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Covariant Description of Dispersion in a Relativistic Thermal Electron Gas

D. B. Melrose

Department of Theoretical Physics, School of Physics, University of Sydney, N.S.W. 2006.

Abstract

A covariant and gauge-invariant theory is used to describe dispersion in an isotropic plasma in terms of two invariant functions. These invariants are evaluated explicitly for a relativistic Maxwellian (Jüttner–Synge) distribution. The relations between forms for the dispersion of a relativistic Maxwellian distribution obtained by some earlier authors and the forms obtained here are demonstrated.

1. Introduction

There is an extensive literature relating to dispersion in a relativistic thermal electron gas (see for example Trubnikov 1958; Silin 1960, 1961; Kurşunoğlu 1960, 1965; Buti 1962; Imre 1962; Prentice 1967, 1968; Hakim and Mangeney 1968, 1971; Misra 1975; Godfrey *et al.* 1975*a*, 1975*b*). For the unmagnetized case, which is the only case considered in this paper, there are two aspects of the problem which require further clarification: the development of a fully covariant and gauge-invariant theory for the dispersion and the interrelations between the diverse forms found for the dispersion functions.

The use of covariant electrodynamics is not necessary for the inclusion of relativistic effects: many of the authors cited above used the conventional Vlasov equation, with relativistic effects included of course, in evaluating the longitudinal $\varepsilon^{\mathrm{l}}(\boldsymbol{k},\omega)$ and transverse $\varepsilon^{\mathrm{t}}(\boldsymbol{k},\omega)$ parts of the dielectric tensor for a relativistic Maxwellian distribution. Four steps may be identified in the development of a fully covariant and gauge-invariant theory, for example, for the purpose of including plasma dispersion in quantum electrodynamics: the basic equations are written in covariant form (Buneman 1958); a 4-dimensional distribution $F(p,x)$ is introduced, along with a covariant version of Vlasov's equation (Goto 1958; Klimontovich 1959; Kurşunoğlu 1961); the plasma response function is defined in covariant and gauge-invariant form (Buneman 1958; Melrose 1973); finally, the dispersion equation is derived in terms of invariants. In the existing covariant treatments a specific gauge (the Lorentz gauge) has been chosen, usually implicitly (the statement by Godfrey *et al.* (1957*b*) that their theory is gauge-independent is not correct), and the dispersion functions and dispersion relations have been evaluated explicitly only after specializing to the rest frame of the plasma. These specializations are unnecessary and must be avoided for the theory to be fully covariant and gauge-invariant.

The dispersion functions for a relativistic, thermal, unmagnetized electron gas have been found in essentially three different ways, and these lead to superficially dissimilar results. Trubnikov (1958) treated the unmagnetized case as a limit of the magnetized case, and obtained dispersion functions involving integrals of Macdonald functions K_ν, with $\nu = 2$ and 3, and with complex argument. Silin's (1960, 1961) method involves evaluation of the imaginary part of the dielectric tensor explicitly and the use of the Kramers–Kronig relation to construct the real part. Kurşunoğlu (1961), Prentice (1967, 1968), Hakim and Mangeney (1968, 1971), Misra (1975) and Godfrey *et al.* (1975*b*) evaluated the dispersion functions directly in terms of a single integral and its derivatives, with the specific integrals chosen by the various authors being somewhat different. The relations between these different forms are not all readily established. In particular Trubnikov's (1958) form is quite unlike any of the others. The form chosen here is essentially that of Godfrey *et al.* (1975*a*); specifically it involves their relativistic plasma dispersion function $T(z,\rho)$.

An outline of the present paper is as follows. In Section 2 a fully covariant and gauge-independent theory (Melrose 1973) is applied to the case of an isotropic plasma, and the response tensor is evaluated explicitly in terms of two invariants $\alpha^{\mathrm{l}}(k)$ and $\alpha^{\mathrm{t}}(k)$. These invariants are related in a simple way to $\varepsilon^{\mathrm{l}}(\boldsymbol{k},\omega)$ and $\varepsilon^{\mathrm{t}}(\boldsymbol{k},\omega)$ in the rest frame of the plasma. The dispersion relations for longitudinal and transverse waves are written down in a covariant form. In Section 3 the invariants are evaluated explicitly for a relativistic Maxwellian distribution. A sequence of plasma dispersion functions $T_n(z,\rho)$ is defined, with $T_0(z,\rho)$ being the function $T(z,\rho)$ of Godfrey *et al.* (1975*a*); the properties of these functions are summarized in Appendix 1. The relations between the form found here for the dispersion functions and earlier results, notably Trubnikov's, are indicated in Section 4.

The 4-tensor notation used is that of Berestetskii *et al.* (1971), Melrose (1973, 1981) and Jackson (1975). Except where stated otherwise, units with $c = 1$ are adopted; not to do so leads to confusion over factors of c which arise from Fourier transforming. The Fourier transform and its inverse are

$$F(k) := \int \mathrm{d}^4x \exp(\mathrm{i}\,kx)\,\tilde{F}(x), \qquad \tilde{F}(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\exp(-\mathrm{i}\,kx)\,F(k),$$

where the symbol $:=$ defines the quantity on the left and kx denotes $k^\mu x_\mu$; it is the differentials $\mathrm{d}^4x = c\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}$ and $\mathrm{d}^4k = c^{-1}\,\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k}$ which differ from those used in a non-covariant theory by factors of c. With $c = 1$ one has $\mu_0 = 1/\varepsilon_0$ in SI units, and μ_0 or $1/\varepsilon_0$ is to be replaced by 4π in gaussian units.

