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Exact Solutions of the Heisenberg Equations and Zitterbewegung of the Electron in a Constant Uniform Magnetic Field

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Abstract

For a free Dirac electron, the Heisenberg equations define an internal dynamical system in the rest frame, isomorphic to a finite three-dimensional oscillator with a compact SO(5) phase space, such that the spin of the electron is the orbital angular momentum of the internal dynamics (Barut and Bracken 1980, 1981*a*). In the present work, the change in this internal dynamics due to an external magnetic field is studied. In order that the internal motion can be distinguished from the centre of mass motion, the solutions of the corresponding Hamilton and Heisenberg equations for the relativistic classical motion and the relativistic quantum mechanical spinless motion are also presented. The solutions for the electron exhibit the effect of the spin terms both in the internal motion and external motion, and we are able to identify the properties of the Zitterbewegung in the external field.

1. Introduction: External and Internal Dynamical Variables

In earlier work, we have presented an interpretation of Dirac's equation for the free electron (Barut and Bracken 1980, 1981*a*). It is the relativistic wave equation describing a quantum system with the internal dynamics of a compact three-dimensional quantum oscillator—the Zitterbewegung (Schrödinger 1930; for other work and references on the Zitterbewegung see Barut and Bracken 1981*a* and Guth 1962). The energy and the orbital angular momentum of this oscillator define the rest mass energy and spin of the electron, that is, of the system as a whole, in the rest frame of its centre of mass (c.m.).

There are two coordinate variables $\boldsymbol{x}, \boldsymbol{Q}$ and two momentum variables $\boldsymbol{p}, \boldsymbol{P}$ involved in the description. All four are Hermitian operators on a Hilbert space. The operator $\boldsymbol{x}$ is interpreted as the coordinate of the charge, and $\boldsymbol{p}$ as the momentum of the c.m. While the existence of the second set of variables $(\boldsymbol{Q}, \boldsymbol{P})$ reflects the fact that there are internal dynamics as well as c.m. dynamics, the interpretation of $\boldsymbol{Q}$ and $\boldsymbol{P}$ has been made clear only for the c.m. frame (Barut and Bracken 1981*a*), defined by $\boldsymbol{p} = \boldsymbol{0}$. There $\boldsymbol{Q}$ and $\boldsymbol{P}$ represent the position and momentum of the charge relative to the c.m.

One of our objectives here is to identify the appropriate c.m. (or external) and relative (or internal) dynamical variables in an arbitrary frame, that is, with no constraint on $\boldsymbol{p}$. But our main aim is to learn something of the behaviour of the system—in particular of the change in the Zitterbewegung, or internal dynamics—when placed in an external electromagnetic field. Because it is the coordinate $\boldsymbol{x}$ of the charge that performs the Zitterbewegung, radiation by the electron is intimately associated with this motion in general, and we would like to understand its radiative and non-radiative modes. We consider what is perhaps the simplest case mathematically, that of a constant and uniform magnetic field.

The dynamical behaviour is most clearly exhibited in the Heisenberg picture, as Schrödinger's (1930) analysis showed in the case of the free electron. Therefore we need to integrate the Heisenberg equations of motion in the presence of the magnetic field. While it is well known that Dirac's equation is exactly soluble in this case (Rabi 1928; Plesset 1930; Huff 1931; Johnson and Lippmann 1949, 1950; Sokolov and Ternov 1953; Jannussis 1966), it appears that the integration of the Heisenberg equations has not been given explicitly before, although Schwinger (1951) solved related equations, expressed in terms of a proper-time variable, and derived from a second-order 'Hamiltonian' (see also Tsai 1978).

Johnson and Lippmann (1949, 1950) discussed the Heisenberg equations of motion for this problem, and obtained a set of constants of the motion. But they did not complete the integration of the equations for the basic non-constant variables. Even when the energy eigenvectors and eigenvalues and a complete set of constants of the motion are known for a system, it is not trivial to obtain in closed form, as functions of the time, the expressions for non-constant variables (supposing such closed forms exist). This is already clear in the case of the free electron (Schrödinger 1930).

Apart from determining the behaviour of the internal dynamics of the electron in the presence of the magnetic field, the integration of the Heisenberg equations in this case is also of interest if one is concerned only with the behaviour of the external (or c.m.) dynamics. The differences between this behaviour and that of a classical (relativistic) point charge in the same field are better illuminated if one works in the Heisenberg rather than the Schrödinger picture in the case of the quantum system (although some care must be exercised not to interpret too literally, in classical terms, formal expressions in the Heisenberg picture involving non-commuting operators). It is also of interest to compare the quantum mechanical behaviour of the electron with that of a spinless particle in the same field, to identify any spin effects on the motion of the c.m. We could not find in the literature the solution of the Heisenberg equations for a spinless charged particle in a constant and homogeneous magnetic field, and so we also present such a solution below.

Many authors have discussed the Heisenberg equations for the electron in the case of a general external electromagnetic field, without presenting solutions. Bunge (1955) and Corben (1961) in particular have attempted to interpret the equations in terms of external and internal dynamics, as we wish to do. Both these authors took $\boldsymbol{x}-\boldsymbol{Q}$ (in our notation) to represent the mean position (or c.m.) of the electron, but according to our ideas, this is only tenable for the free electron, and then only with $\boldsymbol{p} = \boldsymbol{0}$, since $\dot{\boldsymbol{x}}-\dot{\boldsymbol{Q}}$ is otherwise not free from the highly oscillatory time dependence associated with the Zitterbewegung.

Feynman (1962) has remarked of some of the Heisenberg equations for motion in a general field that 'their meaning is not yet completely understood, if at all'. It seems to us that this statement remains true today. But if all variables in the description of the electron, including the rather mysterious 'matrices' of Dirac, can be given a dynamical interpretation in terms of a c.m. motion and an internal motion, then the possibility arises of understanding the meaning of the Heisenberg equations, and perhaps the structure of the electron.

