Aust. J. Phys., 1981, 34, 563–73

The Transformation Properties of the Energy–Momentum Tensor for Dispersive Waves

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Abstract

A covariant formulation of wave dispersion is extended to include dissipation and hence to identify the energy–momentum tensor for dispersive waves. The resulting form is equivalent to that obtained from an averaged-Lagrangian approach. The transformation properties of the wave energy–momentum tensor are discussed. The appearance of unacceptable negative wave energies in frames moving at greater than the phase speed of the waves may be avoided by choosing a physically equivalent second branch of the solution of the dispersion equation in such frames. A quantum mechanical description of dispersive waves as a collection of wave quanta with energy $\hbar\omega$ and momentum $\hbar\boldsymbol{k}$ is compatible with the form of the energy–momentum tensor discussed here. The quantum description also involves the occupation number which plays the role of the distribution function of the wave quanta. The classical counterpart of the occupation number is defined, shown to be an invariant and shown to obey an equation which may be interpreted directly as a transfer equation for a distribution of wave quanta.

1. Introduction

Using a Lagrangian approach, Dewar (1977) has identified the canonical energy–momentum tensor for waves in a dispersive medium. In this paper, I present an alternative derivation of this energy–momentum tensor, discuss its transformation properties and point out some implications for a quantum mechanical description of the waves. The alternative derivation is based on the inclusion of dissipation. Dissipation may be included in two complementary ways. First, it may be a source term. One calculates the rate 4-momentum is transferred to the waves by this source term, and identifies the wave 4-momentum as that wave quantity which is so generated. The other way is to include dissipation as an imaginary part of the dispersion equation. This allows one to identify the continuity equation for 4-momentum directly.

The transformation properties of Dewar's energy–momentum tensor are not obvious in that its normalization is not clear. Here it is identified that the normalization is per unit volume of $\boldsymbol{k}$ space. Specifically, the quantity which transforms as a 4-tensor is Dewar's 'tensor' times $d\boldsymbol{k}/(2\pi)^3$.

My motivation for developing a covariant theory of dispersive waves is in connection with synthesizing quantum electrodynamics and the kinetic theory of plasmas. In such a generalized theory it is essential that the waves be describable as a collection of wave quanta with energy $\hbar\omega$, momentum $\hbar\boldsymbol{k}$ and with a distribution function of the form of a photon occupation number. It is shown here that the classical counterpart of the occupation number, namely the time-component of the wave action, is an invariant. This is required of any acceptable distribution function;

see for example Landau and Lifshitz (1975). It is also shown that this classical quantity satisfies a transfer equation of the appropriate form.

The question of the correct form of the electromagnetic energy–momentum tensor in a material medium is of historical interest through the Abraham–Minkowski controversy (see e.g. Pauli 1958). It is also of current interest, see for example the references cited by Maugin (1980), due in part to the inclusion of electromechanical and other effects and in part to renewed interest in the momentum of waves in a medium (Peierls 1976, 1977; Jones 1978). The discussion in the present paper is not directly relevant to the mainstream of this controversy because the analysis here is developed in momentum space for dispersive waves. However, the canonical energy–momentum tensor implied by such a theory has been shown by Dewar (1977) to reduce to the Minkowski form for nondispersive waves. I comment further on these points in Section 7 below.

2. Covariant Theory of Wave Dispersion

The following is a summary of the covariant formulation of wave dispersion presented by Melrose (1973). The tensor notation is that used by Melrose (1973) and by Dewar (1977).

The response of the medium is described by the linear response tensor $\alpha^{\mu\nu}(k)$ which relates the induced current $J^{\mu}(k)$ to the 4-potential $A^{\mu}(k)$:

$$J^{\mu}(k) = \alpha^{\mu}{}_{\nu}(k)\, A^{\nu}(k)\,. \tag{1}$$

By construction we have

$$k_{\mu}\, \alpha^{\mu}{}_{\nu}(k) = 0\,, \qquad k^{\nu}\, \alpha^{\mu}{}_{\nu}(k) = 0\,. \tag{2a, b}$$

The nondissipative and dissipative parts are the hermitian (H) and antihermitian (A) parts respectively. We are concerned with the wave equation in the form

$$\Lambda^{\mu\nu}(k)\, A_{\nu}(k) = -\mu_0\, c\, J^{\mu}_{\mathrm{ext}}(k)\,, \tag{3}$$

where

$$\Lambda^{\mu\nu}(k) := k^2 g^{\mu\nu} - k^{\mu}k^{\nu} + \mu_0\, c\, \alpha^{\mu\nu(\mathrm{H})}(k) \tag{4}$$

(here the symbols $:=$ and $=:$ define quantities on the left and right respectively) is hermitian by definition, and the extraneous current is identified with the dissipative part of the response:

$$J^{\mu}_{\mathrm{ext}}(k) = \mu_0\, c\, \alpha^{\mu\nu(\mathrm{A})}(k)\, A_{\nu}(k)\,. \tag{5}$$