2. Covariant Theory of Dispersion in an Isotropic Plasma

In this section the covariant and gauge-independent theory of Melrose (1973, 1981) is applied to the case of an isotropic plasma.

The plasma response is described in terms of a response function $\alpha^{\mu\nu}(k)$ which relates the linear induced current $j^\mu(k)$ to the 4-potential $A^\mu(k)$:

$$j^\mu(k) = \alpha^\mu{}_\nu(k)\,A^\nu(k). \tag{1}$$

Charge continuity and gauge-invariance require

$$k_\mu\alpha^{\mu\nu}(k) = 0, \qquad k_\nu\alpha^{\mu\nu}(k) = 0. \tag{2a, b}$$

Including the current (1) in the covariant form of Maxwell's equations leads to the wave equation

$$\Lambda^{\mu\nu}(k)\,A_\nu(k) = -\mu_0 j^{\mu}_{\text{ext}}(k), \tag{3a}$$

with

$$\Lambda^{\mu\nu}(k) = k^2 g^{\mu\nu} - k^\mu k^\nu + \mu_0\,\alpha^{\mu\nu}(k) \tag{3b}$$

and where $j^{\mu}_{\text{ext}}(k)$ is an arbitrary extraneous current which is regarded as a source term.

By definition, a medium is isotropic if there exists an inertial frame, called its rest frame, in which it is isotropic. Let $\bar{u}^\mu$ ($=[\bar{\gamma}, \bar{\gamma}\bar{\boldsymbol{v}}]$) be the 4-velocity of the rest frame. Then $\alpha^{\mu\nu}(k)$ can depend only on the invariants $k\bar{u}$ and k^2, with

$$k\bar{u} = \bar{\gamma}(\omega - \boldsymbol{k}\,.\,\bar{\boldsymbol{v}}), \qquad k^2 = \omega^2 - |\boldsymbol{k}|^2, \tag{4a, b}$$

and second rank tensors constructed from $g^{\mu\nu}$, $\bar{u}^\mu$ and k^μ. The second rank tensors must themselves satisfy equations (2). There are only two such independent symmetric tensors, for example $g^{\mu\nu} - k^\mu k^\nu/k^2$ and $a^{\mu\nu}(k,\bar{u})$, with

$$a^{\mu\nu}(k,u) := g^{\mu\nu} - \frac{k^\mu u^\nu + k^\nu u^\mu}{ku} + \frac{k^2 u^\mu u^\nu}{(ku)^2}. \tag{5}$$

(There is also the antisymmetric tensor $\varepsilon^{\mu\nu\rho\sigma} k_\rho \bar{u}_\sigma$ which needs to be included only if the medium is optically active, which an electron gas is not.) A convenient choice for present purposes is the pair $L^{\mu\nu}(k,\bar{u})$ and $T^{\mu\nu}(k,\bar{u})$, with

$$L^{\mu\nu}(k,u) := \frac{k^2}{k^2-(ku)^2}\Big(a^{\mu\nu}(k,u) - (g^{\mu\nu} - k^\mu k^\nu/k^2)\Big), \tag{6}$$

$$T^{\mu\nu}(k,u) := \frac{1}{k^2-(ku)^2}\Big(-(ku)^2\,a^{\mu\nu}(k,u) + k^2(g^{\mu\nu} - k^\mu k^\nu/k^2)\Big). \tag{7}$$

These are orthogonal in the sense

$$L^{\mu\nu}(k,u)\,T_{\nu\rho}(k,u) = 0, \tag{8}$$

and in the rest frame, the $\mu = i$, $\nu = j$ components of $L^\mu{}_\nu$ and $T^\mu{}_\nu$ reduce to the 3-tensors $k_i k_j/|\boldsymbol{k}|^2$ and $\delta_{ij} - k_i k_j/|\boldsymbol{k}|^2$ respectively. Thus $L^{\mu\nu}$ and $T^{\mu\nu}$ may be regarded as covariant forms of the longitudinal and transverse 3-tensor components.

It follows that for an isotropic plasma there exist invariants $\alpha^{\mathrm{l}}(k)$ and $\alpha^{\mathrm{t}}(k)$, which are functions of k^2 and $k\bar{u}$, such that one has

$$\alpha^{\mu\nu}(k) = \alpha^{\mathrm{l}}(k)\,L^{\mu\nu}(k,\bar{u}) + \alpha^{\mathrm{t}}(k)\,T^{\mu\nu}(k,\bar{u}). \tag{9}$$

On contracting (9) with $L_{\mu\nu}(k,\bar{u})$ and $T_{\mu\nu}(k,\bar{u})$ one identifies

$$\alpha^{\mathrm{l}}(k) = (ku)^4 k^{-4}\,L_{\mu\nu}(k,\bar{u})\,\alpha^{\mu\nu}(k), \tag{10}$$

$$\alpha^{\mathrm{t}}(k) = \tfrac{1}{2}\,T_{\mu\nu}(k,\bar{u})\,\alpha^{\mu\nu}(k). \tag{11}$$

In deriving equations (10) and (11) we use (8) and

$$L^{\mu\nu}(k,u)\,L_{\nu\alpha}(k,u) = k^2(ku)^{-2}\,L^{\mu}{}_{\alpha}(k,u)\,, \tag{12a}$$

$$T^{\mu\nu}(k,u)\,T_{\nu\alpha}(k,u) = T^{\mu}{}_{\alpha}(k,u)\,, \tag{12b}$$

$$L^{\mu}{}_{\mu}(k,u) = k^2(ku)^{-2}\,, \qquad T^{\mu}{}_{\mu}(k,u) = 2\,. \tag{12c, d}$$

