Parametric Instabilities in a Magnetized and Collisional Plasma

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Abstract
The dispersion relation for a magnetized, collisional and hot plasma in the presence of a pump wave is developed for the case where the pump frequency $\omega_p$ is large compared with the cyclotron frequency $\omega_c$ and the plasma frequency $\omega_p$. Formulae for the growth rate, the damping rate for the free electron plasma wave and the threshold power are derived and discussed numerically under different conditions. It is found that in a hot plasma (for magnetic fields with $\omega_c/\omega_p = 1$ and 10) the threshold power $P_T$ is less than or greater than that in a cold plasma for the $(\text{Re}\omega_2)_+$ or $(\text{Re}\omega_2)_-$ modes respectively. In a weak magnetic field $(\omega_c/\omega_p = 0.1)$, $P_T$ does not vary with the direction $\theta$ of the magnetic field for the $(\text{Re}\omega_2)_+$ mode. However, $P_T$ for the $(\text{Re}\omega_2)_-$ mode is a minimum at $\theta = 30^\circ$ and $10^\circ$ for $\omega_c/\omega_p = 1$ and 10 respectively, and it becomes very large $(10^4-10^7$ times its value in a cold unmagnetized plasma) for $\omega_c/\omega_p = 0.1$. The results for the growth rate are found to be just the reverse of those for the threshold power.

Introduction
The study of the parametric decay of an intense electromagnetic (pump) wave into a scattered electromagnetic wave and an electron plasma wave is of much current interest because of the relevance to the problems of heating a magnetically confined plasma and of providing an explanation for anomalous absorption and scattering of radio waves in the ionosphere and guidance for ionospheric modification experiments. Recently various aspects of an electron plasma wave (stimulated Raman scattering) have been investigated by Lee (1974), Forslund et al. (1975), Bodner and Eddleman (1972), Drake et al. (1974), Koch and Albritton (1975), Silin and Starodub (1975), Fuchs (1976), Sodha et al. (1976), Willett and Maraghechi (1978) and many others. In particular, Willett and Maraghechi analysed the problem by neglecting the pressure tensor and collision terms in the electron momentum equation.

In the present work we extend the analysis of Willett and Maraghechi (1978) by including the pressure tensor and collision terms. On solving Maxwell's equations with the electron continuity and momentum equations in a straightforward manner, we obtain the dispersion relation and formulae for the growth rate $\alpha$, the damping rate $\alpha$, for the free electron plasma wave and the threshold power $P_T$. The resulting variations of $\alpha$, $\alpha$ and $P_T$ as functions of the plasma thermal velocity $v$, cyclotron frequency $\omega_c$ and direction $\theta$ of the magnetic field with respect to the electromagnetic wave are shown graphically.
Basic Assumptions and Dispersion Relation

We consider a large amplitude plane polarized electromagnetic pump wave \((\omega_0, k_0)\) propagating in an infinite collisional and hot plasma that is embedded in a uniform static magnetic field \(B_0 = B_0 \hat{b}_0\). To simplify matters, we neglect the ion motion at the outset since it has negligible influence on the high frequency waves under consideration. The resulting parametric decay may be described as a three-wave process involving the pump wave \((\omega_0, k_0)\), a backscattered electromagnetic wave \((\omega_1, k_1)\) and an electron plasma wave \((\omega_2, k_2)\). These waves satisfy the phase matching conditions

\[
\omega_1 = \omega_2 - \omega_0, \quad k_1 = k_2 - k_0,
\]

where \(\Re \omega_1\) is negative and \(\Re \omega_2, \omega_0, k_1, k_2\) and \(k_0\) are positive. The cyclotron frequency \(\omega_c\), plasma frequency \(\omega_p\) and electron plasma frequency \(\Re \omega_2\) are taken to be small compared with the pump frequency \(\omega_0\). To avoid relativistic effects, the amplitude of the electron velocity \(v_0\) is also assumed to be very small in comparison with the speed of light \(c\).

For the parametric decay, the electron number density \(n\), electron gas velocity \(v\) and electric and magnetic fields \(E\) and \(B\) are taken as

\[
n = n^0 + n'_2, \quad v = v^0 + v'_1 + v'_2, \quad E = E^0 + E'_1 + E'_2, \quad B = B^0 + B'_1 + B'_2,
\]

where

\[
A'_{1,2} = A'_{1,2} \exp \{i(k_{1,2} \cdot r - \omega_{1,2} t)\} + \text{c.c.},
\]

\[
v^0 = i v_0 \hat{e}_0 \exp \{i(k_0 \cdot r - \omega_0 t)\} + \text{c.c.},
\]

\[
E^0 = E_0 \hat{e}_0 \exp \{i(k_0 \cdot r - \omega_0 t)\} + \text{c.c.},
\]

\[
B^0 = B_0 \hat{b}_0 + [(m/e)ck_0 v_0 \hat{k}_0 \times \hat{e}_0 \exp \{i(k_0 \cdot r - \omega_0 t)\}] + \text{c.c.}.
\]

Here \(A\) denotes \(n, v, E\) or \(B\), the superscript 0 and the prime denote unperturbed and perturbed states respectively, and the subscripts 1 and 2 indicate the perturbed quantities associated with the backscattered wave and the electron plasma wave.