In the absence of dissipation, particular solutions of the homogeneous wave equation

$$\Lambda^{\mu\nu}(k)\, A_{\nu}(k) = 0 \tag{6}$$

describe waves in a particular wave mode. Equation (6) may be regarded as a set of four coupled linear equations for the four components of $A^{\mu}(k)$. The condition for a solution to exist is that the determinant of the coefficients vanishes. However, the determinant vanishes identically because equations (2b) and (4) imply that $A_{\nu}(k) \propto k_{\nu}$ is always a solution of (6). A non-trivial solution is obtained by setting the matrix of cofactors $\lambda^{\mu}{}_{\nu}(k)$ equal to zero. The matrix terminology used here is summarized in Appendix 1. The vanishing of the determinant implies that $\lambda^{\mu}{}_{\nu}(k)$ is a matrix of rank one, and hence may be written as the outer product of a vector with itself. The relevant vector is the solution of the original equation, i.e. k_ν. Hence we have

$$\lambda^{\mu}{}_{\nu}(k) =: k^\mu k_\nu \lambda(k), \tag{7}$$

which defines $\lambda(k)$. The dispersion equation in covariant form may be identified as

$$\lambda(k) = 0. \tag{8}$$

Any particular solution

$$\omega = \omega_M(\boldsymbol{k}) = -\omega_M(-\boldsymbol{k}) \tag{9}$$

of equation (8) is said to be the dispersion relation for waves in the mode M. The 4-vector constructed from $\omega_M(\boldsymbol{k})$ and $\boldsymbol{k}$ is denoted k^μ_M. The relation (9) of the negative-frequency to the positive-frequency solution is implied by $\lambda(k) = \lambda(-k)$, which in turn follows from the reality condition for Fourier transforms and the fact that $\lambda(k)$ is real.

A solution of (6) can now be obtained to within an arbitrary choice of gauge, normalization and phase. The basic result used to derive the solution is that the matrix of cofactors of the 2×2 minors, denoted $\lambda^{\mu\nu\alpha\beta}(k_M)$, depends only on the two solutions of the wave equation, i.e. the trivial solution k^ν_M and the desired solution $A^\nu_M(k_M)$. The symmetry properties of $\lambda^{\mu\nu\alpha\beta}(k)$ (cf. Appendix 1) then imply

$$\lambda^{\mu\nu\alpha\beta}(k_M) \propto \{k^\mu_M A^\nu_M(k_M) - k^\nu_M A^\mu_M(k_M)\}\{k^\alpha_M A^{*\beta}_M(k_M) - k^\beta_M A^{*\alpha}_M(k_M)\}. \tag{10}$$

It is necessary to make a specific choice of gauge and of normalization to determine the constant of proportionality in (10). A convenient choice is the temporal gauge ($A^0 := 0$), in which case the solution may be written as the polarization 4-vector

$$e^\mu_M(\boldsymbol{k}) \equiv [0, \boldsymbol{e}_M(\boldsymbol{k})], \tag{11}$$

and normalized according to

$$e^\mu_M(\boldsymbol{k})\, e_{M\mu}(\boldsymbol{k}) = -1, \qquad \text{i.e.} \quad \boldsymbol{e}_M(\boldsymbol{k}) . \boldsymbol{e}^*_M(\boldsymbol{k}) = 1, \tag{12}$$

and then the constant of proportionality in (10) is such that we have

$$\begin{aligned}\lambda^{\mu\nu\alpha\beta}(k_M) = -\{c/\omega_M(\boldsymbol{k})\}^2 \lambda^{0\tau}{}_{0\tau}(k_M)\{k^\mu_M e^\nu_M(\boldsymbol{k}) - k^\nu_M e^\mu_M(\boldsymbol{k})\} \\ \times \{k^\alpha_M e^{*\beta}_M(\boldsymbol{k}) - k^\beta_M e^{*\alpha}_M(\boldsymbol{k})\}.\end{aligned} \tag{13}$$

It is convenient to introduce the quantity

$$R_M(\boldsymbol{k}) := \left(\frac{-\lambda^{0\tau}{}_{0\tau}(k)}{\omega\, \partial\lambda(k)/\partial\omega}\right)_{\omega=\omega_M(\boldsymbol{k})}, \tag{14}$$

which may be interpreted as the ratio of electric to total energy in the waves, as is shown below.

Apart from the use of a covariant notation, the foregoing theory is equivalent to the conventional theory of waves in plasmas, for example as presented by Melrose (1980, Ch. 2). That is, for a given medium the expressions found for $\omega_M(\boldsymbol{k})$, $\boldsymbol{e}_M(\boldsymbol{k})$ and $R_M(\boldsymbol{k})$ are identical whether the covariant theory or the noncovariant theory is used.

In the following it is necessary to adopt a specific definition for the amplitude of the waves. It is convenient to use

$$A_M^\mu(k_M) =: a_M(\boldsymbol{k}) \exp\{\mathrm{i}\,\psi_M(\boldsymbol{k})\}\, e_M^\mu(\boldsymbol{k})\, 2\pi c\, \delta(\omega-\omega_M(\boldsymbol{k}))\,, \tag{15}$$

which defines the amplitude $a_M(\boldsymbol{k})$ and the phase $\psi_M(\boldsymbol{k})$, both of which are real. The phase of $e_M^\mu(\boldsymbol{k})$ is as yet unspecified, and one is free to adopt any convenient convention. The relative phases of the components of $e_M^\mu(\boldsymbol{k})$ are completely determined by the Onsager relations once a specific choice of gauge has been made. It is implicit that the δ function in equation (15) includes both the positive- and negative-frequency solutions (9). Because of this, the electric energy $W_M^{[\mathrm{E}]}(\boldsymbol{k})$ in the waves in the range $\mathrm{d}\boldsymbol{k}/(2\pi)^3$ is related to the amplitude $a_M(\boldsymbol{k})$ by (see Appendix 2)

$$W_M^{[\mathrm{E}]}(\boldsymbol{k}) = \varepsilon_0[\{\omega_M(\boldsymbol{k})/c\}a_M(\boldsymbol{k})]^2\,. \tag{16}$$

3. Wave Dissipation

In this section dissipation of the waves is treated by including the extraneous current (5) associated with dissipative processes as a source term in the wave equation.

