda'\}$ followed by the solution of some trivial linear Diophantine-type equations. In this way it became trivial to resolve the Kronecker products listed in Table 2 from just the information given in Table 1 using a small non-programmable hand calculator and carrying out only the simplest of Young tableau operations. The hand calculator was used primarily for checking the results. Inspection of the results in Table 2 shows that if the branching rules are known for all E_8 irreps of power 3 and less, then all those of power 4 can be deduced.

Table 3. Symmetrized third and fourth powers of adjoint irrep of E_8

Plethysm	Evaluation
$(21^7)\otimes\{3\}$	$(63^7)+(43^62)+(3^22^6)+(21^7)$
$(21^7)\otimes\{21\}$	$(543^6)+(43^62)+(42^7)+(3^8)+(3^22^6)+(2^71)+2(21^7)$
$(21^7)\otimes\{1^3\}$	$(4^33^5)+(42^7)+(3^22^6)+(2^71)+(0)$
$(21^7)\otimes\{4\}$	$(84^7)+(64^63)+(543^6)+(4^33^5)+(4^72)+2(42^7)+(3^8)+(2^71)+(0)$
$(21^7)\otimes\{31\}$	$(754^6)+(64^63)+(63^7)+(5^24^53)+(54^7)+2(543^6)+(4^63^2)+(4^33^5)+3(43^62)$ $+2(42^7)+(3^8)+3(3^22^6)+2(2^71)+2(21^7)$
$(21^7)\otimes\{2^2\}$	$(6^24^6)+(64^63)+(54^7)+(543^6)+2(4^33^5)+(4^72)+(43^62)+3(42^7)+(3^8)+(3^22^6)$ $+3(2^71)+2(0)$
$(21^7)\otimes\{21^2\}$	$(65^24^5)+(63^7)+(5^24^53)+(54^7)+2(543^6)+(4^63^2)+(4^33^5)+3(43^62)+(42^7)$ $+2(3^8)+4(3^22^6)+(2^71)+3(21^7)$
$(21^7)\otimes\{1^4\}$	$(5^44^4)+(543^6)+(4^33^5)+(43^62)+(42^7)+(3^8)+(2^71)+(21^7)$

Symmetrized Powers of Adjoint Irrep

Following the methods outlined by Wybourne and Bowick (1977) it was possible from the preceding results to resolve the third and fourth powers of the adjoint irrep of E_8 as listed in Table 3. The resolution of the Kronecker powers of an irrep (Λ) of dimension D_Λ for a group G may be verified by noting that the evaluation of the plethysm (Wybourne 1970)

$$(\Lambda)\otimes\{\mu\} = \sum k_{\Lambda''}\{\Lambda''\} \tag{9}$$

is equivalent to determining the branching rule for the $\{\mu\}$ irrep of U_{D_Λ} under the restriction of $U_{D_\Lambda} \to G$ (Butler and Wybourne 1969), in our case $U_{248} \to E_8$. With this in mind we may simply use the two branching rule identities

$$D_{\{\mu\}} = \sum k_{\Lambda''} D_{\Lambda''} \qquad \text{and} \qquad I^{(n)}_{\{\mu\}} = \rho_n \sum k_{\Lambda''} I^{(n)}_{\Lambda''}. \tag{10}$$

The value of ρ_n may be fixed by knowing the reduction of the vector irrep $\{1\}$ of $U_{D_\Lambda} \to G$. In the present case, the eigenvalues of the second order Casimir operator for U_n, namely

$$C_2(\Lambda) = \tfrac{1}{2}n^{-1} \sum_{i=1}^{n} \Lambda_i(\Lambda_i+n+1-2i) - \tfrac{1}{2}n^{-2}\left(\sum_{i=1}^{n} \Lambda_i\right)^2, \tag{11}$$

were computed and these were then combined with D_Λ to yield the results of Table 4.

