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Barrier Theory based on Half-barrier Penetration

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Abstract

Special methods are investigated for the calculation of reflection and transmission coefficients for a real symmetric potential barrier. These methods, involving half-barrier penetration only, are both of physical and mathematical interest, and they also afford a considerable saving in time when numerical procedures are necessary in the case of analytically nonsoluble models.

1. Introduction

The propagation of waves through an overdense or underdense potential barrier demands the calculation of the field throughout the medium, either analytically or numerically. Some barriers are susceptible to analytical treatment, in terms of the Airy integral, Bessel functions, the parabolic cylinder functions, the Whittaker functions and the hypergeometric function. There is a class of problems where the effect of the barrier on an incident wave can be treated by considering only half-barrier penetration. We refer to the symmetric models in which $q(-z) = q(z)$, where $q(z)$, a real function of z, is the square of the refractive index.

In Section 2 here, we investigate general formulae for the calculation of reflection and transmission coefficients. Their properties are developed, as well as special methods of calculation. Their application, both on the grounds of physical and mathematical interest, is dealt with in Section 3, the more usual profiles being treated as well as some new profiles that lie outside the scope of standard models.

2. Fundamental Formulae

Consider the propagation equation

$$\mathrm{d}^2 w/\mathrm{d}z^2 + k^2 q(z)\, w = 0\,.$$

Here, q is the square of the refractive index; the medium is taken to be symmetric in the sense that $q(-z) = q(z)$; the medium is also lossless, so q is real when z is real. These conditions imply that $w(-z)$ and $w^*(z)$ are solutions when $w(z)$ is a solution, the asterisk denoting the complex conjugate, z being real throughout. If q and its derivatives suffer discontinuities at any point, the continuity of w and $\mathrm{d}w/\mathrm{d}z$ are the imposed boundary conditions. As $z \to +\infty$, let q tend either to a positive constant or to a positive variable function (with zero argument) such that the usual WKBJ solutions exist there.

Define $u(z)$ to be the solution such that

$$\left\{u(z)\Big/ q^{-\frac{1}{4}}\exp\left(-\mathrm{i}k\int_a^z q^{\frac{1}{2}}\,\mathrm{d}t\right)\right\} \to 1 \qquad \text{as} \quad z \to \infty,$$

with $q > 0$ throughout the range of integration. We shall take $u(z)$ to be the upgoing wave (namely, propagating along $+0z$). The general solution is

$$w = A\,u(z) + B\,u^*(z),$$

for real z. For $z > 0$, let $w = T u(z)$, where T is the transmission coefficient. For $z < 0$, the solution is generally

$$w = A\,u(-z) + B\,u^*(-z).$$

Now $u(-z)$ will be the downgoing wave, so more particularly we write

$$w = u^*(-z) + R\,u(-z),$$

where R is the reflection coefficient. At $z = 0$, w and w' are continuous, either because q is regular there or because of the imposed boundary conditions if the derivatives of q suffer discontinuities there. (Throughout, a prime denotes $\mathrm{d}/\mathrm{d}z$.) Hence

$$u^*(0) + R\,u(0) = T u(0), \qquad -u^{*\prime}(0) - R\,u'(0) = T u'(0),$$

or

$$R + u^*/u = T, \qquad R + u^{*\prime}/u' = -T,$$

evaluated at $z = 0$. Hence

$$R = -\tfrac{1}{2}(u^*/u + u^{*\prime}/u'), \qquad T = \tfrac{1}{2}(u^*/u - u^{*\prime}/u'). \tag{1}$$

Simplification gives

$$R = -(uu^*)'/(u^2)' = -|u|\,|u|'/uu', \qquad |R| = \big||u|'\big|/|u'|, \tag{2}$$

evaluated at $z = 0$. If $u(z) = X(z) + \mathrm{i}\,Y(z)$ in terms of real and imaginary parts, then

$$|R| = |XX' + YY'|/\{(X^2+Y^2)(X'^2+Y'^2)\}^{\frac{1}{2}}, \tag{3}$$

evaluated at $z = 0$.

From equations (1), we have

$$RR^* + TT^* = \tfrac{1}{4}\left(\frac{u^*}{u} + \frac{u^{*\prime}}{u'}\right)\left(\frac{u}{u^*} + \frac{u'}{u^{*\prime}}\right) + \tfrac{1}{4}\left(\frac{u^*}{u} - \frac{u^{*\prime}}{u'}\right)\left(\frac{u}{u^*} - \frac{u'}{u^{*\prime}}\right) = 1,$$

as expected, this being the energy conservation equation. Moreover,

$$T/R = -(u^*u' - uu^{*\prime})/(u^*u' + uu^{*\prime}) = -2\mathrm{i}\,\mathrm{Im}(u^*u')/|u^2|',$$

so

$$\arg T - \arg R = \pm\tfrac{1}{2}\pi,$$

a characteristic feature of symmetric models.