For an isotropic plasma the tensor $\Lambda^{\mu\nu}(k)$, defined by (3b), may also be expressed in the form (9), namely

$$\Lambda^{\mu\nu}(k) = \Lambda^{\mathrm{l}}(k)\,L^{\mu\nu}(k,\bar{u}) + \Lambda^{\mathrm{t}}(k)\,T^{\mu\nu}(k,\bar{u})\,. \tag{13}$$

One identifies

$$\Lambda^{\mathrm{l}}(k) = (ku)^2 + \mu_0\,\alpha^{\mathrm{l}}(k)\,, \tag{14}$$

$$\Lambda^{\mathrm{t}}(k) = k^2 + \mu_0\,\alpha^{\mathrm{t}}(k)\,. \tag{15}$$

From Melrose (1973) a covariant form of the dispersion equation can be found by constructing the matrix $\lambda^{\mu\nu}(k)$ of cofactors of $\Lambda^{\mu\nu}(k)$. An implication of equations (2) and of the definition (3b) of $\Lambda^{\mu\nu}(k)$ is that $\lambda^{\mu\nu}(k)$ must be of the form

$$\lambda^{\mu\nu}(k) = k^{\mu}k^{\nu}\,\lambda(k)\,, \tag{16}$$

where $\lambda(k)$ is an invariant. The invariant dispersion equation is then $\lambda(k) = 0$.

Explicit calculation gives

$$\lambda(k) = (ku)^{-2}\,\Lambda^{\mathrm{l}}(k)\{\Lambda^{\mathrm{t}}(k)\}^2\,. \tag{17}$$

The dispersion equation $\lambda(k) = 0$ factorizes into the dispersion equations

$$\Lambda^{\mathrm{l}}(k) = 0\,, \tag{18}$$

$$\Lambda^{\mathrm{t}}(k) = 0 \tag{19}$$

for longitudinal and transverse waves respectively.

In the rest frame $\alpha^{\mathrm{l}}(k)$ and $\alpha^{\mathrm{t}}(k)$ are related to the longitudinal and transverse parts of the dielectric tensor by

$$\varepsilon^{\mathrm{l}}(\boldsymbol{k},\omega) = 1 + \mu_0\,\omega^{-2}\,\alpha^{\mathrm{l}}(k)\,, \tag{20}$$

$$\varepsilon^{\mathrm{t}}(\boldsymbol{k},\omega) = 1 + \mu_0\,\omega^{-2}\,\alpha^{\mathrm{t}}(k)\,. \tag{21}$$

The invariant dispersion relations (18) and (19), with $k\bar{u} = \omega$ and $k^2 = \omega^2 - |\boldsymbol{k}|^2$, then reduce to their familiar forms $\varepsilon^{\mathrm{l}}(\boldsymbol{k},\omega) = 0$ and $|\boldsymbol{k}|^2/\omega^2 = \varepsilon^{\mathrm{t}}(\boldsymbol{k},\omega)$.

3. Relativistic Maxwellian Distribution

In this section the invariants $\alpha^{\mathrm{l}}(k)$ and $\alpha^{\mathrm{t}}(k)$ are evaluated for an arbitrary isotropic distribution of electrons using a covariant version of the Vlasov theory. The case of a relativistic Maxwellian distribution is then treated explicitly.

From the covariant Vlasov theory one obtains

$$\begin{aligned}\alpha^{\mu\nu}(k) &= e^2\int \mathrm{d}^4p\; u^{\mu}\Big(g^{\alpha\nu} - \frac{k^{\alpha}u^{\nu}}{ku}\Big)\frac{\partial F(p)}{\partial p^{\alpha}}\\ &= -\frac{e^2}{m}\int \mathrm{d}^4p\; F(p)\,a^{\mu\nu}(k,u)\,,\end{aligned} \tag{22}$$

with $p^\mu = mu^\mu$ and where the latter form follows after a partial integration with $a^{\mu\nu}(k,u)$ given by the definition (5). The 4-dimensional distribution function $F(p)$ is related to the usual 3-momentum distribution function $f(\boldsymbol{p})$ by

$$\int \mathrm{d}p^0\, F(p) = f(\boldsymbol{p})/\gamma, \tag{23}$$

where γ is the Lorentz factor. It is convenient to define two number densities

$$n := \int \mathrm{d}^4p\, \gamma F(p), \qquad n_0 := \int \mathrm{d}^4p\, F(p), \tag{24a, b}$$

and two plasma frequencies

$$\omega_\mathrm{p}^2 := ne^2/m\varepsilon_0, \qquad \omega_{\mathrm{p}0}^2 := n_0 e^2/m\varepsilon_0. \tag{25a, b}$$

The parameters n and ω_p are the actual number density and plasma frequency respectively in a given frame; these quantities are frame dependent and in the following discussion they will refer specifically to the values in the rest frame. The parameters n_0 and $\omega_{\mathrm{p}0}$ are invariants, and n and ω_p may be related to them for any particular distribution. The use of n or n_0 has caused some confusion in the literature, cf. Section IV of Hakim and Mangeney (1971) for example.