Table 4. Basic properties of some U_{248} irreps

$\{\mu\}$	$D_{\{\mu\}}$	2.248^2C_2	$60C_2.D_{\{\mu\}}/(247.248.249)$
$\{0\}$	1	0	0
$\{1\}$	248	61 503	60
$\{2\}$	30 876	163 336	15 000
$\{1^2\}$	30 628	122 508	14 760
$\{3\}$	2 573 000	185 991	1 882 500
$\{21\}$	5 084 248	184 503	3 690 060
$\{1^3\}$	2 511 496	183 015	1 808 100
$\{4\}$	161 455 750	248 976	158 130 000
$\{31\}$	476 648 250	246 992	463 110 000
$\{2^2\}$	315 223 376	246 000	305 040 000
$\{21^2\}$	469 021 878	245 008	452 039 760
$\{1^4\}$	153 829 130	243 024	147 058 800

Branching Rules for Subgroups

Wybourne and Bowick (1977) have given the branching rules for all power-2 irreps of E_8 for a number of maximal subgroups of interest. Inspection of their Kronecker products for E_8 shows clearly that additional information is required to determine the branching rules for the (63^7), (3^8) and (43^62) power-3 irreps of E_8. If these are known then the branching rules for all the power 4, and possibly all powers, follow.

The problem for the $E_8 \to SU_9$ decomposition was first solved by exploiting the Kronecker product $(21^7)\times(2^71)$ to give the terms in $(43^62)+(3^8)$, and the third symmetrized power $(21^7)\otimes\{3\}$ gave the terms in $(63^7)+(43^62)$. Comparison of these two lists of SU_9 irreps made it possible to assign some of the irreps to (63^7), (43^62) and (3^8), leaving a common residue of terms. The method of elementary multiplets was next used to decide on the distribution of most of the remaining terms among the three E_8 irreps. It was then possible to complete the resolution by solving some trivial Diophantine equations based on the two identities (10) to yield the results shown in Table 5*a*.

The branching rules for $E_8 \to SU_2\times E_7$ proved somewhat simpler to derive, and the results are given in Table 5*b*. The branching rules for $E_8 \to SU_3\times E_6$ are of some interest as they are relevant to the E_6 unified gauge model (Gürsey *et al.* 1975; Achiman and Stech 1978). To obtain these it was noted that the $E_8 \to SU_2\times E_7$ branching rules could be disassembled to give those for $E_8 \to SU_2\times SU_6\times SU_3$ and then the $SU_2\times SU_6$ parts reassembled into irreps of E_6 using the results from Wybourne and Bowick (1977) for $E_6 \to SU_2\times SU_6$. This was trivially done for the (3^8) irrep of E_8 and then the results for (63^7) and (43^62) were obtained by conventional methods. The results are given in Table 5*c*. These were checked both dimensionally and via the Dynkin index. In the latter case it was necessary to first calculate the values of the second and fourth Dynkin indices for $SU_3\times E_6$. Here the tables of McKay and Patera (1977) were most useful.

We may note that since the branching rules $E_8 \to SU_9$ for the power-3 irreps of E_8 are now known it would be possible to use them to obtain all Kronecker products of E_8 involving products with $p(\Lambda) \leqslant 6$. Indeed, we probably now have sufficient results to permit the building up of E_8 products almost without limit.