A special calculation is often useful. Suppose that the solution $u(z)$ under consideration can be expanded as

$$u = A + Bz + Cz^2 + \dots$$

near $z = 0$; this is always possible since $q(z)$ has no singularity at $z = 0$. Then

$$uu^* = AA^* + 2\operatorname{Re}(AB^*)z + \dots, \qquad u^2 = A^2 + 2ABz + \dots,$$

and, at $z = 0$,

$$(uu^*)' = 2\operatorname{Re}(AB^*), \qquad (u^2)' = 2AB.$$

Hence

$$R = -\operatorname{Re}(AB^*)/AB = -(1 + \mathrm{i}\tan\arg B)^{-1} \tag{4}$$

(when A is real) and

$$|R| = |\operatorname{Re}(AB^*)|/|AB| = |\cos\arg(AB^*)|. \tag{5}$$

Clearly, any real factors may be omitted in A and B when these formulae are used, since they cancel directly.

If, instead, we write

$$u = 1 + (B/A)z + \dots,$$

thereby replacing A by 1 and B by B/A, the result (4) yields only a slightly modified value of R, namely

$$R = -\operatorname{Re}(B^*/A^*)/(B/A) = -\operatorname{Re}(AB^*)/A^*B,$$

but the modulus remains the same. Moreover, if B/A equals C/D when expressed in numerator/denominator form in any suitable manner, then

$$|R| = |\operatorname{Re}(C^*/D^*)|/|C/D| = |\operatorname{Re}(C^*D)|/|CD^*|, \tag{6}$$

a formula that is helpful when $|D|$ cannot be directly evaluated, as in Section 3h below.

Again, if

$$u = A\exp(-\mathrm{i}knz) + B\exp(\mathrm{i}knz),$$

when a homogeneous medium of real refractive index n exists in a range of z including $z = 0$, then the calculation of u^2 and uu^* leads to

$$R = -(AB^* - BA^*)/(A^2 - B^2) = -2\operatorname{Im}(AB^*)/(A^2 - B^2). \tag{7}$$

There are several advantages in the use of the results (2) for a symmetrical medium. Firstly, when $q(z)$ is an analytic function of z for all real z, and when w can be expressed in terms of known functions, there is no need to use any connection relations to join an asymptotic form as $z \to +\infty$ to asymptotic forms as $z \to -\infty$, so as to separate out upgoing and downgoing waves on the incident side of the barrier. The series solution at the midpoint of the barrier is all that is required, this solution being that one that corresponds to the upgoing solution as $z \to +\infty$. Secondly, from a numerical point of view, computation time is greatly reduced. Integration can be commenced for an upgoing wave, and this is integrated downwards only half way through the barrier to the centre point $z = 0$. Further integration, and the separation of the solution into upgoing and downgoing waves, are entirely avoided.

3. Applications of Formulae

(*a*) *Complete Transparency of Almost Completely Opaque Slabs*

The phenomenon of the complete transparency ($R = 0, |T| = 1$) of several almost completely opaque slabs can be investigated by a full calculation through all the boundaries involved (Heading 1963, 1975). But the use of the formulae (2) shows that the phenomenon (though outside the range of everyday experience) is simpler than that suggested by the full calculation.

Let the inhomogeneous lossless slab be contained within the range $a < z < a+h$ with $a > 0$, where free space exists below $z = a$ and where h may extend to infinity provided propagating WKBJ solutions exist. If $z = a+s$, let the reflection and transmission properties of the slab be such that

$$\mathrm{e}^{-\mathrm{i}ks} + r\,\mathrm{e}^{\mathrm{i}ks} \leftrightarrow t\,\mathrm{e}^{-\mathrm{i}ks} \quad \text{or} \quad t \times \mathscr{W}$$

($\mathscr{W}$ denoting the WKBJ solution) represent the fields on each side of the slab. If we have $r \approx 1$, the slab is almost opaque, that is, it acts as a potential barrier. With respect to z, the field for $z < a$ may be written as

$$\mathrm{e}^{-\mathrm{i}kz} + r\,\mathrm{e}^{-2\mathrm{i}ka}\,\mathrm{e}^{\mathrm{i}kz}.$$

Now place an identical slab in the range $-(a+h) < z < -a$, with $q(z) = q(-z)$. For this overall symmetrical medium we have $A = 1$ and $B = r\,\mathrm{e}^{-2\mathrm{i}ka}$, and equation (7) gives

$$R = -(r^*\,\mathrm{e}^{2\mathrm{i}ka} - r\,\mathrm{e}^{-2\mathrm{i}ka})/(1 - r^2\,\mathrm{e}^{-4\mathrm{i}ka}). \tag{8}$$

This vanishes when $r/r^* = \mathrm{e}^{4\mathrm{i}ka}$, that is, when a is adjusted to give

$$a = (\arg r + N)/2k,$$

N being any integer so that a is positive. This gives a set of discrete values of a that render the combined system completely transparent whatever the value of $\arg r$.