The operator algebras associated with the pairs $(\boldsymbol{x},\boldsymbol{p})$ and $(\boldsymbol{Q},\boldsymbol{P})$ are best compared if one considers side by side the Lie algebras spanned by the sets

$$(\boldsymbol{x},\boldsymbol{p},\boldsymbol{L},I) \quad \text{and} \quad (\boldsymbol{Q},\boldsymbol{P},\boldsymbol{S},-\beta)\,. \tag{1a, b}$$

Here I is the unit operator, and $-\beta$ is its analogue for the compact SO(5) algebra generated by $\boldsymbol{Q}$ and $\boldsymbol{P}$, while $\boldsymbol{L}$ and $\boldsymbol{S}$ are the respective angular momentum operators. In the Heisenberg picture, at any one time t, any variable from the first set (1a) commutes with any one from the second set (1b), and

$$\begin{gathered}
[x_i,p_j] = \mathrm{i}\hbar\delta_{ij}I\,,\\
[x_i,x_j] = 0\,, \qquad [p_i,p_j] = 0\,,\\
[L_i,x_j] = \mathrm{i}\hbar\varepsilon_{ijk}x_k\,, \qquad [L_i,p_j] = \mathrm{i}\hbar\varepsilon_{ijk}p_k\,,\\
[L_i,L_j] = \mathrm{i}\hbar\varepsilon_{ijk}L_k\,,\\
[I,x_i] = 0\,, \qquad [I,p_i] = 0\,, \qquad [I,L_i] = 0\,,
\end{gathered} \tag{2}$$

while

$$\begin{gathered}
[Q_i,P_j] = \mathrm{i}\hbar\delta_{ij}(-\beta)\,,\\
[Q_i,Q_j] = (\mathrm{i}\lambda^2/\hbar)\varepsilon_{ijk}S_k\,, \qquad [P_i,P_j] = (\mathrm{i}\hbar/\lambda^2)\varepsilon_{ijk}S_k\,,\\
[S_i,Q_j] = \mathrm{i}\hbar\varepsilon_{ijk}Q_k\,, \qquad [S_i,P_j] = \mathrm{i}\hbar\varepsilon_{ijk}P_k\,,\\
[S_i,S_j] = \mathrm{i}\hbar\varepsilon_{ijk}S_k\,,\\
[\beta,Q_i] = -(\mathrm{i}\lambda^2/\hbar)P_i\,, \qquad [\beta,P_i] = (4\mathrm{i}\hbar/\lambda^2)Q_i\,, \qquad [\beta,S_i] = 0\,.
\end{gathered} \tag{3}$$

In addition, certain constitutive and representation relations hold, in particular

$$\begin{gathered}
\boldsymbol{L} = \boldsymbol{x}\wedge\boldsymbol{p}\,, \qquad \boldsymbol{S} = \tfrac{1}{2}\beta\boldsymbol{Q}\wedge\boldsymbol{P}\,, \qquad \beta^2 = I\,,\\
\boldsymbol{Q}\beta = -\beta\boldsymbol{Q} = \tfrac{1}{2}\mathrm{i}(\lambda^2/\hbar)\boldsymbol{P}\,, \qquad \boldsymbol{P}\beta = -\beta\boldsymbol{P} = -2\mathrm{i}(\hbar/\lambda^2)\boldsymbol{Q}\,,\\
\{Q_i,Q_j\} = \tfrac{1}{2}\lambda^2\delta_{ij}I\,, \qquad \{P_i,P_j\} = (2\hbar^2/\lambda^2)\delta_{ij}I\,.
\end{gathered} \tag{4}$$

In equations (3) and (4), λ is a constant with dimensions of length, being the Compton wavelength of the free electron, whose rest mass is accordingly

$$m = \hbar/\lambda c\,. \tag{5}$$

But the primary significance of λ from our point of view is that it is the constant which characterizes the curvature of the phase space associated with the variables $(\boldsymbol{P},\boldsymbol{Q})$, through equations (3) in particular. Its value subsequently determines, in the rest frame of the c.m. of the free electron, the frequency ω_Z $(=2c/\lambda)$ and energy eigenvalues $\pm\frac{1}{2}\hbar\omega_Z$ of the internal compact oscillator, and those energy values are then the rest energies of the electron, in accordance with equation (5) (Barut and Bracken 1981*a*).

Dirac's Hamiltonian for the electron in a time-independent magnetic field $\boldsymbol{B}(\boldsymbol{x})$ $\big(=\mathrm{curl}\,\boldsymbol{A}(\boldsymbol{x})\big)$ is, in these terms,

$$H = (\lambda c/\hbar)\boldsymbol{P}\cdot\boldsymbol{\pi} + (\hbar c/\lambda)\beta\,, \tag{6}$$

where

$$\boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}\,. \tag{7}$$

Noting the relations (4), we can also write

$$H = (\lambda c/\hbar)\boldsymbol{P}\cdot\boldsymbol{\pi} + \frac{2\hbar c}{3\lambda}\left(\frac{\boldsymbol{Q}}{\lambda} - \frac{\mathrm{i}\lambda}{2\hbar}\boldsymbol{P}\right)\cdot\left(\frac{\boldsymbol{Q}}{\lambda} + \frac{\mathrm{i}\lambda}{2\hbar}\boldsymbol{P}\right) - \frac{\hbar c}{\lambda}I\,, \tag{8}$$

which shows suggestively the internal harmonic oscillator dynamics in a factorized form.

Using this Hamiltonian to determine the time derivatives of variables, we find in particular

$$\begin{aligned} \dot{\boldsymbol{x}} &= (\lambda c/\hbar)\boldsymbol{P}, & \dot{\boldsymbol{\pi}} &= (e\lambda c/\hbar)\boldsymbol{P}\wedge\boldsymbol{B}, \\ \dot{\boldsymbol{Q}} &= (\lambda c/\hbar)(\boldsymbol{P} - \beta\boldsymbol{\pi}), & \dot{\boldsymbol{P}} &= (4c/\hbar\lambda^3)(\lambda^2\boldsymbol{\pi}\wedge\boldsymbol{S} - \hbar^2\boldsymbol{Q}), \\ \dot{\beta} &= (4c/\lambda\hbar)\boldsymbol{Q}\cdot\boldsymbol{\pi}, & \dot{\boldsymbol{S}} &= (\lambda c/\hbar)\boldsymbol{\pi}\wedge\boldsymbol{P}, \end{aligned} \tag{9}$$

indicating a complicated dynamical coupling of the two sets of variables (1). We can also see with the help of the anticommutation relations in (4) that

$$\mathrm{i}\hbar\dot{\boldsymbol{Q}} = -2H\boldsymbol{Q} + (\lambda c/\hbar)\beta\boldsymbol{\pi}\wedge\boldsymbol{S}, \tag{10a}$$

$$\mathrm{i}\hbar\dot{\boldsymbol{P}} = -2H\boldsymbol{P} + (2\hbar c/\lambda)\boldsymbol{\pi}, \tag{10b}$$

$$\mathrm{i}\hbar\dot{\beta} = -2H\beta + (2\hbar c/\lambda)I. \tag{10c}$$

The last of these (10c) can be integrated at once, since H is a constant here, to give

$$\beta(t) = (\hbar c/\lambda)H^{-1} + \exp(2\mathrm{i}\,Ht/\hbar)\{\beta(0) - (\hbar c/\lambda)H^{-1}\}\,, \tag{11}$$

whatever the spatial variation of the stationary magnetic field.