Let us introduce the 4-momentum $P_M^\mu(\boldsymbol{k})$ in waves in the range $\mathrm{d}\boldsymbol{k}/(2\pi)^3$ by

$$P_M^\mu(\boldsymbol{k}) := \{W_M(\boldsymbol{k})/c, \boldsymbol{P}_M(\boldsymbol{k})\}\,, \tag{17}$$

where $W_M(\boldsymbol{k})$ and $\boldsymbol{P}_M(\boldsymbol{k})$ are the corresponding (total) energy and 3-momentum in the waves. In a linear medium the waves damp only linearly, i.e. the rate of change of $W_M(\boldsymbol{k})$ must be of the form

$$\mathrm{d}W_M(\boldsymbol{k})/\mathrm{d}t = -\gamma_M(\boldsymbol{k})\, W_M(\boldsymbol{k})\,, \tag{18}$$

where $\gamma_M(\boldsymbol{k})$ is the absorption coefficient. By treating the dissipation explicitly we may calculate $\mathrm{d}W_M(\boldsymbol{k})/\mathrm{d}t$ and hence derive an expression for $-\gamma_M(\boldsymbol{k})\, W_M(\boldsymbol{k})$. In addition, if we use a covariant theory we may calculate the rate of change of $P_M^\mu(\boldsymbol{k})$ and hence use equation (17) to relate $\boldsymbol{P}_M(\boldsymbol{k})$ to $W_M(\boldsymbol{k})$.

A given extraneous current $J_{\mathrm{ext}}^\mu(x)$ generates 4-momentum density in the electromagnetic field at the rate $J_{\mathrm{ext}}^\alpha(x)\, F^\mu{}_\alpha(x)$. On integrating over space and averaging over time this quantity may be equated to $P_M^\mu(\boldsymbol{k})$ integrated over $\mathrm{d}\boldsymbol{k}/(2\pi)^3$. For the particular extraneous current (5), a straightforward calculation (see Appendix 3) leads to

$$\frac{\mathrm{d}P_M^\mu(\boldsymbol{k})}{\mathrm{d}t} = \frac{-2\mathrm{i}\,\mu_0\, c}{\{\omega_M(\boldsymbol{k})\}^2}\, W_M^{[\mathrm{E}]}(\boldsymbol{k})\, k_M^\mu\, e_{M\theta}^*(\boldsymbol{k})\, e_{M\tau}(\boldsymbol{k})\, \alpha^{\theta\tau(\mathrm{A})}(k_M)\,. \tag{19}$$

Comparison of the $\mu = 0$ component of equation (19) with (18) and (17) leads to the identification

$$\gamma_M(\boldsymbol{k}) = \frac{2\mathrm{i}\,\mu_0\, c}{\omega_M(\boldsymbol{k})}\, \frac{W_M^{[\mathrm{E}]}(\boldsymbol{k})}{W_M(\boldsymbol{k})}\, e_{M\theta}^*(\boldsymbol{k})\, e_{M\tau}(\boldsymbol{k})\, \alpha^{\theta\tau(\mathrm{A})}(k_M)\,. \tag{20}$$

By implication $W_M^{[\mathrm{E}]}(\boldsymbol{k})/W_M(\boldsymbol{k})$ is equal to $R_M(\boldsymbol{k})$ as given by equation (14), but this has yet to be established explicitly.

It is convenient to define the quantity

$$N_M(\boldsymbol{k}) := W_M(\boldsymbol{k})/\hbar\omega_M(\boldsymbol{k})\,, \tag{21}$$

where $\hbar$ is a constant with the dimensions of action. Then comparison of equations (17) and (19) implies

$$P^{\mu}_{M}(\boldsymbol{k}) = \hbar k^{\mu}_{M} N_{M}(\boldsymbol{k}). \tag{22}$$

The quantity $\hbar N_M(\boldsymbol{k})$ is the time-component of the wave action 4-current (cf. Whitham 1965; Dewar 1977).

4. Inclusion of Damping in Dispersion Equation

We may also treat dissipation by including it as an imaginary part of the dispersion equation (8). The idea is to include $\alpha^{\mu\nu(\mathrm{A})}(k)$ with $\Lambda^{\mu\nu}(k)$ to form a nonhermitian matrix. The counterpart of $\lambda^{\mu}{}_{\nu}(k)$ then contains an imaginary term proportional to $\alpha^{\mu\nu(\mathrm{A})}(k)$, and terms of higher order in $\alpha^{\mu\nu(\mathrm{A})}(k)$. The wave 4-vector k^{μ} is allowed to have an imaginary part $\mathrm{i}\,k^{\mu}_{\mathrm{I}}$. One solves the resulting dispersion equation by a perturbation approach. To zeroth order in the dissipative processes, equation (8) is reproduced. To first order in the dissipative processes, one has

$$\mathrm{i}\,k^{\mu}_{\mathrm{I}}\frac{\partial\lambda(k)}{\partial k^{\mu}} + \mu_0\frac{k_{\alpha}k^{\beta}}{k^4}\lambda^{\alpha\nu}{}_{\beta\mu}(k)\,\alpha^{\mu}{}_{\nu}{}^{(\mathrm{A})}(k) = 0. \tag{23}$$