The inverse phenomenon is impossible; that is, two almost transparent slabs cannot be placed side by side so as to yield an almost opaque combination. For if $|r|$ is small in equation (8), inspection shows that it is impossible to achieve $|R| \approx 1$.

(*b*) *Homogeneous Slab*

Application to the homogeneous slab $-h < z < h$ enables us to use only two out of the usual four boundary conditions. Let

$$u = \mathrm{e}^{-\mathrm{i}kz}, \quad z \geqslant h; \qquad u = A\,\mathrm{e}^{-\mathrm{i}knz} + B\,\mathrm{e}^{\mathrm{i}knz}, \quad 0 \leqslant z \leqslant h,$$

where n is real and positive. Boundary conditions at $z = h$ give

$$A = \tfrac{1}{2}(n+1)n^{-1}\exp\{\mathrm{i}kh(n+1)\}, \qquad B = \tfrac{1}{2}(n-1)n^{-1}\exp\{-\mathrm{i}kh(n-1)\},$$

so the result (8) reduces to

$$R = \exp(2\mathrm{i}kh)(1-n^2)\sin(2knh)/\{(n^2+1)\sin(2knh) - 2n\mathrm{i}\cos(2knh)\},$$

the standard result (see Heading 1975, p. 88).

(c) Elementary Frequency-dependent Symmetric Barrier reducing to Free Space at Infinity

The differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + k^2\left(1 - \frac{2}{k^2t^2}\right)w = 0$$

may be checked to have the two elementary solutions

$$w = \exp(\pm\mathrm{i}kt)(\mp\mathrm{i}k + t^{-1}).$$

To avoid the singularity, consider a symmetric model governed by

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + k^2\left(1 - \frac{2}{k^2(z+1)^2}\right)w = 0, \qquad z \geqslant 0.$$

Clearly $q \to 1$ as $z \to \infty$. If $k^2 > 2$ then q is underdense (positive) for all z but, if $k^2 < 2$, the medium is overdense (q negative) for $|z| < \sqrt{2}k^{-1} - 1$. Thus we form a frequency-dependent symmetric barrier as for an isotropic ionized medium (with a discontinuity in gradient at $z = 0$).

We take u to be given by

$$u = \exp\{-\mathrm{i}k(z+1)\}\{\mathrm{i}k + (z+1)^{-1}\} \propto \exp(-\mathrm{i}kz)\{1 - z(1+\mathrm{i}k)^{-1}...\},$$

near $z = 0$. Then the formula (5), with $A = 1$ and $B = -\mathrm{i}k - (1+\mathrm{i}k)^{-1}$, yields

$$|R| = (1+k^6)^{-\frac{1}{2}}.$$

For large k^2, we have $|R| \to 0$, the medium effectively becoming free space throughout; but, for small k^2, we have $|R| \to 1$, a potential barrier now being formed. This difference is typical of ionospheric barriers. The critical value of k^2 where q just vanishes at the origin ($k^2 = 2$) forms a dividing line between overdense and underdense barriers.

A similar barrier is given by $q = 1 - 6/k^2(z+1)^2$. We find that

$$u = \exp(-\mathrm{i}kz)\{(z+1)^{-2} + \mathrm{i}k(z+1)^{-1} - \tfrac{1}{3}k^2\},$$

yielding

$$|R| = 3(6+k^2)/\{(9+3k^2+k^4)(36-3k^4+k^6)\}^{\frac{1}{2}},$$

with properties similar to those of the previous model.

A hierarchy of models is produced, given by $q = 1 - A/k^2(z+1)^2$, soluble in terms of the elementary functions whenever $A = n(n+1)$, n being a positive integer. For general values of A,

$$u = (z+1)^{\frac{1}{2}}\,\mathrm{H}^{(2)}_{(A^2+\frac{1}{4})^{1/2}}\big(k(z+1)\big),$$

in terms of the Hankel function of the second kind, but analytically the formulae (2) do not simplify to anything of interest when $z = 0$.