It is of course possible to introduce the dimensionless variables

$$\boldsymbol{\alpha} = (\lambda/\hbar)\boldsymbol{P} = (2\mathrm{i}/\lambda)\beta\boldsymbol{Q}\,, \tag{12}$$

so that H of equation (6) assumes the familiar form in terms of the well-known variables $(\boldsymbol{\alpha}, \beta)$ first introduced by Dirac. Using equations (4), we can write all of $\boldsymbol{Q}$, $\boldsymbol{P}$ and $\boldsymbol{S}$ in terms of $\boldsymbol{\alpha}$ and β, which have representations in terms of 4×4 matrices. From our point of view, the introduction of $\boldsymbol{\alpha}$ in this way obscures the dynamical significance of the set of variables (1b). However, for manipulative purposes, the variables $(\boldsymbol{\alpha}, \beta)$ are more familiar and convenient, and in Section 5 we shall work with them and also the other familiar derived quantities

$$\gamma_5 = -\mathrm{i}\,\alpha_1\alpha_2\alpha_3\,, \qquad \boldsymbol{\sigma} = (2/\hbar)\boldsymbol{S}, \tag{13a, b}$$

translating results into terms of the variables (1) only in Section 6.

2. Centre of Mass and Relative Variables for Free Moving Electron

When $A = \mathbf{0}$, we can follow Schrödinger (1930) to obtain

$$x(t) = x_A(t)+\xi(t), \tag{14}$$

where

$$x_A(t) = x_A(0)+c^2H^{-1}pt, \tag{15a}$$

$$\begin{aligned}\xi(t) &= \exp(2\mathrm{i}\,Ht/\hbar)\,\xi(0) = \xi(0)\exp(-2\mathrm{i}\,Ht/\hbar)\\ &= (\hbar c/\lambda)Q(t)\,\beta(t)\,H^{-1}-\tfrac{1}{2}\mathrm{i}\,\hbar c^2pH^{-2};\end{aligned} \tag{15b}$$

$$p = p(0), \tag{16a}$$

$$x_A(0) = x(0)-\xi(0), \tag{16b}$$

$$\xi(0) = (\hbar c/\lambda)Q(0)\,\beta(0)\,H^{-1}-\tfrac{1}{2}\mathrm{i}\,\hbar c^2pH^{-2}. \tag{16c}$$

We interpret x as the position of the charge and x_A as the position of the c.m., which has momentum p (Barut and Bracken 1981*a*). Then the position of the charge relative to the c.m. is given by

$$Q^{\mathrm{rel}}(t) = \xi(t). \tag{17}$$

It describes the Zitterbewegung, and has the associated highly oscillatory time dependence. From equations (4) and (6) one can see that when $p = \mathbf{0}$, Q^{rel} does indeed reduce to Q.

Turning to the momentum P^{rel} of the charge relative to the c.m., we recall that we have identified it as P when $p = \mathbf{0}$ (Barut and Bracken 1981*a*), and then

$$P^{\mathrm{rel}} = (\hbar/\lambda c)\dot{Q}^{\mathrm{rel}} = (\hbar/\lambda c)\dot{x}. \tag{18}$$

The simplest possibility consistent with that choice and proportional to the relative velocity of the charge is

$$P^{\mathrm{rel}} = (\hbar/\lambda c)\dot{Q}^{\mathrm{rel}} = (-2\mathrm{i}/\lambda c)Q^{\mathrm{rel}}H = P(t)-(\hbar c/\lambda)H^{-1}p, \tag{19}$$

taking into account equations (15) and (4).

Then we have from equation (10b)

$$\dot{P}^{\mathrm{rel}} = -(2\mathrm{i}/\hbar)P^{\mathrm{rel}}H, \tag{20}$$

and we see that

$$\ddot{Q}^{\mathrm{rel}}+\{4(c^2p^2+m^2c^4)/\hbar^2\}Q^{\mathrm{rel}} = \mathbf{0}, \tag{21a}$$

$$\ddot{P}^{\mathrm{rel}}+\{4(c^2p^2+m^2c^4)/\hbar^2\}P^{\mathrm{rel}} = \mathbf{0}, \tag{21b}$$

so that the relative motion is harmonic with angular frequency ω determined by

$$\omega^2 = 4(c^2p^2+m^2c^4)/\hbar^2. \tag{22}$$

(We can treat p as a c-number here as it commutes with Q^{rel}, P^{rel} and H.)

The commutation relations satisfied by the variables $(\boldsymbol{x}_A, \boldsymbol{p})$ and $(\boldsymbol{Q}^{\rm rel}, \boldsymbol{P}^{\rm rel})$ are not simple, apart from

$$[x_{Ai}, p_j] = \mathrm{i}\hbar\delta_{ij} I, \qquad [Q_i^{\rm rel}, p_j] = 0, \qquad [P_i^{\rm rel}, p_j] = 0. \tag{23a, b, c}$$

For example, we have

$$[x_{Ai}, x_{Aj}] = -\mathrm{i}\hbar c^2 H^{-2}\varepsilon_{ijk} S_{Ak} = -[Q_i^{\rm rel}, Q_j^{\rm rel}], \tag{24}$$

where $\boldsymbol{S}_A$ is the constant part of the spin operator $\boldsymbol{S}$ (Schrödinger 1930; Barut and Bracken 1981*a*):

$$\boldsymbol{S}_A = (\lambda/2\hbar c)\boldsymbol{Q}^{\rm rel} \wedge \boldsymbol{P}^{\rm rel} H. \tag{25}$$

Furthermore, we have

$$[Q_i^{\rm rel}, P_j^{\rm rel}] = \mathrm{i}\hbar(\delta_{ij} I - c^2 p_i p_j H^{-2})\{-(\hbar c/\lambda) H^{-1}\}. \tag{26}$$

In particular, it is only in the rest frame of the c.m. that we see explicitly the compact SO(5) structure underlying the internal dynamics. Before treating the electron in a constant and uniform magnetic field, we consider the corresponding problems for a classical particle and a scalar quantum particle, in order that we can properly identify the relative motion in the case of the electron.