Table 5. Branching rules for power-3 irreps of E_8

(Λ)	Branching
	(a) $E_8 \to SU_9$ branching rules
(3^{8})	$\{3^{8}\}+\{3\}+\{3^{3}2^{4}1\}+\{321^{4}\}+\{32^{2}1^{5}\}+\{32^{5}1^{2}\}+\{32^{6}\}+\{3^{2}1^{6}\}+\{31^{3}\}+\{3^{5}2^{3}\}+\{32^{3}1^{3}\}+\{3^{2}2^{3}1^{3}\}+\{3^{2}2^{6}\}+\{31^{6}\}+\{2^{7}1\}+\{21\}+2\{2^{2}1^{5}\}+\{2^{4}1\}+\{2^{4}1^{4}\}+\{21^{4}\}+\{2^{2}1^{2}\}+\{2^{5}1^{2}\}+\{21^{7}\}+\{1^{6}\}+\{1^{3}\}$
$(43^{6}2)$	$\{43^{6}2\}+\{421^{6}\}+\{432^{5}1\}+\{43^{3}2^{4}\}+\{42^{4}1^{3}\}+\{42^{7}\}+\{3^{4}2^{2}1^{2}\}+\{32^{2}1^{2}\}+3\{32^{2}1^{5}\}+3\{32^{5}1^{2}\}+2\{3^{2}2^{6}\}+2\{31^{6}\}+2\{3^{3}2^{4}1\}+2\{321^{4}\}+\{3^{3}21^{4}\}+\{32^{4}1\}+2\{3^{2}2^{3}1^{3}\}+2\{32^{3}1^{3}\}+\{3^{5}2^{3}\}+\{31^{3}\}+\{3^{2}21^{4}\}+\{3^{2}2^{4}1\}+\{32^{6}\}+\{3^{2}1^{6}\}+\{3^{6}21\}+\{321\}+3\{2^{4}1^{4}\}+3\{21^{4}\}+2\{2^{7}1\}+2\{21\}+2\{2^{3}1^{3}\}+3\{2^{2}1^{5}\}+2\{2^{2}1^{2}\}+2\{2^{5}1^{2}\}+\{2^{4}1\}+3\{21^{7}\}+2\{1^{6}\}+2\{1^{3}\}$
$(4^{3}3^{5})$	$\{4^{3}3^{5}\}+\{41^{5}\}+2\{432^{5}1\}+\{4^{2}3^{4}2^{2}\}+\{42^{2}1^{4}\}+\{4^{2}32^{5}\}+\{42^{5}1\}+\{43^{6}2\}+\{421^{6}\}+\{42^{4}1^{3}\}+\{43^{3}2^{4}\}+\{432^{2}1^{4}\}+\{43^{4}2^{2}1\}+\{43^{2}2^{3}1^{2}\}+\{42^{7}\}+\{3^{8}\}+\{3\}+2\{3^{5}2^{3}\}+2\{31^{3}\}+\{3^{5}1^{3}\}+\{3^{2}1^{3}\}+\{3^{4}2^{3}\}+\{3^{2}2^{3}\}+2\{3^{2}1^{6}\}+2\{32^{6}\}+3\{3^{2}2^{3}1^{3}\}+3\{32^{3}1^{3}\}+3\{32^{5}1^{2}\}+3\{32^{2}1^{5}\}+\{3^{3}2^{2}1^{2}\}+\{3^{2}2^{2}1^{2}\}+\{3^{3}21^{4}\}+\{32^{4}1\}+\{3^{4}2^{2}1^{2}\}+\{32^{2}1^{2}\}+2\{3^{3}2^{4}1\}+2\{321^{4}\}+\{3^{2}21^{4}\}+\{3^{2}2^{4}1\}+\{3^{6}21\}+\{321\}+2\{3^{2}2^{6}\}+2\{31^{6}\}+3\{2^{3}1^{3}\}+4\{2^{2}1^{5}\}+3\{2^{4}1^{4}\}+3\{21^{4}\}+2\{2^{7}1\}+2\{21\}+\{2^{4}1\}+\{2^{6}\}+\{2^{3}\}+2\{2^{5}1^{2}\}+2\{2^{2}1^{2}\}+2\{21^{7}\}+2\{1^{6}\}+2\{1^{3}\}+\{0\}$
(543^{6})	$\{543^{6}\}+\{52^{6}1\}+\{53^{5}2^{2}\}+\{53^{2}2^{5}\}+\{43^{6}2\}+\{421^{6}\}+2\{432^{5}1\}+2\{43^{3}2^{4}\}+2\{42^{4}1^{3}\}+\{4^{2}3^{4}2^{2}\}+\{42^{2}1^{4}\}+\{4^{2}32^{5}\}+\{42^{5}1\}+2\{43^{2}2^{3}1^{2}\}+\{432^{2}1^{4}\}+\{43^{4}2^{2}1\}+\{43^{5}1^{2}\}+\{43^{2}1^{5}\}+2\{42^{7}\}+\{3^{5}21\}+\{3^{2}21\}+2\{3^{4}2^{2}1^{2}\}+2\{32^{2}1^{2}\}+2\{3^{2}2^{6}\}+2\{31^{6}\}+3\{3^{3}2^{4}1\}+3\{321^{4}\}+\{3^{2}2^{2}1^{2}\}+\{3^{3}2^{2}1^{2}\}+2\{3^{3}21^{4}\}+2\{32^{4}1\}+3\{3^{2}2^{3}1^{3}\}+3\{32^{3}1^{3}\}+\{3^{3}1^{3}\}+\{3^{3}2^{3}\}+\{3^{6}21\}+\{321\}+\{3^{5}2^{3}\}+\{31^{3}\}+\{32^{6}\}+\{3^{2}1^{6}\}+2\{3^{2}2^{4}1\}+2\{3^{2}21^{4}\}+5\{32^{5}1^{2}\}+5\{32^{2}1^{5}\}+2\{2^{7}1\}+2\{21\}+2\{2^{4}1\}+4\{2^{4}1^{4}\}+4\{21^{4}\}+2\{2^{6}\}+2\{2^{3}\}+4\{2^{2}1^{5}\}+3\{2^{5}1^{2}\}+3\{2^{2}1^{2}\}+4\{2^{3}1^{3}\}+4\{21^{7}\}+3\{1^{6}\}+3\{1^{3}\}$