(*d*) *Power-law Profile*

Consider the symmetric profile

$$\begin{aligned} q &= a^2 z^n, & & z > 0; \\ &= a^2 z^n, & n \text{ even}, \quad & z < 0; \\ &= -a^2 z^n, & n \text{ odd}, \quad & z < 0; \end{aligned}$$

where a is real and positive and n is a positive integer. There is a discontinuity in the nth derivative at $z = 0$ when n is odd. Försterling (1950) has considered the problem (when n is even), employing more usual methods. The solution is

$$u = z^{\frac{1}{2}}\,\mathrm{H}^{(2)}_{1/(n+2)}\left(\frac{2ak}{n+2}z^{\frac{1}{2}(n+2)}\right).$$

The usual method would employ connection formulae joining asymptotic solutions as $z \to +\infty$ to asymptotic solutions as $z \to -\infty$, noting that u contains the factor $\exp\{-2ak\mathrm{i}(n+2)^{-1}z^{\frac{1}{2}(n+2)}\}$ for large positive z, thus representing the upgoing wave. Near the origin, we have

$$\begin{aligned} u &\propto z^{\frac{1}{2}}\exp\{\pi\mathrm{i}/(n+2)\}\,\mathrm{J}_{1/(n+2)}\left(\frac{2ak}{n+2}z^{\frac{1}{2}(n+2)}\right) - z^{\frac{1}{2}}\,\mathrm{J}_{-1/(n+2)}\left(\frac{2ak}{n+2}z^{\frac{1}{2}(n+2)}\right) \\ &= \frac{\exp\{\pi\mathrm{i}/(n+2)\}}{\Gamma(\nu+1)}\left(\frac{ak}{n+2}\right)^{1/(n+2)} z - \frac{1}{\Gamma(-\nu+1)}\left(\frac{ak}{n+2}\right)^{-1/(n+2)} \dots \\ &= Az + B\dots, \end{aligned}$$

say, where $\nu = (n+2)^{-1}$. The result (4) now gives

$$\begin{aligned} R = -(AB^* + BA^*)/2AB &= -\tfrac{1}{2}(1 + A^*/A) \qquad (B \text{ is real}) \\ &= -\tfrac{1}{2}[1 + \exp\{-2\pi\mathrm{i}/(n+2)\}] \\ &= -\exp\{-\pi\mathrm{i}/(n+2)\}\cos\{\pi/(n+2)\}, \end{aligned}$$

and $|R| = \cos\{\pi/(n+2)\}$, corresponding to Försterling's (1950) more lengthy calculation when n is even.

(e) Parabolic Potential Barrier

The profile $q = z^2 - a^2$ (with a^2 real) represents a potential barrier if $a^2 > 0$. The equation

$$\mathrm{d}^2w/\mathrm{d}z^2 + k^2(z^2 - a^2)w = 0$$

has the solution

$$u = D_n\big((2k)^{\frac{1}{2}}\exp(\tfrac{1}{4}\pi i)\,z\big) \sim \{(2k)^{\frac{1}{2}}\exp(\tfrac{1}{4}\pi \mathrm{i})\,z\}^n \exp(-\tfrac{1}{2}k\mathrm{i}z^2),$$

where $n = \frac{1}{2}\mathrm{i}ka^2 - \frac{1}{2}$, in terms of the parabolic cylinder function.

Quoting Whittaker and Watson (1927, p. 347), we have, near $z = 0$,

$$u \propto \frac{\Gamma(\frac{1}{2})z^{-\frac{1}{4}}(k\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}z^2)^{\frac{1}{4}}\{1+O(z)^2\}}{\Gamma(\frac{1}{2}-\frac{1}{2}n)} + \frac{\Gamma(-\frac{1}{2})z^{-\frac{1}{4}}(k\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}z^2)^{\frac{3}{4}}\{1+O(z^2)\}}{\Gamma(-\frac{1}{2}n)},$$

so in the results (4) and (5), discarding any real factors that cancel in R and $|R|$, we may use

$$A = (\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}})^{\frac{1}{4}}/\Gamma(\tfrac{1}{2}-\tfrac{1}{2}n), \qquad B = (\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}})^{\frac{3}{4}}/\Gamma(-\tfrac{1}{2}n).$$

Again, still discarding real factors,

$$\begin{aligned} AB^* &= \mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}/\{\Gamma(\tfrac{3}{4}-\tfrac{1}{4}\mathrm{i}ka^2)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{4}\mathrm{i}ka^2)\} \\ &\propto (1-\mathrm{i})\sin\{\pi(\tfrac{1}{4}+\tfrac{1}{4}\mathrm{i}ka^2)\} \\ &\propto (1-\mathrm{i})\{\cosh(\tfrac{1}{4}\pi ka^2) + \mathrm{i}\sinh(\tfrac{1}{4}\pi ka^2)\}, \end{aligned}$$

so

$$\mathrm{Re}(AB^*) = \exp(\tfrac{1}{4}\pi ka^2).$$

Hence

$$|R| = \frac{|\mathrm{Re}(AB^*)|}{|AB^*|} = \frac{\exp(\frac{1}{4}\pi ka^2)}{[2\{\cosh^2(\frac{1}{4}\pi ka^2)+\sinh^2(\frac{1}{4}\pi ka^2)\}]^{\frac{1}{2}}} = \frac{1}{\{1+\exp(-\pi ka^2)\}^{\frac{1}{2}}},$$

which is the standard result, again achieved only by half-penetration of the barrier.