Solving the linearized equations of continuity and momentum and the Maxwell equations, using the relations (1), (2) and (3), we obtain after some straightforward algebra:

\[
v_0 = eE_0/m\omega_0, \quad \omega_0^2 = \omega_p^2 + c^2 k_0^2, \quad (\Re \omega_1)^2 = \omega_p^2 + c^2 k_1^2,
\]

\[
1 - \frac{\omega_0^2}{\omega_2} \left(1 - \frac{1}{\omega_2} \right) + \left( \frac{c^2 k_0^2 \omega_p^2 \beta}{\omega_2^2 (k_0^2 - k_2^2 c^2)} - \frac{v_0^2 k_0^2 \beta}{\omega_2^2} \right) \left(1 - \frac{i}{\omega_2} \right) \left(1 - \frac{1}{\omega_2} \right) \left(1 - \frac{1}{\omega_2} \right)
\]

\[
= - \frac{\omega_0^2 v_0^2 k_0^2 \beta}{\omega_2^2 Q_1} \left(1 - \frac{i}{\omega_2} \right) \left(1 - \frac{1}{\omega_2} \right) \left(1 - \frac{1}{\omega_2} \right),
\]
where
\[ \beta = (k_2^2 c^2 - \omega_2^2)/(k_2^2 c^2 - \omega_p^2), \quad Q_1 = k_1^2 c^2 - \omega_1^2 + \omega_p^2, \]
\[ v_t^2 = \kappa T/m, \quad \omega_e = eB_0/mc, \quad \omega_2^2 = 4\pi n_0 e^2/m, \quad \tilde{k}_0, \tilde{b}_0 = \cos \theta, \]
and \( v, \kappa \) and \( T \) are the electron collision frequency, Boltzmann constant and absolute temperature respectively. In deriving equation (6) we have neglected the nonresonant anti-Stokes wave \( (\omega_2 + \omega_0, k_2 + k_0) \).

Under the restriction that \( \text{Re} \omega_2, \omega_p \) and \( \omega_e \) are all very much less than \( 2\omega_0 \), and for small \( v \), equation (6) reduces to

\[
1 - \frac{\omega_0^2}{\omega_2^2} (1 - \frac{2iv}{\omega_2}) - \left( \frac{\omega_0^2}{\omega_2^2} + \frac{v_t^2 \omega_2^2}{\omega_2^2} \right) \left( 1 - \frac{iv}{\omega_2} - \frac{\omega_2^2 \cos^2 \theta}{\omega_2^2} + \frac{3iv \omega_2^2 \cos^2 \theta}{\omega_2^3} \right)
= - \frac{\omega_0^2 v_t^2 \omega_2^2}{\omega_2^2 Q_1} \left( 1 - \frac{iv}{\omega_2} - \frac{\omega_2^2 \cos^2 \theta}{\omega_2^2} + \frac{3iv \omega_2^2 \cos^2 \theta}{\omega_2^3} \right). \tag{7}
\]

This is the dispersion relation for the magnetized, collisional and hot plasma in the presence of a large amplitude pump wave. It describes the parametric excitation of a low frequency electron plasma wave \( (\omega_2, k_2) \) and a backscattered electromagnetic wave \( (\omega_1, k_1) \).

**Calculation of \( \alpha_2, \alpha \) and \( P_T \)**

To calculate the damping rate \( \alpha_2 \) for a free electron plasma wave \( (\alpha_2 = -\alpha) \) and the growth rate \( \alpha \) and threshold power \( P_T \) for stimulated Raman backscattering, we assume
\[
\omega_1 = \text{Re} \omega_1 + i\alpha, \quad \omega_2 = \text{Re} \omega_2 + i\alpha. \tag{8}
\]