3. Classical Relativistic Motion in a Constant Uniform Magnetic Field

We take the field and vector potential at $\boldsymbol{x}$ to be

$$\boldsymbol{B} = (0, 0, B), \qquad B \text{ const.}, \tag{27a}$$

$$\boldsymbol{A} = \tfrac{1}{2}\boldsymbol{B} \wedge \boldsymbol{x} = \tfrac{1}{2}B(-x_2, x_1, 0). \tag{27b}$$

The classical relativistic Hamiltonian for a particle of rest mass m and charge e is

$$H = c\{(\boldsymbol{p} - e\boldsymbol{A})^2 + m^2c^2\}^{\frac{1}{2}} \tag{28a}$$

$$= c\{(p_1 + \theta x_2)^2 + (p_2 - \theta x_1)^2 + p_2^3 + m^2c^2\}^{\frac{1}{2}}, \tag{28b}$$

where $\theta = \frac{1}{2}eB$ and $\boldsymbol{p}$ is the canonical momentum conjugate to $\boldsymbol{x}$. The first set of Hamilton's equations gives

$$\dot{x}_\pm = c^2(p_\pm \mp \mathrm{i}\theta x_\pm)H^{-1}, \qquad \dot{x}_3 = c^2 p_3 H^{-1}, \tag{29a, b}$$

where $x_\pm = x_1 \pm \mathrm{i}x_2$ and $p_\pm = p_1 \pm \mathrm{i}p_2$, while the second set gives

$$\dot{p}_\pm = \mp \mathrm{i}c^2\theta(p_\pm \mp \mathrm{i}\theta x_\pm)H^{-1}, \qquad \dot{p}_3 = 0. \tag{30a, b}$$

Since H is constant, equations (29) and (30) are readily solved to give

$$x_\pm(t) = \tfrac{1}{2}\{x_\pm(0) \mp \mathrm{i}\theta^{-1}p_\pm(0)\}$$
$$+ \tfrac{1}{2}\exp(\mp 2\mathrm{i}\,\theta c^2 H^{-1} t)\{x_\pm(0) \pm \mathrm{i}\theta^{-1}p_\pm(0)\}, \tag{31a}$$

$$p_{\pm}(t) = \tfrac{1}{2}\{p_{\pm}(0) \pm \mathrm{i}\,\theta x_{\pm}(0)\}$$
$$+\tfrac{1}{2}\exp(\pm 2\mathrm{i}\,\theta c^2 H^{-1} t)\{p_{\pm}(0) \mp \mathrm{i}\,\theta x_{\pm}(0)\}, \tag{31b}$$

$$x_3(t) = c^2 p_3 H^{-1} t + x_3(0), \qquad p_3(t) = p_3(0). \tag{31c, d}$$

For the components of the gauge-independent kinetic momentum operator $\boldsymbol{\pi}$ of equation (7), we then have

$$\pi_{\pm}(t) = p_{\pm}(t) \mp \mathrm{i}\,\theta x_{\pm}(t) = \exp(\mp 2\mathrm{i}\,\theta c^2 H^{-1} t)\,\pi_{\pm}(0), \tag{32a}$$

$$\pi_3(t) = p_3(t) = \pi_3(0). \tag{32b}$$

We note that

$$p_{\pm}(t) \pm \mathrm{i}\,\theta x_{\pm}(t) = p_{\pm}(0) \pm \mathrm{i}\,\theta x_{\pm}(0), \tag{33}$$

so that $p_{\pm} \pm \mathrm{i}\,\theta x_{\pm}$, like p_3 and H, are also constants of the motion. Another constant is the third component of the canonical angular momentum vector in this gauge:

$$L_3 = x_1 p_2 - x_2 p_1 = \tfrac{1}{2}\mathrm{i}(x_+ p_- - x_- p_+). \tag{34}$$

(In other gauges L_3 as defined would not be constant; see Tassie and Buchdahl 1964 for a general discussion.) However, only four of these five constants are functionally independent, because

$$H^2 - c^2(p_+ + \mathrm{i}\,\theta x_+)(p_- - \mathrm{i}\,\theta x_-) - c^2 p_3^2 - m^2 c^4 + 4\theta c^2 L_3 = 0. \tag{35}$$

It is convenient (cf. Landau 1930) to introduce, in place of (x_1, p_1) and (x_2, p_2), the canonically conjugate pairs (x_L, p_L) and (X_L, P_L), where

$$x_L = \tfrac{1}{2}(x_2 - \theta^{-1} p_1), \qquad p_L = p_2 + \theta x_1, \tag{36a}$$

$$X_L = \tfrac{1}{2}(x_2 + \theta^{-1} p_1), \qquad P_L = p_2 - \theta x_1. \tag{36b}$$

Then we get

$$H = c(P_L^2 + 4\theta^2 X_L^2 + p_3^2 + m^2 c^2)^{\frac{1}{2}}. \tag{37}$$

The variables x_L and p_L can at once be seen to be constants and, with p_3 and H, form a convenient set of functionally independent constants. The constants of equations (33) can be expressed in terms of (x_L, p_L) as

$$p_+ + \mathrm{i}\,\theta x_+ = -2\theta x_L + \mathrm{i}\,p_L, \qquad p_- - \mathrm{i}\,\theta x_- = -2\theta x_L - \mathrm{i}\,p_L, \tag{38a, b}$$

and then L_3 can be expressed in terms of x_L, p_L, p_3 and H via (35).

The motion described by equations (31) is helical about the fixed field line which has x_1 and x_2 coordinates equal to $p_L/2\theta$ and $-x_L$ respectively. The radius r of the helix is constant and given by

$$r = (H^2 - c^2 p_3^2 - m^2 c^4)^{\frac{1}{2}}/2c\,|\theta|. \tag{39}$$

The pitch of the helix (i.e. the distance between successive windings) is constant and equal to $\pi\,|p_3/\theta|$. The particle moves with constant speed v given by

$$v = c(1 - m^2 c^4/H^2)^{\frac{1}{2}}, \tag{40}$$

and the speed parallel to the field $v_{\parallel}$ and the speed perpendicular to the field $v_{\perp}$ are also constant, given by

$$v_{\parallel} = c^2 p_3 H^{-1} \operatorname{sign}(B), \qquad v_{\perp} = cH^{-1}(H^2 - c^2 p_3^2 - m^2 c^4)^{\frac{1}{2}}. \tag{41a,b}$$

The angular frequency ω is constant:

$$\omega = 2|\theta| c^2 H^{-1} = v_{\perp}/r, \tag{42}$$

which is smaller in general than the nonrelativistic angular frequency $|eB|/m$ $(=2|\theta|/m)$, since equations (28) imply $H \geqslant mc^2$. The motion is clockwise (anti-clockwise) about the direction of the field if e is negative (positive).

4. Motion of Scalar Quantum Particle in Same Field

The Hamiltonian operator H is taken to be given by equations (28), where $\boldsymbol{x}$ and $\boldsymbol{p}$ are now Hermitian operators satisfying the canonical commutation relations at any one time. (We work in the Heisenberg picture.) Then H is Hermitian and $H \geqslant mc^2$.

For a general vector potential $\boldsymbol{A}(\boldsymbol{x})$, it is not clear how one could evaluate the commutators of $\boldsymbol{x}$ and $\boldsymbol{p}$ with H as in equation (28a), in order to determine the Heisenberg equations of motion. In the case at hand, one can proceed by making the change of canonical variables as in equations (36), leading to H as in (37).