A straightforward calculation starting from equations (9) and (22) and using (8) and (12) leads to the identifications

$$\alpha^\mathrm{l}(k) = \frac{e^2}{m}\int \mathrm{d}^4p\, F(p)\frac{(k\bar{u})^2}{k^2-(k\bar{u})^2}\left(1 - \frac{2(k\bar{u})(u\bar{u})}{ku} + \frac{k^2(u\bar{u})^2}{(ku)^2}\right), \tag{26}$$

$$\alpha^\mathrm{t}(k) = \frac{e^2}{m}\int \mathrm{d}^4p\, F(p)\left\{1 + \frac{k^2}{2(ku)^2} - \frac{k^2}{2\{k^2-(k\bar{u})^2\}}\left(1 - \frac{2(k\bar{u})(u\bar{u})}{ku} + \frac{k^2(u\bar{u})^2}{(ku)^2}\right)\right\}. \tag{27}$$

It follows that to evaluate $\alpha^\mathrm{l}(k)$ and $\alpha^\mathrm{t}(k)$ explicitly for a specific distribution $F(p)$ we need to evaluate the integrals (24) and

$$\int \mathrm{d}^4p\, F(p)\frac{k^2}{(ku)^2}, \quad \int \mathrm{d}^4p\, F(p)\frac{u\bar{u}}{ku}, \quad \int \mathrm{d}^4p\, F(p)\left(\frac{u\bar{u}}{ku}\right)^2.$$

A relativistic Maxwellian distribution is the Jüttner–Synge (Jüttner 1911; Synge 1957)

$$F(p) = \frac{n\rho}{2\pi K_2(\rho)\, m^2}\,\delta(p^2-m^2)\,\theta(p\bar{u})\exp(-\rho p\bar{u}/m), \tag{28}$$

where ρ ($=mc^2/T$ in ordinary units) is the inverse of the temperature in units of mc^2 ($\approx 5\times 10^9$ K) and θ is the step function. The integrals are evaluated in Appendix 2 in terms the relativistic plasma dispersion function of Godfrey *et al.* (1975*a*)

$$T(z,\rho) := \int_{-1}^{1} \frac{\mathrm{d}v\, \exp(-\rho\gamma)}{v-z}. \tag{29}$$

It is found convenient to introduce the functions

$$T_n(z,\rho) := \int_{-1}^{1} \frac{\mathrm{d}v\, \gamma^n \exp(-\rho\gamma)}{v-z}, \tag{30}$$

with $T(z,\rho) = T_0(z,\rho)$, and to derive recursion relations which enable one to reduce all the integrals to ones involving $T(z,\rho)$ and $\partial T(z,\rho)/\partial z$.

In the rest frame the parameter

$$z := k\bar{u}/\{(k\bar{u})^2 - k^2\}^{\frac{1}{2}} \tag{31}$$

reduces to the phase speed, i.e. to $\omega/|\boldsymbol{k}|c$ in ordinary units. The relevant integrals are

$$\int d^4p\, F(p)\frac{k^2}{(ku)^2} = -\frac{n\rho}{2K_2(\rho)}\frac{k^2}{(k\bar{u})^2}z^2\left(\frac{2}{1-z^2}K_0(\rho) + \frac{z}{1-z^2}T(z,\rho)\right), \tag{32}$$

$$\int d^4p\, F(p)\frac{u\bar{u}}{ku} = \frac{n\rho}{2K_2(\rho)}\frac{1}{k\bar{u}}\left\{\frac{2K_1(\rho)}{\rho} - \frac{1}{\rho^2}\left(z\,T(z,\rho) - (1-z^2)\frac{\partial T(z,\rho)}{\partial z}\right)\right\}, \tag{33}$$

$$\int d^4p\, F(p)\left(\frac{u\bar{u}}{ku}\right)^2 = -\frac{n\rho}{2K_2(\rho)}\frac{z^2}{(k\bar{u})^2}\left(\frac{2}{(1-z^2)^2}K_0(\rho) + \frac{2}{1-z^2}\frac{K_1(\rho)}{\rho} + \frac{z}{1-z^2}T(z,\rho)\right). \tag{34}$$

We also have the identity

$$n/n_0 = \omega_p^2/\omega_{p0}^2 = K_2(\rho)/K_1(\rho). \tag{35}$$

If we combine these results, equations (26) and (27) give

$$\alpha^l(k) = \frac{\omega_p^2}{\mu_0}\frac{z^2}{1-z^2}\left\{\frac{\rho}{2K_2(\rho)}\left(2K_0(\rho) + z\,T(z,\rho)\right) + \frac{1-z^2}{\rho\,K_2(\rho)}\left(z\,T(z,\rho) - (1-z^2)\frac{\partial T(z,\rho)}{\partial z}\right)\right\}, \tag{36}$$

$$\alpha^t(k) = -\frac{\omega_p^2}{\mu_0}\left\{\frac{K_1(\rho)}{K_2(\rho)} - \frac{1-z^2}{2\rho\,K_2(\rho)}\left(z\,T(z,\rho) - (1-z^2)\frac{\partial T(z,\rho)}{\partial z}\right)\right\}, \tag{37}$$

respectively. The dispersion relations (18) and (19) for longitudinal and transverse waves then become

$$(k\bar{u})^2 = -\omega_p^2\frac{z^2}{1-z^2}\left\{\frac{\rho}{2K_2(\rho)}\left(2K_0(\rho) + z\,T(z,\rho)\right) + \frac{1-z^2}{\rho\,K_2(\rho)}\left(z\,T(z,\rho) - (1-z^2)\frac{\partial T(z,\rho)}{\partial z}\right)\right\}, \tag{38}$$

$$k^2 = \omega_p^2\left\{\frac{K_1(\rho)}{K_2(\rho)} - \frac{1-z^2}{2\rho\,K_2(\rho)}\left(z\,T(z,\rho) - (1-z^2)\frac{\partial T(z,\rho)}{\partial z}\right)\right\}, \tag{39}$$

respectively. In the rest frame, equations (38) and (39) reproduce the results obtained by Godfrey *et al.* (1975*b*): one sets $k\bar{u} = \omega$, $k^2 = \omega^2 - |\boldsymbol{k}|^2c^2$ and $z = \omega/|\boldsymbol{k}|c$ in ordinary units.