(63^{7})	$\{63^{7}\}+\{53^{2}2^{5}\}+\{53^{5}2^{2}\}+\{43^{5}1^{2}\}+\{43^{2}1^{5}\}+\{43^{3}2^{4}\}+\{42^{4}1^{3}\}+\{43^{2}2^{3}1^{2}\}+\{432^{5}1\}+\{42^{7}\}+\{3^{6}\}+\{3^{3}\}+\{3^{3}1^{3}\}+\{3^{3}2^{3}\}+\{3^{4}2^{2}1^{2}\}+\{32^{2}1^{2}\}+\{3^{3}21^{4}\}+\{32^{4}1\}+\{3^{2}2^{3}1^{3}\}+\{32^{3}1^{3}\}+2\{32^{5}1^{2}\}+2\{32^{2}1^{5}\}+\{3^{2}2^{4}1\}+\{3^{2}21^{4}\}+\{3^{2}2^{6}\}+\{31^{6}\}+\{2^{5}1^{2}\}+\{2^{2}1^{2}\}+\{2^{6}\}+\{2^{3}\}+\{2^{4}1^{4}\}+\{21^{4}\}+3\{2^{3}1^{3}\}+\{2^{2}1^{5}\}+\{21^{7}\}+2\{1^{6}\}+2\{1^{3}\}+\{0\}$
	(b) $E_8 \to SU_2 \times E_7$ branching rules[A]
(3^{8})	${}^{4}(2^{7})+{}^{3}[(3^{2}2^{5})+(2^{5}1^{2})+(21^{6})]+{}^{2}[(3^{4}2^{3})+(32^{5}1)+(2^{7})+(1^{6})]+{}^{1}[(3^{6}2)+(2^{5}1^{2})+(21^{6})]$
$(43^{6}2)$	${}^{5}(21^{6})+{}^{4}[(32^{5}1)+(2^{7})+(1^{6})]+{}^{3}[(42^{6})+(3^{6}2)+(3^{2}2^{5})+(2^{6})+2(2^{5}1^{2})+(21^{6})+(0)]+{}^{2}[(43^{6})+(3^{5}21)+(3^{4}2^{3})+2(32^{5}1)+2(2^{7})+2(1^{6})]+{}^{1}[(43^{4}2^{2})+(3^{6}2)+(3^{2}2^{5})+(2^{6})+(2^{5}1^{2})+2(21^{6})]$
$(4^{3}3^{5})$	${}^{5}(2^{5}1^{2})+{}^{4}[(3^{4}2^{3})+(32^{5}1)+(2^{7})+(1^{6})]+{}^{3}[(43^{4}2^{2})+(3^{6}2)+2(3^{2}2^{5})+(2^{6})+(2^{5}1^{2})+2(21^{6})]+{}^{2}[(4^{2}3^{4}2^{2})+(43^{6})+(3^{5}21)+(3^{4}2^{3})+2(32^{5}1)+2(2^{7})+(1^{6})]+{}^{1}[(4^{3}3^{4})+(43^{5}1)+(42^{6})+(3^{6}2)+(3^{2}2^{5})+2(2^{5}1^{2})+(0)]$
(543^{6})	${}^{6}(1^{6})+{}^{5}[(2^{6})+(2^{5}1^{2})+(21^{6})+(0)]+{}^{4}[(3^{5}21)+2(32^{5}1)+(2^{7})+2(1^{6})]+{}^{3}[(43^{5}1)+(43^{4}2^{2})+(42^{6})+(3^{6}2)+(3^{2}2^{5})+2(2^{6})+2(2^{5}1)+2(21^{6})+(0)]+{}^{2}[(53^{5}2)+(4^{2}3^{4}2)+(43^{6})+(3^{6})+(3^{5}21)+(3^{4}2^{3})+3(32^{5}1)+(2^{7})+2(1^{6})]+{}^{1}[(543^{5})+(43^{5}1)+(43^{4}2^{2})+(42^{6})+(3^{6}2)+(3^{2}2^{5})+(2^{6})+(2^{5}1^{2})+(21^{6})]$
(63^{7})	${}^{7}(0)+{}^{6}(1^{6})+{}^{5}[(2^{6})+(21^{6})]+{}^{4}[(3^{6})+(32^{5}1)+(1^{6})]+{}^{3}[(43^{5}1)+(42^{6})+(2^{6})+(2^{5}1^{2})+(0)]+{}^{2}[(53^{5}2)+(3^{5}21)+(32^{5}1)+(1^{6})]+{}^{1}[(63^{6})+(43^{4}2^{2})+(2^{6})+(21^{6})]$