(f) Symmetric Potential Barriers by WKBJ Method

Let the even function q be negative for $|z| < a$, constituting a potential barrier. For large k, the upgoing WKBJ solution for $z > a$ is

$$u = q^{-\frac{1}{4}}\exp\left(-\mathrm{i}k\int_a^z q^{\frac{1}{2}}\,\mathrm{d}s\right),$$

where $\arg q = 0$. The formal tracing of this solution around the transition point $z = a$ in a negative sense by the methods described by Heading (1974, 1975) gives a dominant solution within the barrier

$$u = \exp(\tfrac{1}{4}\pi\mathrm{i})\,|q|^{-\frac{1}{4}}\exp\left(-k\int_a^z |q|^{\frac{1}{2}}\,\mathrm{d}s\right).$$

We note that we have $uu^* \propto u^2$ within the barrier, so the results (2) give $|R| = 1$, with no correction term being produced. But the result (1) for T gives

$$T = (u^*u' - uu^{*\prime})/2uu' = \mathrm{i}\,\mathrm{Im}(u^*u')/uu'.$$

Now $\mathrm{Im}(u^*u') = $ const. for all real z, this being equivalent to energy conservation. Its value at $z = 0$ is equal to its value for $z > a$, where it equals $-k$. Hence

$$T = \left\{\mathrm{i}(-k)\Big/\mathrm{i}(-k)\exp\left(-2k\int_a^0 |q|^{\frac{1}{2}}\,\mathrm{d}s\right)\right\} = \exp\left(-k\int_{-a}^{a} |q|^{\frac{1}{2}}\,\mathrm{d}s\right).$$

Finally, the energy relation between $|R|$ and $|T|$ yields

$$|R| = (1 - |T|^2)^{\frac{1}{2}} \approx 1 - \tfrac{1}{2}\exp\left(-2k\int_{-a}^{a} |q|^{\frac{1}{2}}\,\mathrm{d}s\right),$$

which is a standard result, obtained only by dominant penetration from above through half the barrier.

(*g*) *Generalized Potential Barriers*

Consider the equation

$$\mathrm{d}^2w/\mathrm{d}z^2 + k^2(z^{2N-2} - a^N z^{N-2})w = 0,$$

where N is an integer greater than 2. When N is even ($N \geqslant 4$), the function q is symmetric in z, with a maximum at $z = 0$, and two minima on either side. When N is odd, q is not symmetric, possessing only one minimum and a point of inflection at $z = 0$ ($N \geqslant 5$). The investigations of Heading (1974) have produced the reflection coefficients in both cases. Here, in the present investigation, when N is odd we replace $q(z)$ by $q(-z)$ when $z \leqslant 0$ to achieve a symmetrical model. Then for all values of N we have a symmetric model, with a continuous derivative at $z = 0$ except when $N = 3$. In terms of the Whittaker function, the solution required is

$$u = z^{\frac{1}{2}-\frac{1}{2}N}\,\mathrm{W}_{\mathrm{i}L,1/2N}(2\,\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}kz^N/N)$$

$$\sim z^{\frac{1}{2}-\frac{1}{2}N}(2\,\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}kz^N/N)^{\mathrm{i}L}\exp(-\mathrm{i}kz^N/N)$$

when z is large and positive, where $L = a^N k/2N$.

Using formulae given by Whittaker and Watson (1927), and writing $m = 1/2N$ and $s = 2\,\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}kz^N/N$, we have (M being a modified Whittaker function)

$$u = z^{\frac{1}{2}-\frac{1}{2}N}\left(\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-\mathrm{i}L)}\mathrm{M}_{\mathrm{i}L,m}(s) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-\mathrm{i}L)}\mathrm{M}_{\mathrm{i}L,-m}(s)\right)$$

$$= z^{\frac{1}{2}-\frac{1}{2}N}\left(\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-\mathrm{i}L)}s^{\frac{1}{2}+m}\mathrm{e}^{-\frac{1}{2}s}(1+\ldots) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-\mathrm{i}L)}s^{\frac{1}{2}-m}\mathrm{e}^{-\frac{1}{2}s}(1+\ldots)\right)$$

$$= \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-\mathrm{i}L)}\left(\frac{2\,\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}k}{N}\right)^{\frac{1}{2}+m} z + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-\mathrm{i}L)}\left(\frac{2\,\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}k}{N}\right)^{\frac{1}{2}-m} + \ldots.$$