4. Alternative Forms of Dispersion Function

In this section the results obtained in Section 3 are compared with those of earlier authors. The relation to the forms obtained by Godfrey *et al.* (1975*b*) is obvious and it is not difficult to establish the relations to the results of Silin (1960, 1961), Prentice (1967, 1968), Hakim and Mangeney (1968, 1971) and Misra (1975). However, Trubnikov's (1958) forms are quite different and some subtle transformations are required to establish the equivalence of his results to the others.

The results obtained by Prentice (1967, 1968), Hakim and Mangeney (1968, 1971) and some earlier authors cited by them may be related to the foregoing by firstly writing $T_n(z,\rho)$ in the form (cf. equation 30 with $\phi = m\sinh\chi$, $\gamma = \cosh\chi$ and $v = \tanh\chi$)

$$T_n(z,\rho) = -2z\int_0^\infty \mathrm{d}\chi\,\frac{(\cosh\chi)^n\exp(-\rho\cosh\chi)}{(z^2-1)\cosh^2\chi+1}. \tag{40}$$

For example, Hakim and Mangeney (1971) expressed their dispersion relations in terms of the function I_0 and its derivative $\partial I_0/\partial\omega$, with

$$I_0 = \frac{\rho}{2|\boldsymbol{k}|K_2(\rho)}\frac{\partial}{\partial\rho}\left(\frac{T(z,\rho)}{\rho}\right) \tag{41}$$

and $\partial I_0/\partial\omega = |k|^{-1}\,\partial I_0/\partial z$; these functions and also the function defined by Misra (1975) may be revaluated in terms of $T(z,\rho)$ and $\partial T(z,\rho)/\partial z$ using the results given in Appendix 1. A lengthy calculation then shows that Hakim and Mangeney's dispersion relations are equivalent to (38) and (39).

Silin (1960, 1961) evaluated the imaginary part of $\varepsilon^{\mathrm{t}}(\boldsymbol{k},\omega)$ and then used the Kramers–Kronig relations to derive the real part. The imaginary (Im) part of equation (22) arises from the resonant part of the denominator:

$$\operatorname{Im}\alpha^{\mu\nu}(k) = \pi e^2\int \mathrm{d}^4p\; u^\mu u^\nu\,\delta(ku)\;k^\alpha\frac{\partial F(p)}{\partial p^\alpha}. \tag{42}$$

Then equation (9) implies

$$\operatorname{Im}\alpha^{\mathrm{l}}(k) = \pi e^2\frac{(k\bar{u})^2}{k^2-(k\bar{u})^2}\int \mathrm{d}^4p\,(u\bar{u})^2\,\delta(ku)\,k^\alpha\frac{\partial F(p)}{\partial p^\alpha}, \tag{43}$$

$$\operatorname{Im}\alpha^{\mathrm{t}}(k) = \frac{\pi e^2}{2}\int \mathrm{d}^4p\left(1-\frac{k^2(u\bar{u})^2}{k^2-(k\bar{u})^2}\right)\delta(ku)\,k^\alpha\frac{\partial F(p)}{\partial p^\alpha}. \tag{44}$$

These are evaluated explicitly for the distribution (28) in Appendix 2: the results are

$$\operatorname{Im}\alpha^{\mathrm{l}}(k) = \frac{\omega_{\mathrm{p}}^2}{\mu_0}\frac{\pi z}{K_2(\rho)}\frac{\gamma_0^2-1}{\gamma^2}\left(\tfrac{1}{2}\rho\gamma_0^2+\gamma_0+\frac{1}{\rho}\right)\exp(-\rho\gamma_0), \tag{45}$$

$$\operatorname{Im}\alpha^{\mathrm{t}}(k) = \frac{\omega_{\mathrm{p}}^2}{\mu_0}\frac{\pi z}{2K_2(\rho)}\left(\frac{1}{\gamma_0}+\frac{1}{\rho\gamma_0^2}\right)\exp(-\rho\gamma_0), \tag{46}$$

when $\gamma_0 := (1-z^2)^{-\frac{1}{2}}$ is real, with $\mathrm{Im}\,\alpha^{\mathrm{l}}(k)$ and $\mathrm{Im}\,\alpha^{\mathrm{t}}(k)$ identically zero for $z^2 \geqslant 1$. The Kramers–Kronig relations then imply

$$\alpha^{\mathrm{l}}(k) = \frac{\omega_{\mathrm{p}}^2}{\mu_0}\frac{z}{K_2(\rho)}\int_{-1}^{1}\frac{\mathrm{d}v}{v-z}\frac{\gamma^2-1}{\gamma^2}\left(\tfrac{1}{2}\rho\gamma^2+\gamma+\frac{1}{\rho}\right)\exp(-\rho\gamma), \tag{47}$$

$$\alpha^{\mathrm{t}}(k) = \frac{\omega_{\mathrm{p}}^2}{\mu_0}\frac{z}{2K_2(\rho)}\int_{-1}^{1}\frac{\mathrm{d}v}{v-z}\left(\frac{1}{\gamma}+\frac{1}{\rho\gamma^2}\right)\exp(-\rho\gamma). \tag{48}$$

These integrals may be expressed in terms of the integrals $T_n(z,\rho)$ and thence in terms of $T(z,\rho)$ and $\partial T(z,\rho)/\partial z$ using the results in Appendix 1. In this way one readily establishes that with Silin's (1960, 1961) method equations (36) and (37) are reproduced.