[A] The left superscript in this table corresponds to the dimension of the relevant SU_2 irrep.

Table 5 (*Continued*)

(Λ)	Branching
	(*c*) $E_8 \to SU_3 \times E_6$ *branching rules*
(3^8)	$\{32\}[(1:1)+(1^4:2)]+\{31\}[(1^5:1)+(1^2:2)]+\{3^2\}(0:2)+\{3\}(0:2)+\{21\}[2(0:0)+2(0:2)+2(21^4:2)+(1^3:3)]+\{2\}[2(1:1)+(1^4:2)+(1:3)]+\{2^2\}[2(1^5:1)+(1^2:2)+(1^5:3)]+\{1\}[2(1^5:1)+(2:2)+2(1^2:2)+(1^5:3)+(21^3:3)]+\{1^2\}[2(1:1)+(2^5:2)+2(1^4:2)+(1:3)+(2^21^3:3)]+\{0\}[2(0:2)+(21^4:2)+(21:3)+(2^41:3)]$
(43^62)	$\{43\}(1^5:1)+\{41\}(1:1)+\{42\}[(0:0)+(0:2)]+\{3^2\}[(0:0)+(0:2)+(21^4:2)]+\{3\}[(0:0)+(0:2)+(21^4:2)]+\{32\}[3(1:1)+(2^5:2)+2(1^4:2)+(1:3)]+\{31\}[3(1^5:1)+(2:2)+2(1^2:2)+(1^5:3)]+\{21\}[3(0:0)+5(0:2)+5(21^4:2)+2(1^3:3)+(21:3)+(2^41:3)+(0:4)]+\{2\}[3(1:1)+(2^5:2)+3(1^4:2)+2(1:3)+(2^21^3:3)]+\{2^2\}[3(1^5:1)+(2:2)+3(1^2:2)+2(1^5:3)+(21^3:3)]+\{1\}[5(1^5:1)+2(2:2)+4(1^2:2)+3(1^5:3)+2(21^3:3)+(32^4:3)+(1^2:4)]+\{1^2\}[5(1:1)+2(2^5:2)+4(1^4:2)+3(1:3)+2(2^21^3:3)+(31^4:3)+(1^4:4)]+\{0\}[(0:0)+3(0:2)+3(21^4:2)+2(1^3:3)+(21:3)+(2^41:3)+(21^4:4)]$
(4^33^5)	$\{43\}[(1^5:1)+(1^2:2)]+\{41\}[(1:1)+(1^4:2)]+\{42\}[(0:0)+(0:2)+(21^4:2)]+\{4^2\}(1:1)+\{4\}(1^5:1)+\{3^2\}[(0:0)+2(0:2)+(21^4:2)+(1^3:3)]+\{3\}[(0:0)+2(0:2)+(21^4:2)+(1^3:3)]+\{32\}[3(1:1)+3(1^4:2)+(2^5:2)+2(1:3)+(2^21^3:3)]+\{31\}[3(1^5:1)+3(1^2:2)+(2:2)+2(1^5:3)+(21^3:3)]+\{21\}[2(0:0)+6(0:2)+6(21^4:2)+2(21:3)+2(2^41:3)+3(1^3:3)+(0:4)+(21^4:4)]+\{2\}[(1^5:1)+3(1:1)+3(1^4:2)+2(2^5:2)+(2^21^3:3)+3(1:3)+(31^4:3)+(1^4:4)]+\{2^2\}[(1:1)+3(1^5:1)+3(1^2:2)+2(2:2)+(21^3:3)+3(1^5:3)+(32^4:3)+(1^2:4)]+\{1\}[(1:1)+3(1^5:1)+5(1^2:2)+2(2:2)+3(21^3:3)+4(1^5:3)+(32^4:3)+(1^2:4)+(2:4)+(2^31^2:4)]+\{1^2\}[1^5:1)+3(1:1)+5(1^4:2)+2(2^5:2)+3(2^21^3:3)+4(1:3)+(32^4:3)+(1^4:4)+(2^5:4)+(21^2:4)]+\{0\}[2(0:0)+2(0:2)+4(21^4:2)+2(1^3:3)+(21:3)+(2^41:3)+(3:3)+(3^5:3)+(0:4)+(21^4:4)+(2^21^2:4)]$