Hence, in using the results (4) and (5), we may take (apart from omitting real factors)

$$A = \exp\{\tfrac{1}{2}\pi \mathrm{i}(\tfrac{1}{2}-m)\}/\Gamma(\tfrac{1}{2}+m-\mathrm{i}L), \qquad B = \exp\{\tfrac{1}{2}\pi \mathrm{i}(\tfrac{1}{2}+m)\}/\Gamma(\tfrac{1}{2}-m-\mathrm{i}L),$$

which yields

$$R = \left|\mathrm{Re}\left(\frac{\exp(-\pi \mathrm{i}m)}{\Gamma(\tfrac{1}{2}+m-\mathrm{i}L)\,\Gamma(\tfrac{1}{2}-m+\mathrm{i}L)}\right)\right| \Big/ \left|\frac{1}{\Gamma(\tfrac{1}{2}+m-\mathrm{i}L)\,\Gamma(\tfrac{1}{2}-m+\mathrm{i}L)}\right|$$

$$= \frac{|\,\mathrm{Re}(\exp(-\pi \mathrm{i}m)\cos\{\pi(m-\mathrm{i}L)\})\,|}{|\cos\{\pi(m-\mathrm{i}L)\}\,|} = \frac{\exp(\pi L) + \exp(-\pi L)\cos(\pi/N)}{[2\{\cosh(2\pi L)+\cos(\pi/N)\}]^{\frac{1}{2}}},$$

as found by Heading (1974), but now including the symmetric barrier when N is odd. Again, this result has been obtained without the use of the asymptotic forms as $z \to -\infty$, as was used by Heading (1974) in a more restricted model.

We may also treat the related model governed by

$$\mathrm{d}^2w/\mathrm{d}z^2 + k^2(z^{2N-2} + b^N z^{N-2})w = 0;$$

with $b^N > 0$ and $N > 2$. If N is odd, let $q(-z) = q(z)$ for $z < 0$, to form a symmetric model. The profile q possesses a minimum at $z = 0$, but no other stationary points. We use the previous solution, with a^N replaced by $-b^N$ in L. Thus

$$|R| = \frac{\exp(-\pi L) + \exp(\pi L)\cos(\pi/N)}{[2\{\cosh(2\pi L)+\cos(\pi/N)\}]^{\frac{1}{2}}},$$

with $L = b^N k/2N$. When $b = 0$, this reduces to the model with the power-law profile considered in subsection (*d*) above.

(*h*) *Symmetrical Epstein Barrier*

The soluble Epstein profile given by

$$q = 1 + \frac{(\kappa_1^2-1)\mathrm{e}^{2\alpha z}}{1+\mathrm{e}^{2\alpha z}} - \frac{4\kappa\,\mathrm{e}^{2\alpha z}}{(1+\mathrm{e}^{2\alpha z})^2},$$

considered by Brekhovskikh (1960) and Budden (1961), and the soluble hyperbolic profile given by

$$q = 1 - (\kappa^2 + \kappa_2^2 \sinh \alpha z)/\cosh^2 \alpha z,$$

considered by Heading (1967), are symmetric when $\kappa_1 = 1$ and $\kappa_2 = 0$, the governing differential equation being

$$\mathrm{d}^2w/\mathrm{d}z^2 + k^2(1 - \kappa^2 \operatorname{sech}^2 \alpha z)w = 0.$$

We shall omit the symbol α since this is removed by a trivial change in the value of k. The required solution is

$$u = \mathrm{e}^{-\mathrm{i}kz}(1+\mathrm{e}^{-2z})^N\,{}_2\mathrm{F}_1(N+\mathrm{i}k, N; 1+\mathrm{i}k; -\mathrm{e}^{-2z}) \tag{9}$$

$$\to \mathrm{e}^{-\mathrm{i}kz} \quad \text{as} \quad z \to +\infty,$$

the hypergeometric function having the value unity when its argument vanishes. Here, N is one root of the quadratic equation

$$N^2-N+k^2\kappa^2 = 0.$$

The calculation of R and $|R|$ is now not so straightforward as in the previous examples because, as $z \to 0$, the argument of the hypergeometric function tends to -1, a point on its circle of convergence. To this end, we quote formulae from Erdélyi *et al.* (1953, pp. 104, 105):

$$\mathrm{F}(a,b;c;t) = (1-t)^{-b}\,\mathrm{F}\big(c-a,\, b;\, c;\, t/(t-1)\big), \tag{10}$$

$$\mathrm{F}(a,1-a;b;\tfrac{1}{2}) = 2^{1-b}\,\Gamma(b)\,\Gamma(\tfrac{1}{2})/\Gamma(\tfrac{1}{2}a+\tfrac{1}{2}b)\,\Gamma(\tfrac{1}{2}b-\tfrac{1}{2}a+\tfrac{1}{2}), \tag{11}$$

$$\mathrm{d}\{\mathrm{F}(a,b;c;z)\}/\mathrm{d}z = (ab/c)\,\mathrm{F}(a+1,b+1;c+1;z), \tag{12}$$

$$(c-b)\,\mathrm{F}(a,b-1;c;z) + \{2b-c+(a-b)z\}\,\mathrm{F}(a,b;c;z) + b(z-1)\,\mathrm{F}(a,b+1;c;z) = 0, \tag{13}$$

$$(c-a-1)\,\mathrm{F}(a,b;c;z) + a\,\mathrm{F}(a+1,b;c;z) - (c-1)\,\mathrm{F}(a,b;c-1;z) = 0. \tag{14}$$