Then it is clear that x_L, p_L, p_3 and H are constants, just as in the classical case. Equations (38) and hence (33) remain valid, and L_3 in (34) is also constant, but (35) is now replaced by

$$H^2 - c^2(p_L^2 + 4\theta^2 x_L^2 + p_3^2 + m^2 c^2) + 4\theta c^2 L_3 = 0. \tag{43}$$

Now H in equation (37) can be written in the form

$$H = c\{4|\theta|\hbar(a^\dagger a + \tfrac{1}{2}) + p_3^2 + m^2 c^2\}^{\frac{1}{2}}, \tag{44}$$

where $a^\dagger$ and a are the usual (boson) raising and lowering operators for the 'number operator' $N\ (=a^\dagger a)$:

$$a = \tfrac{1}{2}|\hbar\theta|^{-\frac{1}{2}}(P_L - 2\mathrm{i}|\theta|X_L), \qquad a^\dagger = \tfrac{1}{2}|\hbar\theta|^{-\frac{1}{2}}(P_L + 2\mathrm{i}|\theta|X_L), \tag{45a}$$

$$Na = a(N-1), \qquad Na^\dagger = a^\dagger(N+1). \tag{45b}$$

Then it follows that

$$aH = c\{4|\theta|\hbar(N + \tfrac{3}{2}) + p_3^2 + m^2 c^2\}^{\frac{1}{2}} a = Ka; \tag{46a}$$

$$K = (H^2 + 4|\theta|\hbar c^2)^{\frac{1}{2}}. \tag{46b}$$

From the definition (45a) of a we have then

$$(P_L - 2\mathrm{i}|\theta|X_L)H = K(P_L - 2\mathrm{i}|\theta|X_L), \tag{47}$$

so that

$$\begin{aligned} \mathrm{i}\hbar(\dot{P}_L - 2\mathrm{i}|\theta|\dot{X}_L) &= [(P_L - 2\mathrm{i}|\theta|X_L), H] \\ &= (K - H)(P_L - 2\mathrm{i}|\theta|X_L). \end{aligned} \tag{48}$$

Since K is a function of H and hence, like H itself, is a constant, we have at once

$$P_L(t) - 2\mathrm{i}\,|\theta|\,X_L(t) = \exp\{\mathrm{i}\hbar^{-1}(H-K)t\}\{P_L(0) - 2\mathrm{i}\,|\theta|\,X_L(0)\}. \tag{49}$$

Similarly (or by Hermitian conjugation) we have

$$P_L(t) + 2\mathrm{i}\,|\theta|\,X_L(t) = \{P_L(0) + 2\mathrm{i}\,|\theta|\,X_L(0)\}\exp\{-\mathrm{i}\hbar^{-1}(H-K)t\}. \tag{50}$$

If θ is positive, equations (33), (36), (49) and (50) imply

$$x_+(t) = \tfrac{1}{2}\{x_+(0) - \mathrm{i}\,\theta^{-1}p_+(0)\} + \tfrac{1}{2}\exp\{\mathrm{i}\hbar^{-1}(H-K)t\}\{x_+(0) + \mathrm{i}\,\theta^{-1}p_+(0)\}, \tag{51a}$$

$$p_+(t) = \tfrac{1}{2}\{p_+(0) + \mathrm{i}\,\theta x_+(0)\} + \tfrac{1}{2}\exp\{\mathrm{i}\hbar^{-1}(H-K)t\}\{p_+(0) - \mathrm{i}\,\theta x_+(0)\}, \tag{51b}$$

$$x_-(t) = \tfrac{1}{2}\{x_-(0) + \mathrm{i}\,\theta^{-1}p_-(0)\} + \tfrac{1}{2}\{x_-(0) - \mathrm{i}\,\theta^{-1}p_-(0)\}\exp\{-\mathrm{i}\hbar^{-1}(H-K)t\}, \tag{51c}$$

$$p_-(t) = \tfrac{1}{2}\{p_-(0) - \mathrm{i}\,\theta x_-(0)\} + \tfrac{1}{2}\{p_-(0) + \mathrm{i}\,\theta x_-(0)\}\exp\{-\mathrm{i}\hbar^{-1}(H-K)t\}. \tag{51d}$$

We also have

$$\mathrm{i}\hbar\dot{x}_3 = [x_3, H] = \mathrm{i}\hbar c^2 p_3 H^{-1}, \tag{52}$$

where p_3 is constant, so that

$$x_3(t) = c^2 p_3 H^{-1} t + x_3(0), \qquad p_3(t) = p_3(0). \tag{53a, b}$$

Equations (51) and (53) are to be compared with the classical results (31). Note that the operator appearing in the exponent in (51) is

$$\begin{aligned}\hbar^{-1}(K-H) &= \hbar^{-1}\{(H^2 + 4|\theta|\hbar c^2)^{\frac{1}{2}} - H\}\\ &= \hbar^{-1}H\{(1+4|\theta|\hbar c^2 H^{-2})^{\frac{1}{2}} - 1\}\\ &= 2|\theta|c^2H^{-1} - 2\hbar\theta^2c^4H^{-3} + \ldots,\end{aligned} \tag{54}$$

which is to be compared with the classical angular frequency $2|\theta|c^2H^{-1}$ in (31a) and (31b). The expansion in (54) is only valid if

$$4|\theta|\hbar c^2 H^{-2} < 1, \tag{55}$$

and, since $H \geqslant mc^2$, a sufficient condition is

$$4|\theta|\hbar/m^2c^2 < 1, \quad \text{i.e.} \quad \hbar\,|eB|/m < \tfrac{1}{2}mc^2. \tag{56}$$

For the spin 0 pion, this condition is satisfied for $|B| < 1{\cdot}6 \times 10^{18}$ G, which is certainly true for fields produced in the laboratory where $|B| \lesssim 10^6$ G (Garstang 1977). Then the second term in the series (54) is extremely small compared with the leading term, while higher order terms are completely negligible.