(543^6)	$\{54\}(0:0)+\{51\}(0:0)+\{53\}(1:1)+\{52\}(1^5:1)+\{4^2\}(1:1)+\{4\}(1^5:1)+\{43\}[2(1^5:1)+(1^2:2)+(2:2)]+\{41\}[2(1:1)+(1^4:2)+(2^5:2)]+\{42\}[2(0:0)+2(0:2)+2(21^4:2)]+\{3^2\}[(0:0)+2(0:2)+2(21^4:2)+(21:3)]+\{3\}[(0:0)+2(0:2)+2(21^4:2)+(2^41:3)]+\{32\}[(1^5:1)+4(1:1)+2(2^5:2)+3(1^4:2)+3(1:3)+(2^21^3:3)+(31^4:3)]+\{31\}[(1:1)+4(1^5:1)+2(2:2)+3(1^2:2)+3(1^5:3)+(21^3:3)+(32^4:3)]+\{21\}[4(0:0)+6(0:2)+8(21^4:2)+2(21:3)+2(2^41:3)+4(1^3:3)+(3:3)+(3^5:3)+2(0:4)+2(21^4:4)]+\{2\}[(1^5:1)+3(1:1)+3(2^5:2)+4(1^2:2)+3(1:3)+2(2^21^3:3)+(31^4:3)+(1^4:4)+(2^5:4)]+\{2^2\}[(1:1)+3(1^5:1)+3(2:2)+4(1^2:2)+3(1^5:3)+2(21^3:3)+(32^4:3)+(1^2:4)+(2:4)]+\{1\}[2(1:1)+3(1^5:1)+3(2:2)+5(1^2:2)+6(1^5:3)+3(21^3:3)+2(32^4:3)+2(1^2:4)+(2:4)+(2^31^2:4)+(1^5:5)]+\{1^2\}[2(1^5:1)+3(1:1)+3(2^5:2)+5(1^4:2)+6(1:3)+3(2^21^3:3)+2(31^4:3)+2(1^4:4)+(2^5:4)+(21^2:4)+(1:5)]+\{0\}[4(0:2)+4(21^4:2)+2(21:3)+2(2^41:3)+2(1^3:3)+2(0:4)+2(21^4:4)+(1^3:5)]$
(63^7)	$\{63\}(0:0)+\{53\}(1:1)+\{52\}(1^5:1)+\{43\}[(1^5:1)+(2:2)]+\{41\}[(1:1)+(2^5:2)]+\{42\}[(0:0)+(0:2)+(21^4:2)]+\{3^2\}[(0:0)+(21^4:2)+(3:3)]+\{3\}[(0:0)+(21^4:2)+(3^5:3)]+\{32\}[(1^5:1)+(1:1)+(2^5:2)+(1^4:2)+(1:3)+(31^4:3)]+\{31\}[(1:1)+(1^5:1)+(2:2)+(1^2:2)+(1^5:3)+(32^4:3)]+\{21\}[(0:0)+2(0:2)+3(21^4:2)+(21:3)+(2^41:3)+(0:4)+(21^4:4)]+\{2\}[(1^5:1)+(2^5:2)+(1^4:2)+(1:3)+(2^21^3:3)+(2^5:4)]+\{2^2\}[(1:1)+(2:2)+(1^2:2)+(1^5:3)+(21^3:3)+(2:4)]+\{1\}[(1:1)+(1^5:1)+(2:2)+(1^2:2)+2(1^5:3)+2(21^3:3)+(32^4:3)+(1^2:4)+(1^5:5)]+\{1^2\}[(1^5:1)+(1:1)+(2^5:2)+(1^4:2)+2(1:3)+2(2^21^3:3)+(31^4:3)+(1^4:4)+(1:5)]+\{0\}[(0:0)+(0:2)+(21^4:2)+2(1^3:3)+(21^4:4)+(0:4)+(0:6)]$