In keeping with formula (10), the solution (9a) may be rewritten as

$$u = \mathrm{e}^{-ikz}\,\mathrm{F}\big(1-N,\, N;\, 1+\mathrm{i}k;\, \mathrm{e}^{-2z}/(\mathrm{e}^{-2z}+1)\big),$$

where the point $z = 0$ now lies within the circle of convergence. In fact, $\mathrm{e}^{-2z}/(\mathrm{e}^{-2z}+1)$ tends to zero as $z \to \infty$, and tends to $\tfrac{1}{2}(1-z) \to \tfrac{1}{2}$ as $z \to 0$. Hence from the formula (12) the first two terms of u as an expansion in terms of z are given by

$$\begin{aligned} u &= \Big(1-\mathrm{i}kz\Big)\Big(\mathrm{F}(1-N,\, N;\, 1+\mathrm{i}k;\, \tfrac{1}{2}) - \frac{z(1-N)N}{2(1+\mathrm{i}k)}\mathrm{F}(2-N,\, 1+N;\, 2+\mathrm{i}k;\, \tfrac{1}{2})\Big) \\ &= \mathrm{F}(1-N,\, N;\, 1+\mathrm{i}k;\, \tfrac{1}{2}) \\ &\quad + z\Big(-\mathrm{i}k\,\mathrm{F}(1-N,\, N;\, 1+\mathrm{i}k;\, \tfrac{1}{2}) - \frac{(1-N)N}{2(1+\mathrm{i}k)}\mathrm{F}(2-N,\, 1+N;\, 2+\mathrm{i}k;\, \tfrac{1}{2})\Big), \end{aligned}$$

so we may take $|R| = |\mathrm{Re}\,B|/|B|$, where

$$B = -\mathrm{i}k - \frac{(1-N)N}{2(1+\mathrm{i}k)}\,\frac{\mathrm{F}(2-N,\, 1+N;\, 2+\mathrm{i}k;\, \tfrac{1}{2})}{\mathrm{F}(1-N,\, N;\, 1+\mathrm{i}k;\, \tfrac{1}{2})}. \tag{15}$$

The formula (11) gives the value of the denominator, but not that of the numerator. For this, we must derive a new formula not given by Erdélyi *et al.* (1953); to this end we have quoted the formulae (13) and (14).

In (13) substitute $a = 1+N$, $b = 1-N$, $c = 2+\mathrm{i}k$ and $z = \tfrac{1}{2}$, giving

$$\tfrac{1}{2}(1-N)\,\mathrm{F}(1+N,\, 2-N;\, 2+\mathrm{i}k;\, \tfrac{1}{2}) = (1+N+\mathrm{i}k)\,\mathrm{F}(1+N,\, -N;\, 2+\mathrm{i}k;\, \tfrac{1}{2}) - (N+\mathrm{i}k)\,\mathrm{F}(1+N,\, 1-N;\, 2+\mathrm{i}k;\, \tfrac{1}{2}). \tag{16}$$

Similarly in (14) substitute $a = -N$, $b = 1+N$, $c = 2+ik$ and $z = \frac{1}{2}$, giving

$$-N\,\mathrm{F}(1-N,\,1+N;\,2+\mathrm{i}k;\,\tfrac{1}{2})$$
$$= -(1+\mathrm{i}k+N)\,\mathrm{F}(-N,\,1+N;\,2+\mathrm{i}k;\,\tfrac{1}{2}) + (1+\mathrm{i}k)\,\mathrm{F}(-N,\,1+N;\,1+\mathrm{i}k;\,\tfrac{1}{2}).$$

Then equation (16) yields after reduction

$$\tfrac{1}{2}(1-N)\,\mathrm{F}(1+N,\,2-N;\,2+\mathrm{i}k;\,\tfrac{1}{2}) = -(\mathrm{i}k/N)(1+N+\mathrm{i}k)\,\mathrm{F}(1+N,\,-N;\,2+\mathrm{i}k;\,\tfrac{1}{2})$$
$$+\{(N+\mathrm{i}k)(1+\mathrm{i}k)/N\}\,\mathrm{F}(-N,\,1+N;\,1+\mathrm{i}k;\,\tfrac{1}{2}).$$

This enables us to write B given by equation (15) as

$$B = -\mathrm{i}k + \frac{\mathrm{i}k(1+N+\mathrm{i}k)}{1+\mathrm{i}k}\frac{\mathrm{F}(1+N,\,-N;\,2+\mathrm{i}k;\,\frac{1}{2})}{\mathrm{F}(1-N,\,N;\,1+\mathrm{i}k;\,\frac{1}{2})} - (N+\mathrm{i}k)\frac{\mathrm{F}(-N,\,1+N;\,1+\mathrm{i}k;\,\frac{1}{2})}{\mathrm{F}(1-N,\,N;\,1+\mathrm{i}k;\,\frac{1}{2})}, \tag{17}$$

all of which can be evaluated by the formula (11) (the first two elements in every hypergeometric function add up to unity). This, of course, was the object of the reduction using equations (13) and (14).