On inserting the dispersion relation $\omega = \omega_M(\boldsymbol{k})$ in equation (23), one may use (2a, b) and (13) to simplify the second term in (23):

$$\begin{aligned}&[\{k_{\alpha}k^{\beta}/k^4\}\lambda^{\alpha\nu}{}_{\beta\mu}(k)\,\alpha^{\mu}{}_{\nu}{}^{(\mathrm{A})}(k)]_{\omega=\omega_M(\boldsymbol{k})}\\ &\qquad = -\{c/\omega_M(\boldsymbol{k})\}^2\lambda^{0\tau}{}_{0\tau}(k_M)\,e_{M\mu}(\boldsymbol{k})\,e_{M\nu}(\boldsymbol{k})\,\alpha^{\mu\nu(\mathrm{A})}(k_M).\end{aligned} \tag{24}$$

On comparing equations (20) and (24), one finds that (23) may be written in the form

$$W^{[\mathrm{E}]}_{M}(\boldsymbol{k})\left(2k^{\mu}_{\mathrm{I}}\frac{\partial\lambda(k)}{\partial k^{\mu}}\frac{\omega}{\lambda^{0\tau}{}_{0\tau}(k)}\right)_{\omega=\omega_M(\boldsymbol{k})} = -\gamma_M(\boldsymbol{k})\,W_M(\boldsymbol{k}). \tag{25}$$

The imaginary part $\mathrm{i}\,k^{\mu}_{\mathrm{I}}$ implies that the amplitude of the waves varies secularly as $\exp(k_{\mathrm{I}}x)$, and hence that the energy varies as $\exp(2k_{\mathrm{I}}x)$. Consider boundary conditions such that the damping is entirely temporal, i.e. waves uniformly excited everywhere initially. This corresponds to the time-component of k^{μ}_{I} being $-\gamma_M(\boldsymbol{k})/2c$ and the space-components of k^{μ}_{I} being zero. Then equation (25) allows us to identify $W_M(\boldsymbol{k})$:

$$W_M(\boldsymbol{k}) = W^{[\mathrm{E}]}_{M}(\boldsymbol{k})/R_M(\boldsymbol{k}), \tag{26}$$

with $R_M(\boldsymbol{k})$ given by (14). This establishes the interpretation of $R_M(\boldsymbol{k})$ as the ratio of electric to total energy.

The rules of partial differentiation imply

$$\left(\frac{\partial\lambda(k)}{\partial k_{\mu}}\Big/\frac{\partial\lambda(k)}{\partial\omega}\right)_{\omega=\omega_M(\boldsymbol{k})} = [c, \boldsymbol{v}_{M\mathrm{g}}(\boldsymbol{k})] =: v^{\mu}_{M\mathrm{g}}(\boldsymbol{k}), \tag{27}$$

where

$$\boldsymbol{v}_{M\mathrm{g}}(\boldsymbol{k}) := \partial\omega_M(\boldsymbol{k})/\partial\boldsymbol{k} \tag{28}$$

is the group velocity.

For a homogeneous medium we may re-interpret $2k_{\text{I}}^{\mu}$ in equation (25) as the 4-gradient ∂^{μ} for secular variations. Comparison of equations (25) and (19), together with (21), (22) and (27), leads to the continuity equation for wave energy–momentum:

$$\partial_{\mu} T_M^{\mu\nu}(\boldsymbol{k}) = -\gamma_M(\boldsymbol{k}) P_M^{\nu}(\boldsymbol{k}), \tag{29}$$

with

$$T_M^{\mu\nu}(\boldsymbol{k}) = \hbar N_M(\boldsymbol{k}) v_{M\text{g}}^{\mu}(\boldsymbol{k}) k_M^{\nu}. \tag{30}$$

The result (30) is equivalent to Dewar's (1977) result; cf. his equations (93) and (148).

The components of $T_M^{\mu\nu}(\boldsymbol{k})$ have the following interpretation in terms of 3-tensor quantities:

$T_M^{00}(\boldsymbol{k})$ is $1/c$ times the total wave energy $\quad W_M(\boldsymbol{k}) = \hbar\omega_M(\boldsymbol{k}) N_M(\boldsymbol{k})$;

$T_M^{0j}(\boldsymbol{k})$ is the jth component of the wave momentum $\quad \boldsymbol{P}_M(\boldsymbol{k}) = \hbar\boldsymbol{k} N_M(\boldsymbol{k})$;

$T_M^{i0}(\boldsymbol{k})$ is $1/c$ times the ith component of the wave energy flux $\quad \boldsymbol{F}_M(\boldsymbol{k}) = \boldsymbol{v}_{M\text{g}}(\boldsymbol{k}) W_M(\boldsymbol{k})$;

$T_M^{ij}(\boldsymbol{k})$ is the ijth component of the wave stress 3-tensor.

5. Transformation Properties

The quantity $T_M^{\mu\nu}(\boldsymbol{k})$ is the energy–momentum 4-tensor for waves in the elemental range $\text{d}\boldsymbol{k}/(2\pi)^3$. Under a Lorentz transformation only the product $T_M^{\mu\nu}(\boldsymbol{k})\,\text{d}\boldsymbol{k}/(2\pi)^3$ transforms as a 4-tensor.