Trubnikov's (1958) forms were obtained from the zero-$\boldsymbol{B}$ limit of the magnetized case. In the present notation his forms become

$$\alpha^{\mathrm{t}}(k) = \frac{\mathrm{i}\,\omega_{\mathrm{p0}}^2}{\mu_0}\frac{\rho}{K_2(\rho)}\int_0^\infty \mathrm{d}\eta\,\frac{K_2(\rho\varpi)}{\varpi^2}, \tag{49}$$

$$\alpha^{\mathrm{t}}(k)-\alpha^{\mathrm{l}}(k) = \frac{\mathrm{i}\,\omega_{\mathrm{p0}}^2}{\mu_0}\frac{\rho^2}{K_2(\rho)}\frac{1}{z^2}\int_0^\infty \mathrm{d}\eta\,\eta^2\frac{K_3(\rho\varpi)}{\varpi^3}, \tag{50}$$

with

$$\varpi^2 := 1-2\mathrm{i}\,\eta+\eta^2(1-z^2)/z^2. \tag{51}$$

Writing $\eta = x+\mathrm{i}\,y$, one may deform the contour of integration to one along which ϖ^2 is real: this is from the origin along the y-axis to $y = z^2/(1-z^2)$, and thence parallel to the x-axis to infinity. The zeros of ϖ^2 lie outside the rectangular region enclosed by the old and new contours. Then changing the y integration to one over

$$u := z-y(1-z^2)/z, \tag{52}$$

one obtains

$$\alpha^{\mathrm{t}}(k) = -\frac{\omega_{\mathrm{p0}}^2}{\mu_0}\frac{\rho}{K_2(\rho)}\left(\frac{z}{1-z^2}\int_0^z \mathrm{d}u\,\frac{K_2(\rho R)}{R^2}+\mathrm{i}\int_0^\infty \mathrm{d}x\,\frac{K_2(\rho\varpi)}{\varpi^2}\right), \tag{53}$$

$$\begin{aligned}\alpha^{\mathrm{t}}(k)-\alpha^{\mathrm{l}}(k) = \frac{\omega_{\mathrm{p}}^2}{\mu_0}\frac{\rho^2}{K_2(\rho)}\Bigg\{&\frac{z}{(1-z^2)^3}\int_0^z \mathrm{d}u\,(u-z)^2\frac{K_3(\rho R)}{R^3}-\frac{2}{1-z^2}\\ &\times\int_0^\infty \mathrm{d}x\,x\frac{K_3(\rho\varpi)}{\varpi^3}+\frac{\mathrm{i}}{z^2}\int_0^\infty \mathrm{d}x\left(x^2-\frac{z^4}{(1-z^2)^2}\right)\frac{K_3(\rho\varpi)}{\varpi^3}\Bigg\},\end{aligned} \tag{54}$$

with

$$R^2 := \frac{1-u^2}{1-z^2}, \qquad \varpi^2 := \frac{1}{1-z^2}+\frac{x^2(1-z^2)}{z^2}. \tag{55}$$

The important other result used in establishing the equivalence to equations (36) and (37) is an alternative form of $T(z,\rho)$ given by equation (A11) of Appendix 1; some other details are summarized in equations (A23)–(A26) in Appendix 3.

5. Discussion and Conclusions

The main results presented in this paper are a manifestly covariant and gauge-invariant treatment of dispersion in an isotropic plasma, and the explicit demonstration of the equivalence of some apparently quite different forms for the dispersion functions of a relativistic thermal electron gas. My motivation in carrying out this investigation is connected with the development of a theory which synthesizes the kinetic theory of plasmas and quantum electrodynamics. The results reported here for an isotropic medium are directly applicable to the quantum case, and the demonstration of the equivalence of existing classical results is a relevant preliminary step to the investigation of the dispersive properties of a relativistic, quantum, thermal electron gas.

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Appendix 1. Plasma Dispersion Functions

The functions $T_n(z,\rho)$ may be defined by either equation (30) or (40):

$$T_n(z,\rho) = \int_{-1}^{1} \frac{\mathrm{d}v}{v-z}\,\gamma^n \exp(-\rho\gamma) = -2z\int_0^\infty \frac{\mathrm{d}\chi\,(\cosh\chi)^n \exp(-\rho\cosh\chi)}{(z^2-1)\cosh^2\chi+1}. \quad \text{(A1)}$$

The former leads trivially to the identities

$$T_{n+m}(z,\rho) = (-)^m(\partial^m/\partial\rho^m)\,T_n(z,\rho), \quad \text{(A2)}$$

and the latter to

$$(1-z^2)\,T_{n+2}(z,\rho) = T_n(z,\rho)+2z\,C_n(\rho), \quad \text{(A3)}$$

with

$$C_n(\rho) := \int_0^\infty d\chi\,(\cosh\chi)^n \exp(-\rho\cosh\chi)$$
$$= (-)^n(\partial^n/\partial\rho^n)\,K_0(\rho), \quad n \geqslant 0 \tag{A4}$$
$$= Ki_{|n|}(\rho), \qquad n \leqslant 0,$$

with $Ki_0(\rho) := K_0(\rho)$ and

$$Ki_n(\rho) := \int_\rho^\infty dz\; Ki_{n-1}(z)\,. \tag{A5}$$

The identity (Abramowitz and Stegun 1965, p. 483)

$$r\,Ki_{r+1}(\rho) = -\rho\,Ki_r(\rho) + (r-1)Ki_{r-1}(\rho) + \rho\,Ki_{r-2}(\rho) \tag{A6a}$$

then implies

$$\rho\,C_{n+2}(\rho) = (n+1)C_{n+1}(\rho) + \rho\,C_n(\rho) + n\,C_{n-1}(\rho)\,. \tag{A6b}$$