Application to Unified Gauge Theories

The application of the exceptional groups to possible unified gauge theories has been considered by a number of authors. Gürsey and his associates (Gürsey *et al.* 1975; Gürsey and Sikivie 1976; Sikivie and Gürsey 1977) have considered models based on E_7, while a detailed study of a vector-like E_7 model has been made by Ramond (1976, 1977). The weak interaction angle $\sin^2\theta_W = \frac{2}{3} + O(e^2/g_c^2)$ is strongly in conflict with experiment (note that we include D'yakonov's (1977) correction to earlier published values of $\frac{1}{2}$). The vector-like E_7 model is also in serious error in its failure to recognize properly the consequences of choosing a symplectic-type irrep (1^6) of E_7 to embed fermions. Consistent usage requires that the representation be doubled; a model using such a doubled representation has been discussed by Gell-Mann *et al.* (1978). Nonvector-like E_6 gauge models remain consistent with most experimental data (Achiman and Stech 1978).

Gell-Mann *et al.* (1978) have determined all Lie groups G that can be written in the direct product to

$$G \supset SU_3^c \times G^{fl},$$

where $SU_3^c \times G^{fl}$ is a maximal subgroup of G, with G^{fl} the group of flavours and SU_3^c the assumed exact colour gauge group of QCD. In choosing G they restricted their attention to groups whose fermion representations involved only triplets, anti-triplets and singlets of colour. This restriction excluded E_8 from their considerations. Possibly such a restriction is excessively severe since octets and sextets of colour arise naturally in SO_8 supergravity models (Gell-Mann 1977; Gell-Mann *et al.* 1978).