It would appear from a comparison of equations (31) and (51) that the principal quantum effect is to change the angular frequency of the helical motion at a given energy from $2|\theta|c^2H^{-1}$ to $\hbar^{-1}(K-H)$ ($\approx 2|\theta|c^2H^{-1}-2\hbar\theta^2c^4H^{-3}$ for weak fields). However, such reasoning is misleading. In the first place, when the energy is definite the system is in a stationary state and the expectation values of all the variables x_+, x_- etc. are constant. We need to consider a superposition of states corresponding to different energy values if we are to see any non-stationary effects at all, and in such a superposition the interpretation of equations (51) is not so clear. In the second place, we can easily obtain, in addition to (47), the result

$$H(P_L-2\mathrm{i}\,|\theta|\,X_L) = (P_L-2\mathrm{i}\,|\theta|\,X_L)G\,; \tag{57a}$$

$$G = (H^2-4|\theta|\hbar c^2)^{\frac{1}{2}}, \tag{57b}$$

and thence for $\theta > 0$ that, in place of equation (51a) for example,

$$\begin{aligned} x_+(t) &= \tfrac{1}{2}\{x_+(0)-\mathrm{i}\,\theta^{-1}p_+(0)\} \\ &\quad +\tfrac{1}{2}\{x_+(0)+\mathrm{i}\,\theta^{-1}p_+(0)\}\exp\{\mathrm{i}\hbar^{-1}(G-H)t\}\,. \end{aligned} \tag{58}$$

Now for weak fields in particular we have

$$\begin{aligned} \hbar^{-1}(H-G) &= 2|\theta|c^2H^{-1}+2\hbar\theta^2c^4H^{-3}+\ldots \\ &\neq \hbar^{-1}(K-H)\,. \end{aligned} \tag{59}$$

Thus a naive interpretation of the results (51) is that the frequency of the motion is *decreased*, while the same reasoning applied to the results in a form like (58) would suggest that the frequency is *increased*. This highlights the dangers of trying to interpret results like (51) too literally in classical terms.

We prefer to write the solutions of the Heisenberg equations in the form (51) rather than (58), because K is always a well-defined Hermitian operator, whereas G is not when the condition (56) is violated (that is, for very strong fields). However, for reasons of notational and manipulative convenience, it is advantageous to work formally with (in place of G and K)

$$H_\pm = (H^2\pm 4\theta\hbar c^2)^{\frac{1}{2}}, \tag{60}$$

whatever the sign of θ. Then equations (51) can be replaced by (for either sign of θ)

$$\begin{aligned} x_\pm(t) &= \tfrac{1}{2}\{x_\pm(0)\mp\mathrm{i}\,\theta^{-1}p_\pm(0)\} \\ &\quad +\tfrac{1}{2}\exp\{\mathrm{i}\hbar^{-1}(H-H_\pm)t\}\{x_\pm(0)\pm\mathrm{i}\,\theta^{-1}p_\pm(0)\}, \end{aligned} \tag{61a}$$

$$\begin{aligned} p_\pm(t) &= \tfrac{1}{2}\{p_\pm(0)\pm\mathrm{i}\,\theta x_\pm(0)\} \\ &\quad +\tfrac{1}{2}\exp\{\mathrm{i}\hbar^{-1}(H-H_\pm)t\}\{p_\pm(0)\mp\mathrm{i}\,\theta x_\pm(0)\}\,. \end{aligned} \tag{61b}$$

For the kinetic momentum we then have from equations (61) and (53)

$$\pi_\pm(t) = \exp\{\mathrm{i}\hbar^{-1}(H-H_\pm)t\}\,\pi_\pm(0), \qquad \pi_3(t) = \pi_3(0), \tag{62a, b}$$

to be compared with (32).

The basic dynamical variables in this problem are most conveniently taken to be x_L, p_L, x_3, p_3, a and $a^\dagger$. Eigenvectors of $N\,(=a^\dagger a)$ can be constructed by introducing a normalized vacuum vector $|\,0\rangle$, which is annihilated by $a(0)$:

$$|\,n\rangle = (n!)^{-\frac{1}{2}}\{a^\dagger(0)\}^n|\,0\rangle, \qquad n = 0,1,2,\ldots. \tag{63}$$

These vectors $|\,n\rangle$ must be labelled also by eigenvalues of some other operators (such as p_L and p_3) which together with N form a complete set of commuting operators. Thus we can define vectors $|\,n,k_L,k_3\rangle$, where $n = 0,1,2,\ldots,\; -\infty < k_L,k_3 < \infty$, with

$$p_L\,|\,n,k_L,k_3\rangle = k_L\,|\,n,k_L,k_3\rangle, \tag{64a}$$

$$p_3\,|\,n,k_L,k_3\rangle = k_3\,|\,n,k_L,k_3\rangle, \tag{64b}$$

$$N\,|\,n,k_L,k_3\rangle = n\,|\,n,k_L,k_3\rangle, \tag{64c}$$

and hence from equation (44)

$$H\,|\,n,k_L,k_3\rangle = c\{4|\,\theta\,|\,\hbar(n+\tfrac{1}{2})+k_3^2+m^2c^2\}^{\frac{1}{2}}|\,n,k_L,k_3\rangle. \tag{65}$$

5. Motion of Dirac Electron in Same Field

The Dirac Hamiltonian H of equation (6) satisfies

$$H^2 = c^2(\boldsymbol{\alpha}\,.\,\boldsymbol{\pi})^2+m^2c^4, \tag{66}$$

so that $H^2 \geqslant m^2c^4$. In this case however, H is not positive definite: positive and negative energies appear symmetrically on either side of the gap of width $2mc^2$.

If we again make the change of variables as in (36), having chosen the potential as in (27), we find

$$H = c(2\theta\alpha_1 X_L+\alpha_2 P_L+\alpha_3 p_3)+mc^2\beta, \tag{67}$$

and it is at once evident that x_L and p_L (and so by equations 38, $p_\pm \pm\mathrm{i}\,\theta x_\pm$) are again constants of the motion, together with H and p_3.

Also constant is J_3, the third component of the total (canonical) angular momentum vector in this gauge:

$$J_3 = L_3+S_3 = \tfrac{1}{2}\mathrm{i}(x_+p_- - x_-p_+)+\tfrac{1}{4}\hbar(\alpha_+\alpha_- - 2), \tag{68}$$

where $\alpha_\pm = \alpha_1 \pm\mathrm{i}\,\alpha_2$. There are further constants associated with the spin degrees of freedom (Johnson and Lippmann 1949, 1950), in particular (i) the zeroth and third components of the polarization four-vector

$$\zeta_0 = \boldsymbol{\sigma}\,.\,\boldsymbol{\pi}, \qquad \boldsymbol{\zeta} = \gamma_5\boldsymbol{\pi}+mc\beta\boldsymbol{\sigma}, \tag{69a, b}$$

which satisfy

$$c^2\zeta_0^2 = H^2-m^2c^4, \qquad \zeta_3^2 = p_3^2+m^2c^2; \tag{70a, b}$$

(ii) the third component of the vector $\boldsymbol{T}$,

$$\boldsymbol{T} = \beta\boldsymbol{\pi}\wedge\boldsymbol{\sigma}, \tag{71}$$

which satisfies

$$c^2T_3^2 = H^2 - p_3^2 - m^2c^4\,; \tag{72}$$

and (iii) the third component of the vector $\boldsymbol{G}$,

$$\boldsymbol{G} = mc\boldsymbol{\sigma} + \mathrm{i}\gamma_5\beta\boldsymbol{\pi}\wedge\boldsymbol{\sigma}, \tag{73}$$

which satisfies

$$c^2G_3^2 = H^2 - p_3^2. \tag{74}$$

(In the notation of Johnson and Lippmann (1949, 1950) $\zeta_0 = I/c$, $\zeta_3 = L/c$ and $T_3 = T/c$; they did not discuss the constant G_3. Note that in the free particle case, when $\boldsymbol{\pi} = \boldsymbol{p}$, then ζ_0 and all components of $\boldsymbol{J}$, $\boldsymbol{\zeta}$, $\boldsymbol{T}$ and $\boldsymbol{G}$ are constants.)