Under the restriction of weak coupling \( (\nu_0 \to 0) \), equation (7) leads to the following equations for \( \omega_2 \) and \( \alpha_2 \):
\[
(1 - v_t^2 c^{-2})(\text{Re} \omega_2)^6 - (\omega_0^2 + \omega_2^2 - v_t^2 c^{-2} \omega_2^2 \cos^2 \theta + 5v_0 v_t^2 c^{-2})(\text{Re} \omega_2)^4
+ \{\omega_2^2 \omega_0^2 \cos^2 \theta - 3v_0(2\omega_0^2 + \omega_p^2 - 3v_t^2 c^{-2} \omega_2^2 \cos^2 \theta) \}(\text{Re} \omega_2)^2 + 3v_0 \omega_p^2 \omega_2^2 \cos^2 \theta = 0, \tag{9a}
\]
\[
\alpha_2 = \frac{1}{2v_0} \left( \frac{v_t^2 c^{-2}(\text{Re} \omega_2)^4 + (2\omega_0^2 + \omega_p^2 - 3v_t^2 c^{-2} \omega_2^2 \cos^2 \theta)(\text{Re} \omega_2)^2 - 3\omega_p^2 \omega_2^2 \cos^2 \theta}{3(1 - v_t^2 c^{-2})(\text{Re} \omega_2)^4 - 2(\omega_0^2 + \omega_p^2 - v_t^2 c^{-2} \omega_2^2 \cos^2 \theta)(\text{Re} \omega_2)^2 + \omega_p^2 \omega_2^2 \cos^2 \theta} \right). \tag{9b}
\]

Inserting equations (8) into the dispersion relation (7) and applying the approximation \( \alpha \ll \text{Re} \omega_2 \ll \text{Re} \omega_1 \), we obtain
\[
\alpha^2 + v_0(P + Q) = R, \tag{10a}
\]
where
\[
P = \frac{3 - 2v_t^2 c^{-2} - (\omega_p^2 + 2\omega_0^2)(\text{Re} \omega_2)^{-2}}{2\{1 - v_t^2 c^{-2} - \omega_p^2 \omega_2^2 \cos^2 \theta(\text{Re} \omega_2)^{-4}\}}, \tag{10b}
\]
\[ \begin{align*}
Q &= \frac{21 v_0^4}{4c^2(\text{Re} \omega_2)(\omega_0 - \text{Re} \omega_2)} \left(1 - v_t^2 c^{-2} \right) (\text{Re} \omega_2)^2 - \omega_c^2 (1 - v_t^2 c^{-2} \cos^2 \theta) \left(1 - v_t^2 c^{-2} \right) - 2v_0^2 \omega_c^2 \cos^2 \theta (\text{Re} \omega_2)^{-4}, \quad (10c) \\
R &= \frac{1}{\text{Re} \omega_2^2} (10d) \\
\text{The growth rate for instability above the threshold is given by the positive root of equation (10a) while the growth rate just above the threshold is} \\
\alpha \approx (R/vP)(1 - Q/P). \quad (11) \\
\text{Substituting } \alpha = 0 \text{ in equation (11), we obtain the minimum threshold power } P_T \text{ (} \alpha v_0^2), \text{ in terms of its value in a cold unmagnetized plasma, as} \\
P_T = \frac{\omega_p \text{Re} \omega_2 (\omega_0 - \text{Re} \omega_2)}{\omega_0 - \omega_p} \left(3 - 2v_t^2 c^{-2} - \omega_c^2 + 2\omega_c^2 \text{Re} \omega_2 \right) - 2 \frac{(1 - v_t^2 c^{-2} \text{Re} \omega_2)^2}{(1 - v_t^2 c^{-2} \text{Re} \omega_2)^2 - \omega_c^2 (1 - v_t^2 c^{-2} \cos^2 \theta)} \quad (12) \\
\end{align*} \]

Fig. 1. Variation of the damping rate \( \alpha \) of the free electron plasma wave as a function of the direction \( \theta \) of the magnetic field for (a) the \((\text{Re} \omega_2)_+ \) mode and (b) the \((\text{Re} \omega_2)_- \) mode. The curves are for the parameter values \( v/v_0 = 10^{-3}, \omega_0/v_0 = 10^2, \omega_c/v_0 = 10^{-2} \) and the indicated values of \( \omega_c/v_0 \), with \( (v_t/c)^2 = 0 \) and 0.2 for the full and dashed curves respectively in each case.

Results and Discussion

At the beat frequencies \( \omega_0 - \omega_1 \) and \( \omega_0 - \omega_2 \) the electron plasma wave and the backscattered wave are resonantly excited and the energy is added to these modes (at the expense of the pump wave) at the resonance if their natural damping rates are small. The nonresonant modes are not important in the backscattering process.
Fig. 2. Variation of the growth rate $\omega$ as a function of $\theta$ for (a) the (Re $\omega_2$)$_+ \text{ mode}$ and (b) the (Re $\omega_2$)$_- \text{ mode}$. The curves are for $\omega_0/\omega_p = 10^2$, $\omega_0/\omega_p = 10^2$, $v_0/c = 10^{-2}$ and the indicated values of $\omega_2/\omega_p$, with $(v_i/c)^2 = 0$ and 0.2 for the full and dashed curves respectively in each case.