In the case of $E_8 \supset SU_3^c \times E_6$, we would have in the fermionic (21^7) irrep of E_8 a colour octet of flavourless states, 27 coloured quarks and antiquarks and 78 colourless leptons. In the bosonic (21^7) irrep there would be the expected 8 gluons of the colour octet, 27 leptoquarks and antiquarks and 78 colourless vector bosons. Again we note that in any supersymmetric theory, even for $N = 1$, we have colour octets of both fermions and bosons. Detailed discussion of possible E_8 models is premature and must await further developments in theory and experiment.

Concluding Remarks

The problem of resolving Kronecker products of E_8 irreps and branching rules to the principal maximal subgroups of E_8 has been solved to the extent of possible applications in physics. Sufficient information has been given to permit a considerable extension if required. It would seem to be important that physicists have available to them at least a sketch of the basic properties of E_8, and this has been the aim of the present work.

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References

Achiman, Y., and Stech, B. (1978). *Phys. Lett.* B **77**, 789.
Butler, P. H., Haase, R. W., and Wybourne, B. G. (1978). *Aust. J. Phys.* **31**, 131.
Butler, P. H., Haase, R. W., and Wybourne, B. G. (1979). *Aust. J. Phys.* **32**, 137.

Butler, P. H., and Wybourne, B. G. (1969). *J. Phys. (Paris)* **30**, 655, 795.
D'yakonov, D. I. (1977). *Yad. Fiz.* **26**, 845; *Sov. J. Nucl. Phys.* **26**, 443.
Dynkin, E. B. (1952*a*). *Mat. Sb.* **30**(72), 349; *Am. Math. Soc. Transl.* **6**(2), 111.
Dynkin, E. B. (1952*b*). *Tr. Mosk. Mat. Obsc.* **1**, 39; *Am. Math. Soc. Transl.* **6**(2), 245.
Gell-Mann, M. (1977). Invited Paper at American Physical Society Meeting, Washington, April 1977.
Gell-Mann, M., Ramond, P., and Slansky, R. (1978). *Rev. Mod. Phys.* **50**, 721.
Gürsey, F., Ramond, P., and Sikivie, P. (1975). *Phys. Rev. Lett.* **12**, 2166.
Gürsey, F., and Sikivie, P. (1976). *Phys. Rev. Lett.* **36**, 775.
McKay, W., and Patera, J. (1977). Centre de Recherches Mathematiques, Université de Montreal, Rep. No. CRM–742.
McKay, W., Patera, J., and Sankoff, D. (1976*a*). *In* 'Computers in the Study of Nonassociative Rings and Algebras' (Eds R. E. Beck and B. Kolman) (Academic: New York).
McKay, W., Patera, J., and Sharp, R. T. (1976*b*). *J. Math. Phys. (New York)* **17**, 1371.
Malćev, A. I. (1944). *Izv. Akad. Nauk. SSSR Ser. Mat.* **8**, 143; English transl. (1962) in *Am. Math. Soc. Transl.* **9**(1).
Patera, J., and Sankoff, D. (1973). 'Tables of Branching Rules for Representations of Simple Lie Algebras' (Montreal Univ. Press).
Patera, J., Sharp, R. T., and Winternitz, P. (1976). *J. Math. Phys. (New York)* **17**, 1872.
Patera, J., Sharp, R. T., and Winternitz, P. (1977). *J. Math. Phys. (New York)* **18**, 1518.
Qubanchi, A. H. A. Al- (1978). Ph.D. Thesis, University of Southampton.
Ramond, P. (1976). *Nucl. Phys.* B **110**, 214.
Ramond, P. (1977). *Nucl. Phys.* B **126**, 509.
Sikivie, P., and Gürsey, F. (1977). *Phys. Rev.* D **16**, 816.
Wybourne, B. G. (1970). 'Symmetry Principles and Atomic Spectroscopy' (Wiley–Interscience: New York).
Wybourne, B. G. (1978). *J. Math. Phys. (New York)* **19**, 529.
Wybourne, B. G. (1979). Powers of the fundamental representation of a semisimple Lie group. *J. Phys.* A (in press).
Wybourne, B. G., and Bowick, M. J. (1977). *Aust. J. Phys.* **30**, 259.

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