These constants are not all functionally independent, and in particular (cf. equation 43)

$$-4\theta c^2J_3 = H^2 - m^2c^4 - c^2(p_L^2 + 4\theta^2x_L^2 + p_3^2), \tag{75a}$$

$$\mathrm{i}mc^2T_3 = c\zeta_3\zeta_0 - p_3H. \tag{75b}$$

We turn now to the solution of the Heisenberg equations of motion for the non-constant operators. Using H as in (67) we get

$$[X_L, H] = \mathrm{i}\hbar c\alpha_2, \qquad [P_L, H] = -2\mathrm{i}\,\theta\hbar c\alpha_1, \tag{76a}$$

$$[\alpha_1, H] = -2H\alpha_1 + 4\theta cX_L, \qquad [\alpha_2, H] = -2H\alpha_2 + 2cP_L. \tag{76b}$$

We now introduce (cf. equations 60)

$$H_\pm = H(1 \pm 4\theta\hbar c^2H^{-2})^{\frac{1}{2}}. \tag{77}$$

Just as in the scalar case, one or the other of these operators—depending on the sign of θ—is not a well-defined Hermitian operator if the field is so strong that the inequality (56) is violated. (For an electron, the critical field strength is $2{\cdot}2\times10^{13}$ G.) However, the introduction of these operators is very convenient for formal manipulation. Any of the results we obtain which involve $H_\pm$ can be re-expressed, if necessary, in terms of the operator

$$H(1 + 4|\theta|\hbar c^2H^{-2})^{\frac{1}{2}}, \tag{78}$$

which is always well-defined. Next we define

$$\begin{aligned} U_\pm &= \{-c(P_L \mp 2\mathrm{i}\,\theta X_L) \mp \tfrac{1}{2}\mathrm{i}(H - H_\pm)\alpha_\pm\}H^{-1} \\ &= \{\theta cx_\pm \pm \mathrm{i}\,cp_\pm \mp \tfrac{1}{2}\mathrm{i}(H - H_\pm)\alpha_\pm\}H^{-1}, \end{aligned} \tag{79a}$$

$$\begin{aligned} V_\pm &= \{-c(P_L \mp 2\mathrm{i}\,\theta X_L) \mp \tfrac{1}{2}\mathrm{i}(H + H_\pm)\alpha_\pm\}H^{-1} \\ &= \{\theta cx_\pm \pm \mathrm{i}\,cp_\pm \mp \tfrac{1}{2}\mathrm{i}(H + H_\pm)\alpha_\pm\}H^{-1}. \end{aligned} \tag{79b}$$

Noting that

$$H^2 - H_\pm^2 = \mp 4\theta\hbar c^2, \tag{80}$$

we get from equations (76)

$$[U_\pm, H] = -(H-H_\pm)U_\pm, \tag{81}$$

or, equivalently,

$$U_\pm H = H_\pm U_\pm ; \tag{82}$$

and

$$[V_\pm, H] = -(H+H_\pm)V_\pm, \tag{83}$$

or

$$V_\pm H = -H_\pm V_\pm . \tag{84}$$

It follows from (82) and the definition (77) of $H_\pm$ that $U_\pm$ is an operator which shifts one eigenvector of H into another without changing the sign of the energy. On the other hand, equation (84) shows that $V_\pm$ not only shifts the energy, but also changes its sign. In the language used by Schrödinger (1930) and others since, $U_\pm$ is an even operator and $V_\pm$ an odd operator.

Now equation (82) says $U_\pm H = H(1 \pm 4\theta\hbar c^2 H^{-2})^{\frac{1}{2}} U_\pm$, and hence

$$U_\pm H(1 \mp 4\theta\hbar c^2 H^{-2})^{\frac{1}{2}}$$
$$= H(1 \pm 4\theta\hbar c^2 H^{-2})^{\frac{1}{2}}\{1 \mp 4\theta\hbar c^2 H^{-2}(1 \pm 4\theta\hbar c^2 H^{-2})^{-1}\}^{\frac{1}{2}} U_\pm = HU_\pm ,$$

that is

$$HU_\pm = U_\pm H_\mp . \tag{85}$$

Similarly, we get

$$HV_\pm = -V_\pm H_\mp . \tag{86}$$

From the definitions (79), and using (82) and (84)–(86), we see that

$$\pm \mathrm{i}\alpha_\pm = (H_\pm)^{-1}(U_\pm - V_\pm)H = U_\pm + V_\pm , \tag{87}$$

$$\begin{aligned} 2(\theta c x_\pm \pm \mathrm{i}\, c p_\pm) &= (H_\pm)^{-1}\{(H+H_\pm)U_\pm - (H-H_\pm)V_\pm\}H \\ &= (H+H_\pm)U_\pm + (H-H_\pm)V_\pm \\ &= U_\pm(H+H_\mp) + V_\pm(H-H_\mp). \end{aligned} \tag{88}$$

It then follows that we can also write

$$U_\pm = H^{-1}\{\theta c x_\pm \pm \mathrm{i}\, c p_\pm \mp \tfrac{1}{2}\mathrm{i}\,\alpha_\pm(H-H_\mp)\}, \tag{89a}$$

$$V_\pm = H^{-1}\{\theta c x_\pm \pm \mathrm{i}\, c p_\pm \mp \tfrac{1}{2}\mathrm{i}\,\alpha_\pm(H+H_\mp)\}, \tag{89b}$$

so that $U_\pm$ and $V_\pm$ are the Hermitian conjugates of $U_\mp$ and $V_\mp$ respectively.