On differentiating $T_n(z,\rho)$ with respect to z and partially integrating the result, one finds

$$(1-z^2)\,\partial T_n(z,\rho)/\partial z = -2\rho\,C_{n+1}(\rho) + 2n\,C_n(\rho) + z\{-\rho\,T_{n+1}(z,\rho) + n\,T_n(z,\rho)\}\,. \tag{A7}$$

These results enable one to rederive the following relations for $T(z,\rho) = T_0(z,\rho)$ quoted by Godfrey *et al.* (1975*a*) (after correcting an omission in their version of the first of these):

$$(1-z^2)\frac{\partial^2}{\partial\rho^2}T(z,\rho) = 2z\,K_0(\rho) + T(z,\rho)\,, \tag{A8}$$

$$z(1-z^2)^3\frac{\partial^2}{\partial z^2}T(z,\rho) - (1-z^2)^2(1+2z^2)\frac{\partial T(z,\rho)}{\partial z} - \rho^2 z^3\,T(z,\rho)$$
$$= 2z^2\rho^2\,K_0(\rho) + 2(1-z^2)\rho\,K_1(\rho)\,, \tag{A9}$$

$$z\frac{\partial T(z,\rho)}{\partial\rho} - \frac{1-z^2}{\rho}\frac{\partial T(z,\rho)}{\partial z} = 2K_1(\rho)\,. \tag{A10}$$

Godfrey *et al.* (1975*a*) also quoted the alternative form

$$T(z,\rho) = -\frac{2\rho}{1-z^2}\int_0^z du\,\frac{K_1(\rho R)}{R} + i\pi\exp(-\rho\gamma_0)\,. \tag{A11}$$

with $R^2 := (1-u^2)/(1-z^2)$ and $\gamma_0 := (1-z^2)^{-\frac{1}{2}}$. A standard identity for the Macdonald functions then implies

$$T_1(z,\rho) = \frac{2\rho}{1-z^2}\int_0^z du\;K_0(\rho R) - i\pi\gamma_0\exp(-\rho\gamma_0)\,. \tag{A12}$$

The function considered by Misra (1975) is

$$I := \int_{-\infty}^{\infty} \frac{\mathrm{d}x \exp\{-\rho(1+x^2)^{\frac{1}{2}}\}}{(1+x^2)^{\frac{1}{2}}(x-y)}, \tag{A13}$$

with $y = x^2/(1-z^2)^{\frac{1}{2}}$. The substitution $x = \sinh\chi$ leads to the identification

$$I = (1-z^2)^{\frac{1}{2}} T(z,\rho). \tag{A14}$$

Misra (1975) derived an expansion which is equivalent to

$$\operatorname{Re} T(z,\rho) = \frac{2}{z}\sum_{n=0}^{\infty} \frac{(2n)!}{2^n n!}\left(\frac{1-z^2}{\rho z^2}\right)^n K_n(\rho). \tag{A15}$$

Appendix 2. Evaluation of Integrals

The integrals (32)–(34) may be evaluated in an arbitrary frame, but it is simpler to use the rest frame, where we have $p\bar{u} = m\gamma$ and $pk = \gamma m(\omega - \boldsymbol{k}\,.\,\boldsymbol{v})$. On inserting (28), using (23), evaluating some elementary integrals and converting from an integral over $|\boldsymbol{p}|$ to one over v, and using $\mathrm{d}|\boldsymbol{p}|\,|\boldsymbol{p}|^2 = m^3\,\mathrm{d}v\, v^2\gamma^5$, one finds

$$\int \mathrm{d}^4p\, F(p)\left\{\frac{k^2}{(ku)^2}, \frac{u\bar{u}}{ku}, \left(\frac{u\bar{u}}{ku}\right)^2\right\} = \frac{n\rho}{2K_2(\rho)}\int_0^1 \mathrm{d}v\, v^2\gamma^4 \exp(-\rho\gamma)$$

$$\times\left\{\frac{1-z^2}{\gamma^2 v}\left(\frac{1}{v-z}-\frac{1}{v+z}\right), \frac{z}{v\omega}\ln\frac{v+z}{v-z}, -\frac{z^2}{\omega}\left(\frac{1}{v-z}-\frac{1}{v+z}\right)\right\}. \tag{A16}$$

After a partial integration of the middle integral, all three may be reduced to sums of integrals of the form (A1) and (A4), with

$$\int_0^1 \mathrm{d}v\, \gamma^n \exp(-\rho\gamma) = C_{n-2}(\rho). \tag{A17}$$

After some algebra the final results (32)–(34) are obtained on re-expressing ω and z in invariant forms $\omega = k\bar{u}$ and $z = k\bar{u}/\{(k\bar{u})^2 - k^2\}^{\frac{1}{2}}$.