Fig. 3. Variation of the threshold power $P_T$ as a function of $\theta$ for (a) the (Re $\omega_2$)$_+ \text{ mode}$ and (b) the (Re $\omega_2$)$_- \text{ mode}$. The curves are for $\omega_0/\omega_p = 10^2$ and the indicated values of $\omega_2/\omega_p$, with $(v_i/c)^2 = 0$ and 0.2 for the full and dashed curves respectively in each case.
The mode frequencies, which are independent of the wave number \( k_z \) and the collision frequency \( v \), are determined by the two roots of equation (9a) (at \( v = 0 \)), namely

\[
(\text{Re} \omega)^2 = \left[ (\omega_c^2 + \omega_p^2 - v_i^2 c^{-2} \omega_c^2 \cos^2 \theta) \pm \sqrt{(\omega_c^2 + \omega_p^2 - v_i^2 c^{-2} \omega_c^2 \cos^2 \theta)^2 - 4(1 - v_i^2 c^{-2}) \omega_p^2 \omega_c^2 \cos^2 \theta} \right] / 2(1 - v_i^2 c^{-2}).
\]  

(13)

The calculated results for the two modes (\( \text{Re} \omega \)) and (\( \text{Re} \omega \)) are plotted in Figs 1, 2 and 3, which show the variation with \( \theta \) of \( \alpha_2 \), \( \alpha \) and \( P_T \) (in terms of their values for Langmuir mode excitation in an unmagnetized cold plasma) at parameter values of \( \omega_c/\omega_p = 0.1, 1 \) and 10 and \( v_i^2/c^2 = 0 \) and 0.2, with \( v_0/c = 10^{-2} \), \( v/\omega_p = 10^{-3} \) and \( \omega_0/\omega_p = 10^2 \).

From Fig. 1 it can be seen that, for the (\( \text{Re} \omega \)) mode (Fig. 1a), the damping rate \( \alpha_2 \) is very weakly dependent on \( \theta \) in a cold plasma and increases slightly with \( \theta \) as the plasma temperature is raised; whereas, for the (\( \text{Re} \omega \)) mode (Fig. 1b), \( \alpha_2 \) is independent of \( \theta \) for \( \omega_c/\omega_p = 0.1 \) but an increase in plasma temperature and magnetic field \( (\omega_c/\omega_p) \) results in a marked decrease of \( \alpha_2 \) with \( \theta \).

Fig. 2a shows that the growth rate \( \alpha \) for the (\( \text{Re} \omega \)) mode is independent of \( \theta \) in a weak magnetic field \( (\omega_c/\omega_p = 0.1) \) but increases with \( \theta \) as the magnetic field is strengthened. At small angles the value of \( \alpha \) for \( \omega_c/\omega_p = 10 \) is less than for \( \omega_c/\omega_p = 0.1 \) and 1 but, after about \( \theta = 26^\circ \), \( \alpha \) is greatest for the strongest magnetic field. An increase in plasma temperature results in an increase in \( \alpha \) for all magnetic field strengths at all angles. In contrast, for the (\( \text{Re} \omega \)) mode (Fig. 2b), the growth rate is negligible for \( \omega_c/\omega_p = 0.1 \) and a maximum at \( 10^\circ \) for \( \omega_c/\omega_p = 1 \) and 10. An increase in the plasma temperature results in an increase in \( \alpha \) for the strongest magnetic field but in a large decrease in \( \alpha \) for \( \omega_c/\omega_p = 1 \) at small angles, which becomes less marked with increasing \( \theta \) up to about \( 70^\circ \), when the growth rate becomes independent of \( \theta \).

From Fig. 3a it can be seen that, for the (\( \text{Re} \omega \)) mode, the threshold power \( P_T \) is high for a strong magnetic field, and an increase in the plasma temperature reduces \( P_T \) for \( \omega_c \geq \omega_p \) but has the reverse effect for \( \omega_c < \omega_p \). In the case of the (\( \text{Re} \omega \)) mode (Fig. 3b), for \( \omega_c/\omega_p = 0.1 \), \( P_T \) is found to be \( 10^5 - 10^7 \) times its value in an unmagnetized cold plasma (not shown in the figure) while, for \( \omega_c/\omega_p = 1 \) and 10 (in a hot plasma), \( P_T \) is a minimum at angles of \( 30^\circ \) and \( 10^\circ \) respectively. At small angles an increase in the plasma temperature results in an increase in \( P_T \) up to about \( \theta = 80^\circ \). Beyond this angle the plasma temperature has no effect.

The formulae derived here for the dispersion relation and the threshold and growth rate of the (\( \text{Re} \omega \)) modes for backscattering, when evaluated at \( v = 0 \) and \( v_i = 0 \), agree with the results obtained by Willett and Maraghechi (1978) and, at \( \theta = 90^\circ \), also reduce to those obtained by Lee (1974).

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References


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