Consider a specific Lorentz transformation from an unprimed frame to a primed frame with relative velocity $\boldsymbol{\beta}c$ and Lorentz factor $\gamma := (1-\beta^2)^{-\frac{1}{2}}$. In the primed frame we have

$$\omega_M'(\boldsymbol{k}') = \gamma\{\omega_M(\boldsymbol{k}) - k_{\parallel}\beta c\}, \tag{31a}$$

$$k_{\parallel}' = \gamma\{k_{\parallel} - \omega_M(\boldsymbol{k})\beta/c\}, \tag{31b}$$

$$\boldsymbol{k}_{\perp}' = \boldsymbol{k}_{\perp}, \tag{31c}$$

where components parallel and perpendicular to $\boldsymbol{\beta}$ are indicated accordingly. Let $\boldsymbol{\beta}_{M\text{g}}$ denote $\boldsymbol{v}_{M\text{g}}(\boldsymbol{k})/c$ for simplicity, and let $\boldsymbol{\beta}_{M\text{g}}'$ denote the corresponding quantity in the primed frame with

$$\boldsymbol{v}_{M\text{g}}'(\boldsymbol{k}') = \partial\omega_M'(\boldsymbol{k}')/\partial\boldsymbol{k}'.$$

Using

$$(\partial k_{\parallel}'/\partial k_{\parallel})_{\boldsymbol{k}\perp} = \gamma(1-\beta\beta_{M\text{g}\parallel}), \tag{32}$$

one finds

$$\text{d}\boldsymbol{k}' = \text{d}\boldsymbol{k}\,\gamma(1-\beta\beta_{M\text{g}\parallel}). \tag{33}$$

The transformation properties of the group velocity follow from

$$\left(\frac{\partial\omega_M'(\boldsymbol{k}')}{\partial k_{\parallel}'}\right)_{\boldsymbol{k}'\perp} = \left(\frac{\partial\omega_M'(\boldsymbol{k}')}{\partial k_{\parallel}}\right)_{\boldsymbol{k}\perp} \Bigg/ \left(\frac{\partial k_{\parallel}'}{\partial k_{\parallel}}\right)_{\boldsymbol{k}\perp}, \tag{34}$$

$$\left(\frac{\partial\omega_M'(\boldsymbol{k}')}{\partial \boldsymbol{k}_{\perp}'}\right)_{k_{\parallel}'} = \left(\frac{\partial\omega_M'(\boldsymbol{k}')}{\partial \boldsymbol{k}_{\perp}}\right)_{k_{\parallel}} - \left(\frac{\partial\omega_M'(\boldsymbol{k}')}{\partial k_{\parallel}}\right)_{\boldsymbol{k}\perp}\left(\frac{\partial k_{\parallel}'}{\partial \boldsymbol{k}_{\perp}}\right)_{k_{\parallel}}, \tag{35}$$

where equation (31c) and the rules of partial differentiation have been used. One finds

$$\beta'_{Mg\parallel} = \frac{\beta_{Mg\parallel}-\beta}{1-\beta\beta_{Mg\parallel}}, \qquad \boldsymbol{\beta}'_{Mg\perp} = \frac{\boldsymbol{\beta}_{Mg\perp}}{\gamma(1-\beta\beta_{Mg\parallel})}. \tag{36a, b}$$

Thus the group velocity transforms according to Einstein's law for the addition of velocities (Ko and Chuang 1977). It follows that

$$u^{\mu}_{Mg} := [\gamma_{Mg}\, c, \gamma_{Mg}\, \boldsymbol{\beta}_{Mg}\, c] \tag{37}$$

is a 4-vector, where $\gamma_{Mg} := (1-|\boldsymbol{\beta}_{Mg}|^2)^{-\frac{1}{2}}$ is the group Lorentz factor. The group Lorentz factor transforms according to

$$\gamma'_{Mg} = \gamma_{Mg}\, \gamma(1-\beta\beta_{Mg\parallel}). \tag{38}$$

It follows by inspection of equations (27), (33), (37) and (38) that $v^{\mu}_{Mg}(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}/(2\pi)^3$ transforms as a 4-vector. From the fact that $T^{\mu\nu}_{M}(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}/(2\pi)^3$ transforms as a 4-tensor it then follows from equation (30) that $N_M(\boldsymbol{k})$ is an invariant.

The transformation properties of $T^{\mu\nu}_{M}(\boldsymbol{k})$, as given by (30), are now evident: $N_M(\boldsymbol{k})$ is an invariant; $v^{\mu}_{Mg}(\boldsymbol{k})$ does not transform as a 4-vector but transforms such that its form $[c, \boldsymbol{v}_{Mg}(\boldsymbol{k})]$ is the same in all frames with $\boldsymbol{v}_{Mg}(\boldsymbol{k})$ transforming by the law for addition of velocities; the final factor k^{ν}_{M} in (30) transforms as a 4-vector.

In specific situations the signs of $\omega_M(\boldsymbol{k})$, $\boldsymbol{k}$ or $\boldsymbol{v}_{Mg}(\boldsymbol{k})$ may be opposite between the new and the old frames. No conceptual difficulties arise with changes in sign of $\boldsymbol{k}$ or $\boldsymbol{v}_{Mg}(\boldsymbol{k})$. However, a negative value of $\omega_M(\boldsymbol{k})$ would imply a negative energy. Such an unacceptable result is avoided by requiring that one always choose the frequencies to be positive. This convention implies that in a frame where $\omega'_M(\boldsymbol{k}')$ is negative one is to describe the waves in terms of the other solution $-\omega'_M(-\boldsymbol{k}')$, cf. equation (9), which then must be positive. That is, in this frame one regards the waves as having positive frequency and wave vector $-\boldsymbol{k}'$; they then have the complex conjugated polarization vector $e'^{*}_{M}(-\boldsymbol{k}')$. Put another way, negative-frequency forward-propagating waves are to be re-interpreted as positive-frequency backward-propagating waves. No conceptual difficulties then arise.