The derivation of equations (45) and (46) from (43) and (44) is also facilitated by choosing the rest frame. For an arbitrary isotropic distribution we have $\boldsymbol{k}\,.\,\partial f(\boldsymbol{p})/\partial\boldsymbol{p} = (\boldsymbol{k}\,.\,\boldsymbol{v}/v)\,\partial f(\boldsymbol{p})/\partial|\boldsymbol{p}|$, and with $\cos\theta := \boldsymbol{k}\,.\,\boldsymbol{v}/|\boldsymbol{k}|\,v$, equations (43) and (44) with (23) give

$$\operatorname{Im}\alpha^{\mathrm{l}}(k) = -\pi e^2 z^2 2\pi \int_{-1}^{1} \mathrm{d}(\cos\theta)\int_0^{\infty} \mathrm{d}|\boldsymbol{p}|\,|\boldsymbol{p}|^2\delta(\omega - |\boldsymbol{k}|\,v\cos\theta)\frac{\omega}{v}\frac{\partial f(\boldsymbol{p})}{\partial|\boldsymbol{p}|}, \tag{A18}$$

$$\operatorname{Im}\alpha^{\mathrm{t}}(k) = -\frac{\pi e^2}{2}2\pi\int_{-1}^{1}\mathrm{d}(\cos\theta)\int_0^{\infty}\mathrm{d}|\boldsymbol{p}|\,|\boldsymbol{p}|^2\{(1-z^2)-\gamma^{-2}\}$$

$$\times\,\delta(\omega - |\boldsymbol{k}|\,v\cos\theta)(\omega/v)\{\partial f(\boldsymbol{p})/\partial|\boldsymbol{p}|\}. \tag{A19}$$

The $\cos\theta$ integral may be performed over the δ function and the remaining integral converted to one over $\gamma = (1 + |\boldsymbol{p}|^2/m^2)^{\frac{1}{2}}$, with $\gamma > \gamma_0 := (1-z^2)^{-\frac{1}{2}}$ required by the δ function. Then after inserting (cf. the distribution 28),

$$f(\boldsymbol{p}) = \frac{n\rho}{4\pi K_2(\rho)\, m^3}\exp(-\rho\gamma), \tag{A20}$$

one finds

$$\operatorname{Im}\alpha^{\mathrm{l}}(k) = \frac{\omega_{\mathrm{p}}^2}{4\pi}\frac{\pi z^3\rho^2}{2K_2(\rho)}\int_{\gamma_0}^{\infty}\mathrm{d}\gamma\,\gamma^2\exp(-\rho\gamma)\,, \tag{A21}$$

$$\operatorname{Im}\alpha^{\mathrm{t}}(k) = \frac{\omega_{\mathrm{p}}^2}{4\pi}\frac{\pi z\rho^2}{4K_2(\rho)}\int_{\gamma_0}^{\infty}\mathrm{d}\gamma\,(\gamma^2/\gamma_0^2-1)\exp(-\rho\gamma)\,. \tag{A22}$$

The remaining integrals are elementary and equations (45) and (46) follow.

Appendix 3. Reduction of Trubnikov's Forms

The reduction of Trubnikov's (1958) forms (49) and (50) to the forms (36) and (37) involves the steps leading to (53) and (54) and the following. First the u integrals may be evaluated in terms of (A11) or (A12) by using the identities

$$\frac{\partial}{\partial z}\int_0^z \mathrm{d}u\,\frac{K_1(\rho R)}{R} = K_1(\rho) - \frac{z\rho}{1-z^2}\int_0^z \mathrm{d}u\,K_2(\rho R)\,, \tag{A23}$$

$$\int_0^z \mathrm{d}u\,\frac{K_1(\rho R)}{R} = z\,K_1(\rho) + \rho\int_0^{\rho}\mathrm{d}u\,K_2(\rho R) - \frac{\rho}{1-z^2}\int_0^{\rho}\mathrm{d}u\,\frac{K_2(\rho R)}{R^2}\,, \tag{A24}$$

$$\int_0^z \mathrm{d}u\,\frac{K_3(\rho R)}{R} = \int_0^z \mathrm{d}u\,\frac{K_1(\rho R)}{R} + \frac{4}{\rho}\int_0^z \mathrm{d}u\,\frac{K_2(\rho R)}{R^2}\,, \tag{A25}$$

$$\int_0^z \mathrm{d}u\,\frac{K_2(\rho R)}{R^2} = z\,K_2(\rho) - \frac{\rho}{1-z^2}\int_0^z \mathrm{d}u\,u^2\frac{K_3(\rho R)}{R^3}\,, \tag{A26}$$

which follow from recurrence relations for K_ν and from partial integrations.

The x integrals in (53) and (54) may be evaluated in terms of the standard integral (Gradshteyn and Ryzhik 1965, 6.596.4)

$$\int_0^{\infty}\mathrm{d}x\,x^{2\mu+1}\frac{K_\nu(\alpha(x^2+y^2)^{\frac{1}{2}})}{(x^2+y^2)^{\frac{1}{2}\nu}} = \frac{2^{\mu}\Gamma(\mu+1)}{\alpha^{\mu+1}z^{\nu-\mu-1}}K_{\nu-\mu-1}(\alpha y) \tag{A27}$$

for $\mu > 1$. The imaginary terms in (53) and (54) may then be re-expressed in terms of exponentials using (Gradshteyn and Ryzhik 1965, 8.468)

$$K_{n+\frac{1}{2}}(\rho\gamma_0) = \left(\frac{\pi}{2\rho\gamma_0}\right)^{\frac{1}{2}}\exp(-\rho\gamma_0)\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!(2\rho\gamma_0)^k}\,. \tag{A28}$$

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