Upon substitution of the formula (11) into equation (17), the first two terms vanish, yielding simply

$$B = -(N+\mathrm{i}k)\frac{\Gamma(1-\frac{1}{2}N+\frac{1}{2}\mathrm{i}k)\,\Gamma(\frac{1}{2}+\frac{1}{2}N+\frac{1}{2}\mathrm{i}k)}{\Gamma(\frac{1}{2}-\frac{1}{2}N+\frac{1}{2}\mathrm{i}k)\,\Gamma(1+\frac{1}{2}N+\frac{1}{2}\mathrm{i}k)}$$

$$= -\frac{2\,\Gamma(1-\frac{1}{2}N+\frac{1}{2}\mathrm{i}k)\,\Gamma(\frac{1}{2}+\frac{1}{2}N+\frac{1}{2}\mathrm{i}k)}{\Gamma(\frac{1}{2}-\frac{1}{2}N+\frac{1}{2}\mathrm{i}k)\,\Gamma(\frac{1}{2}N+\frac{1}{2}\mathrm{i}k)}.$$

Now we have $N = \frac{1}{2}+(\frac{1}{4}-k^2\kappa^2)^{\frac{1}{2}}$. If this is real, write $N = \frac{1}{2}+M$ but, if complex, write $N = \frac{1}{2}+\mathrm{i}P$. Moreover, if B is written as X/Y (with the factor -2 omitted), the formula (6) may be used. Still discarding all real factors when they occur, we have when $N = \frac{1}{2}+M$

$$XY^* \propto \Gamma(1-\tfrac{1}{2}N+\tfrac{1}{2}\mathrm{i}k)\,\Gamma(\tfrac{1}{2}N-\tfrac{1}{2}\mathrm{i}k)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}N+\tfrac{1}{2}\mathrm{i}k)\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}N-\tfrac{1}{2}\mathrm{i}k)$$
$$\propto [\sin\{\tfrac{1}{2}\pi(N-\mathrm{i}k)\}\sin\{\tfrac{1}{2}\pi(1+N+\mathrm{i}k)\}]^{-1}$$
$$\propto \{\sin\pi N - \sin\pi\mathrm{i}k\}^{-1} \propto \sin\pi N + \mathrm{i}\sinh\pi k.$$

Hence

$$|R| = \frac{|\sin\pi N|}{(\sin^2\pi N+\sinh^2\pi k)^{\frac{1}{2}}} = \left(\frac{1+\cos 2\pi M}{\cosh 2\pi k+\cos 2\pi M}\right)^{\frac{1}{2}}. \tag{18a}$$

Similarly, when $N = \frac{1}{2}+\mathrm{i}P$

$$XY^* \propto \Gamma(\tfrac{3}{4}-\tfrac{1}{2}\mathrm{i}P+\tfrac{1}{2}\mathrm{i}k)\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mathrm{i}P+\tfrac{1}{2}\mathrm{i}k)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mathrm{i}P-\tfrac{1}{2}\mathrm{i}k)\,\Gamma(\tfrac{1}{4}-\tfrac{1}{2}\mathrm{i}P-\tfrac{1}{2}\mathrm{i}k)$$
$$\propto [\sin\{\tfrac{1}{2}\pi(\tfrac{1}{2}+\mathrm{i}P-\mathrm{i}k)\}\sin\{\tfrac{1}{2}\pi(\tfrac{3}{2}+\mathrm{i}P+\mathrm{i}k)\}]^{-1} \propto \cosh\pi P + \mathrm{i}\sinh\pi k,$$

yielding

$$|R| = \frac{\cosh \pi P}{(\cosh^2\pi P + \sinh^2\pi k)^{\frac{1}{2}}} = \left(\frac{1+\cosh 2\pi P}{\cosh 2\pi k + \cosh 2\pi P}\right)^{\frac{1}{2}}. \tag{18b}$$

The results (18) are the usual values, normally calculated from the solution (9) by using the circuit relations for the analytic continuation of the hypergeometric function when $z \to -\infty$. The present calculation is longer than the normal one, though it illustrates an interesting analytic calculation, a feature brought about by the necessity of knowing the values of the various hypergeometric functions at the value $\frac{1}{2}$ of their arguments.

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