6. Transfer Equation

The present theory permits a classical description of the waves in terms of the invariant $N_M(\boldsymbol{k})$. In an inhomogeneous weakly dissipative medium the desired form of the transfer equation in a theory in which $N_M(\boldsymbol{k}, \boldsymbol{x}, t)$ is regarded as a distribution function for wave quanta is

$$\left(\frac{\partial}{\partial t} + \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\cdot\frac{\partial}{\partial \boldsymbol{x}} + \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t}\cdot\frac{\partial}{\partial \boldsymbol{k}} + \gamma_M(\boldsymbol{k})\right) N_M(\boldsymbol{k}, \boldsymbol{x}, t) = \left(\frac{\partial N_M(\boldsymbol{k})}{\partial t}\right)_{\text{source}}, \tag{39}$$

where the right-hand member is a source term. To complete the argument that the present classical theory is compatible with a quantum theory described in this way, we need to show that equation (39) is implied by the classical theory.

In Whitham's (1965) averaged Lagrangian approach, the relevant continuity equation is for the wave action. It is, see for example Dewar (1977),

$$\partial_{\mu}(v^{\mu}_{Mg}\, N_M) = 0, \tag{40}$$

where the independent variables are the 4-vectors x^μ and k^μ. On regarding $\omega = \omega_M(\boldsymbol{k}, \boldsymbol{x}, t)$ as a function of $\boldsymbol{k}$ and using

$$\left.\frac{\partial}{\partial \boldsymbol{x}}\right|_{\omega,k} = \left.\frac{\partial}{\partial \boldsymbol{x}}\right|_{k} + \left.\frac{\partial}{\partial \boldsymbol{k}}\right|_{x} \cdot \left.\frac{\partial \boldsymbol{k}}{\partial \boldsymbol{x}}\right|_{\omega}, \qquad \left.\frac{\partial \omega}{\partial \boldsymbol{k}}\right|_{x} \cdot \left.\frac{\partial \boldsymbol{k}}{\partial \boldsymbol{x}}\right|_{\omega} = -\left.\frac{\partial \omega}{\partial \boldsymbol{x}}\right|_{k},$$

we find equation (40) reproduces (39), without the damping and source terms, and with

$$\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \partial\omega_M(\boldsymbol{k}, \boldsymbol{x}, t)/\partial\boldsymbol{k}, \qquad \mathrm{d}\boldsymbol{k}/\mathrm{d}t = -\partial\omega_M(\boldsymbol{k}, \boldsymbol{x}, t)/\partial\boldsymbol{x}, \qquad (41\mathrm{a,b})$$

which are just the Hamiltonian equations for a ray. The damping and source terms are not included in the Lagrangian approach. The damping is given by the theory in Sections 2 and 3, and specific source terms are given by other specific theories.

7. Discussion and Conclusions

It is sometimes convenient to use a quantum mechanical description of waves even in treating classical plasmas; see for example Pines and Schrieffer (1962), Harris (1969), Tsytovich (1970) and Melrose (1980). In such a description one defines the wave quanta by ascribing energy $\hbar\omega$ and momentum $\hbar\boldsymbol{k}$ to them. It is usually assumed that the level of excitation of the waves may be described in terms of the occupation number for the quanta. However, it should be shown that there exists a classical quantity which may be interpreted as the occupation number. It is shown here that the wave action, as defined by Whitham (1965), satisfies the requirements to be so interpreted: it is an invariant and it satisfies the transfer equation (39).

The form of the energy–momentum tensor is given by equation (30), namely

$$T_M^{\mu\nu}(\boldsymbol{k}) = \hbar N_M(\boldsymbol{k})\, v_{M\mathrm{g}}^{\mu}(\boldsymbol{k})\, k_M^{\nu},$$

where M labels the wave mode. This form is determined by two properties, namely that the velocity of energy propagation be the group velocity and that the ratio of the wave momentum to the wave energy be $\boldsymbol{k}/\omega$. The former property seems to be implicit in any theory based on an expansion in plane waves; see for example Bekefi (1966). The latter property is implicit in a quantum mechanical description in terms of wave quanta with energy $\hbar\omega$ and momentum $\hbar\boldsymbol{k}$. Peierls (1976) and Dewar (1977) have remarked that the identification of the ratio of the wave momentum to the wave energy as $\boldsymbol{k}/\omega$ is consistent with the Minkowski form of the electromagnetic energy–momentum tensor in a medium. Dewar (1977) showed that a form equivalent to (30) reduces to the Minkowski form for the special case of a nondispersive medium. The form (30) may be regarded or defined as the appropriate generalization of the Minkowski tensor to dispersive waves.

The identification of the energy–momentum tensor (30) with the Minkowski tensor should not be interpreted as an argument against the Abraham or any other form of the tensor in any general context. The important point is that in any classical or quantum theory of waves in a dispersive medium, the description in terms of plane waves virtually determines that the generalization of the Minkowski tensor is the relevant energy–momentum tensor. This tensor is not symmetric, and as pointed out by Dewar (1977), this implies that the energy–momentum tensor for the background medium must depend on the presence of the waves and also be nonsymmetric. The particular separation of the total energy–momentum tensor into a wave part (30) and a background part may not be appropriate in other connections. In the interpretation of the momentum in waves (Peierls 1976, 1977; Jones 1978) a given experiment may well correspond to an identification of the wave subsystem which is different from that made in deriving (30). Put another way, the identification of the ratio of the wave momentum to energy as $\boldsymbol{k}/\omega$ may be regarded as a definition of 'wave momentum', but the quantity measured in a specific experiment may well correspond to a different definition of wave momentum. This possibility arises because the background system, as implicitly defined here, contains an energy–momentum component dependent on the presence of the waves, and a variety of separations into the wave and background systems is possible. The particular separation implicit in (30) is appropriate in theories for dispersive waves, and in particular in any quantum mechanical treatment.

In summary, it may be concluded from the work of Dewar (1977) and from the results obtained here that:

(1) The energy–momentum tensors for dispersive waves derived (a) by Whitham's (1965) averaged-Lagrangian approach and (b) using a covariant formulation of the theory of weakly damped waves are equivalent, and (c) that in the nondispersive case they are equivalent to the Minkowski tensor.

(2) These tensors are compatible with the interpretation of the waves as a collection of wave quantity with energy $\hbar\omega(\boldsymbol{k})$, momentum $\hbar\boldsymbol{k}$ and occupation number $N_M(\boldsymbol{k})$.

(3) For waves with phase speeds less than the speed of light there exist inertial frames in which given waves have frequencies of opposite signs. No conceptual problem arises in such situations provided one appeals to the equivalence of positive- and negative-frequency solutions. One is free to adopt the convention that the physical branch is that which corresponds to positive frequency in the frame of interest.

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Appendix 1. Matrix Results

Let $A^\mu{}_\nu$ be a 4×4 hermitian matrix ($A^{*\mu\nu} = A^{\nu\mu}$). Its determinant A, matrix of cofactors $a^\mu{}_\nu$ and the set of cofactors of the 2×2 signed minors are defined by

$$A := (1/4!)\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\zeta\tau}A^\mu{}_\alpha A^\nu{}_\beta A^\zeta{}_\gamma A^\tau{}_\delta, \tag{A1}$$

$$a^{\mu}{}_{\nu} := (1/3!)\varepsilon^{\mu\alpha\beta\gamma}\varepsilon_{\nu\eta\theta\zeta}A^{\eta}{}_{\alpha}A^{\theta}{}_{\beta}A^{\zeta}{}_{\gamma}, \tag{A2}$$

$$a^{\mu\nu}{}_{\alpha\beta} := (1/2!)\varepsilon^{\mu\nu\theta\eta}\varepsilon_{\alpha\beta\zeta\tau}A^{\zeta}{}_{\theta}A^{\tau}{}_{\eta}, \tag{A3}$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is the permutation symbol with $\varepsilon^{0123} = -\varepsilon_{0123} = 1$. We also have the identities

$$a^{\mu\nu}{}_{\alpha\beta} = -a^{\nu\mu}{}_{\alpha\beta} = -a^{\mu\nu}{}_{\beta\alpha}, \tag{A4}$$

$$a^{\mu\nu\alpha\beta} = a^{*\alpha\beta\mu\nu}, \tag{A5}$$

$$A^{\mu}{}_{\zeta}a^{\zeta}{}_{\nu} = A\,\delta^{\mu}_{\nu}, \tag{A6}$$

$$A^{\mu}{}_{\zeta}a^{\zeta\tau}{}_{\alpha\beta} = \delta^{\mu}_{\alpha}a^{\tau}{}_{\beta} - \delta^{\mu}_{\beta}a^{\tau}{}_{\alpha}, \tag{A7}$$

$$A\,a^{\mu\nu}{}_{\alpha\beta} = a^{\mu}{}_{\alpha}a^{\nu}{}_{\beta} - a^{\mu}{}_{\beta}a^{\nu}{}_{\alpha}. \tag{A8}$$

In Section 2 the following results have been used. First, if A vanishes then equation (A8) implies that $a^{\mu}{}_{\nu}$ is of rank one and (A6) then implies that the vector v^{μ} defined by writing

$$a^{\mu}{}_{\nu} = gv^{\mu}v_{\nu} \qquad (A = 0) \tag{A9}$$

satisfies

$$A^{\mu}{}_{\nu}v^{\nu} = 0. \tag{A10}$$

Next, if g also vanishes then equation (A7) implies

$$A^{\mu}{}_{\zeta}a^{\zeta\tau}{}_{\alpha\beta} = 0 \qquad (g = 0). \tag{A11}$$

There then must exist two vectors v_1^{μ} and v_2^{μ} satisfying equation (A10); and (A4), (A5) and (A11) then imply

$$\alpha^{\mu\nu}{}_{\alpha\beta} \propto (v_1^{\mu}v_2^{\nu} - v_1^{\nu}v_2^{\mu})(v_{1\alpha}^{*}v_{2\beta}^{*} - v_{1\beta}^{*}v_{2\alpha}^{*}). \tag{A12}$$

For completeness I state the following explicit expressions for λ (cf. equation 7) and $\lambda^{\mu\nu}{}_{\alpha\beta}$ in the case of the matrix $\Lambda^{\mu}{}_{\nu}$ of Section 2:

$$\lambda = (1/6k^2)\{(\Lambda^{\alpha}{}_{\alpha})^3 - 3\Lambda^{\alpha}{}_{\alpha}\Lambda^{\beta}{}_{\gamma}\Lambda^{\gamma}{}_{\beta} + 2\Lambda^{\alpha}{}_{\beta}\Lambda^{\beta}{}_{\gamma}\Lambda^{\gamma}{}_{\alpha}\}, \tag{A13}$$

$$\begin{aligned}\lambda^{\mu\nu}{}_{\alpha\beta} = {} & \tfrac{1}{2}(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta})\{(\Lambda^{\eta}{}_{\eta})^2 - \Lambda^{\eta}{}_{\theta}\Lambda^{\theta}{}_{\eta}\} \\ & - \Lambda^{\theta}{}_{\theta}(\delta^{\mu}_{\alpha}\Lambda^{\nu}{}_{\beta} - \delta^{\nu}_{\alpha}\Lambda^{\mu}{}_{\beta} - \delta^{\mu}_{\beta}\Lambda^{\nu}{}_{\alpha} + \delta^{\nu}_{\beta}\Lambda^{\mu}{}_{\alpha}) \\ & + \delta^{\mu}_{\alpha}\Lambda^{\nu}{}_{\eta}\Lambda^{\eta}{}_{\beta} - \delta^{\nu}_{\alpha}\Lambda^{\mu}{}_{\eta}\Lambda^{\eta}{}_{\beta} - \delta^{\mu}_{\beta}\Lambda^{\nu}{}_{\eta}\Lambda^{\eta}{}_{\alpha} + \delta^{\nu}_{\beta}\Lambda^{\mu}{}_{\eta}\Lambda^{\eta}{}_{\alpha} \\ & + \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} - \Lambda^{\mu}{}_{\beta}\Lambda^{\nu}{}_{\alpha}. \end{aligned} \tag{A14}$$

Appendix 2. Electric Energy $W_M^{[E]}(\boldsymbol{k})$

The time-averaged electric energy in a field described by the vector potential $A(x)$ is

$$\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T}\mathrm{d}t\int \mathrm{d}\boldsymbol{x}\,\tfrac{1}{2}\varepsilon_0\,|\boldsymbol{E}(x)|^2$$

$$= \lim_{T\to\infty}\frac{1}{cT}\int \mathrm{d}^4x\,\frac{\varepsilon_0}{2c^2}\left|\frac{\partial A(x)}{\partial t}\right|^2 = \lim_{T\to\infty}\frac{1}{cT}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\varepsilon_0}{2c^2}|\omega A(k)|^2, \tag{A15}$$

where in the second step the truncation is implicit and in the third step the power theorem for Fourier transforms has been used. On inserting the space components of the amplitude (15) for waves in the mode M, the resulting expression is equated to the integral of $W_M^{[E]}(\boldsymbol{k})$:

$$\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} W_M^{[E]}(\boldsymbol{k}) = \lim_{T\to\infty} \frac{1}{cT} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\varepsilon_0}{2c^2} \omega^2 \,|\, \boldsymbol{A}_M(k_M) \,|^2 \,. \tag{A16}$$

The square of the δ function which appears when equation (15) is inserted in (A16) is rewritten using

$$\{\delta(\omega)\}^2 = \lim_{T\to\infty} \frac{T}{2\pi} \delta(\omega) \,. \tag{A17}$$

The negative- and positive-frequency solutions (9) contribute equally to the $k^0 = \omega/c$ integral over the δ function. Then using (12), equation (A16) becomes

$$\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} W_M^{[E]}(\boldsymbol{k}) = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \varepsilon_0 \{\omega_M(\boldsymbol{k})\, a_M(\boldsymbol{k})/c\}^2 \,, \tag{A18}$$

which implies (16).

Appendix 3. Derivation of Equation (19)

The time-averaged rate in which 4-momentum is given to the electromagnetic field by an extraneous current $J^{\mu}_{\mathrm{ext}}(x)$ is

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\, J^{\alpha}_{\mathrm{ext}}(x)\, F^{\mu}{}_{\alpha}(x)$$

$$= \lim_{T\to\infty} \frac{1}{cT} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} J^{\alpha}_{\mathrm{ext}}(k) \{\mathrm{i}\, k_{\alpha} A^{*\mu}(k) - \mathrm{i}\, k^{\mu} A^{*}_{\alpha}(k)\} \,, \tag{A19}$$

where the steps are the same as in equation (A15), with the Maxwell tensor given by

$$F^{\mu\nu}(k) = -\mathrm{i}\, k^{\mu} A^{\nu}(k) + \mathrm{i}\, k^{\nu} A^{\mu}(k) \tag{A20}$$

in terms of the 4-potential. On inserting equation (5), we find the term $\mathrm{i}\, k_{\alpha} A^{*\mu}(k)$ in (A19) gives zero in view of (2a). On equating $A^{\mu}(k)$ with $A^{\mu}_M(k_M)$, given by (15), equation (A19) is identified with the rate 4-momentum is given to waves in the mode M, and hence equated to the integral of $\mathrm{d}\, P^{\mu}_M(\boldsymbol{k})/\mathrm{d}t$:

$$\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \frac{\mathrm{d}P^{\mu}_M(\boldsymbol{k})}{\mathrm{d}t} = \lim_{T\to\infty} \frac{1}{cT} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \{-\mathrm{i}\, \tilde{\mu}_0\, c k^{\mu} A^{*}_{\zeta}(k)\, A_{\tau}(k)\, \alpha^{\zeta\tau(\mathrm{A})}(k)\}_{k=k_M} \,. \tag{A21}$$

The remaining steps in the derivation of equation (19) are the same as the steps leading from (A16) to (A18).

Manuscript received 10 November 1980, accepted 18 March 1981