Now from (81) and (83) we have

$$\mathrm{i}\hbar \dot{U}_\pm = -(H-H_\pm)U_\pm , \qquad \mathrm{i}\hbar \dot{V}_\pm = -(H+H_\pm)V_\pm , \tag{90a, b}$$

and so

$$U_\pm(t) = \exp\{\mathrm{i}\hbar^{-1}(H-H_\pm)t\}\, U_\pm(0) = U_\pm(0)\exp\{-\mathrm{i}\hbar^{-1}(H-H_\mp)t\}, \tag{91a}$$

$$V_\pm(t) = \exp\{\mathrm{i}\hbar^{-1}(H+H_\pm)t\}\, V_\pm(0) = V_\pm(0)\exp\{-\mathrm{i}\hbar^{-1}(H+H_\mp)t\}. \tag{91b}$$

Noting equations (87), (88) and (33) we can then deduce the time dependence of $x_\pm$, $p_\pm$ and $\alpha_\pm$:

$$\begin{aligned}
x_\pm(t) &= \tfrac{1}{2}\{x_\pm(0) \mp \mathrm{i}\,\theta^{-1}p_\pm(0)\} + (1/4c\theta)(H+H_\pm)\,U_\pm(t) + (1/4c\theta)(H-H_\pm)\,V_\pm(t) \\
&= \tfrac{1}{2}\{x_\pm(0) \mp \mathrm{i}\,\theta^{-1}p_\pm(0)\} \\
&\quad + \tfrac{1}{4}\{1+H(H_\pm)^{-1}\}\exp\{\mathrm{i}\hbar^{-1}(H-H_\pm)t\} \\
&\qquad \times \{x_\pm(0) \pm \mathrm{i}\,\theta^{-1}p_\pm(0) \mp (\mathrm{i}/2c\theta)(H-H_\pm)\alpha_\pm(0)\} \\
&\quad + \tfrac{1}{4}\{1-H(H_\pm)^{-1}\}\exp\{\mathrm{i}\hbar^{-1}(H+H_\pm)t\} \\
&\qquad \times \{x_\pm(0) \pm \mathrm{i}\,\theta^{-1}p_\pm(0) \mp (\mathrm{i}/2c\theta)(H+H_\pm)\alpha_\pm(0)\},
\end{aligned} \tag{92}$$

$$\begin{aligned}
p_\pm(t) &= \tfrac{1}{2}\{p_\pm(0) \pm \mathrm{i}\,\theta x_\pm(0)\} \mp (\mathrm{i}/4c)(H+H_\pm)\,U_\pm(t) \mp (\mathrm{i}/4c)(H-H_\pm)\,V_\pm(t) \\
&= \tfrac{1}{2}\{p_\pm(0) \pm \mathrm{i}\,\theta x_\pm(0)\} \\
&\quad + \tfrac{1}{4}\{1+H(H_\pm)^{-1}\}\exp\{\mathrm{i}\hbar^{-1}(H-H_\pm)t\} \\
&\qquad \times \{p_\pm(0) \mp \mathrm{i}\,\theta x_\pm(0) - (1/2c)(H-H_\pm)\alpha_\pm(0)\} \\
&\quad + \tfrac{1}{4}\{1-H(H_\pm)^{-1}\}\exp\{\mathrm{i}\hbar^{-1}(H+H_\pm)t\} \\
&\qquad \times \{p_\pm(0) \mp \mathrm{i}\,\theta x_\pm(0) - (1/2c)(H+H_\pm)\alpha_\pm(0)\},
\end{aligned} \tag{93}$$

$$\begin{aligned}
\alpha_\pm(t) &= \mp \mathrm{i}\{U_\pm(t) + V_\pm(t)\} \\
&= (H_\pm)^{-1}\exp\{\mathrm{i}\hbar^{-1}(H-H_\pm)t\}\{\tfrac{1}{2}(H_\pm - H)\alpha_\pm(0) \mp \mathrm{i}\,\theta c x_\pm(0) + c p_\pm(0)\} \\
&\quad + (H_\pm)^{-1}\exp\{\mathrm{i}\hbar^{-1}(H+H_\pm)t\}\{\tfrac{1}{2}(H_\pm + H)\alpha_\pm(0) \pm \mathrm{i}\,\theta c x_\pm(0) - c p_\pm(0)\}.
\end{aligned} \tag{94}$$

We also have, using equation (67),

$$\dot{x}_3 = -\mathrm{i}\hbar^{-1}[x_3, H] = c\alpha_3, \qquad \dot{p}_3 = -\mathrm{i}\hbar^{-1}[p_3, H] = 0, \tag{95a, b}$$

$$\mathrm{i}\hbar\dot{\alpha}_3 = [\alpha_3, H] = -2H\alpha_3 + 2cp_3, \tag{95c}$$

from which we get by successive integrations

$$\begin{aligned}
x_3(t) = x_3(0) &+ \tfrac{1}{2}\mathrm{i}\hbar c\{H^{-1}\alpha_3(0) - cp_3(0)H^{-2}\} + cp_3(0)H^{-1}t \\
&- \tfrac{1}{2}\mathrm{i}\hbar c H^{-1}\exp(2\mathrm{i}\hbar^{-1}Ht)\{\alpha_3(0) - cp_3(0)H^{-1}\},
\end{aligned} \tag{96a}$$

$$p_3(t) = p_3(0), \tag{96b}$$

$$\alpha_3(t) = cp_3(0)H^{-1} + \exp(2\mathrm{i}\hbar^{-1}Ht)\{\alpha_3(0) - cp_3(0)H^{-1}\}. \tag{96c}$$

In comparing solutions (92), (93), (96a) and (96b) with those for the scalar particle (equations 31), we see in particular that the last of the three contributions to $x_\pm(t)$, and to $p_\pm(t)$, and the last of the four contributions to $x_3(t)$ have no counterpart in the earlier cases. These are 'Zitterbewegung' terms, and are characterized by angular frequencies of the order of magnitude $2mc^2/\hbar$. The other contributions are either non-periodic, or are characterized by angular frequencies of the order of magnitude $|eB|/m$. For very strong fields, these frequencies become comparable (cf. the inequality 56), but the Zitterbewegung terms are further distinguished by the fact that, just as for the free particle, they are all odd operators, anticommuting with the sign of the energy $H(H^2)^{-\frac{1}{2}}$. Thus the operator $\alpha_3(0)-cp_3(0)H^{-1}$ anticommutes with H, while the Zitterbewegung terms in $x_\pm$ and $p_\pm$ are those involving the odd operators $V_\pm$. These terms would therefore make no contribution to the expectation values of $x_\pm$, $p_\pm$, x_3 and p_3 in positive energy states. However, they do have observable effects, and will contribute to the expectation value of, for example, x_+p_- in such states.

The even parts of $x_\pm$ and x_3 show a time dependence similar to the variables $x_\pm$ and x_3 for the scalar particle, so the appearance of the exponent $\mathrm{i}\hbar^{-1}(H-H_\pm)t$ in place of the classical exponent $\mp 2\mathrm{i}\,\theta c^2 H^{-1}t$ can be called a quantum effect, which is not influenced by the presence or absence of spin.

Since all dynamical variables for the electron can be constructed from $\boldsymbol{x}$, $\boldsymbol{p}$, $\boldsymbol{\alpha}$ and β (see equation 11), we can now determine the time dependence of any variable we choose. In particular, we could consider the motion of the spin and magnetic moment operators in the field, but that is an interesting story in its own right which we have considered in part elsewhere (Barut and Bracken 1981*b*).

From a dynamical point of view, the variables x_L, p_L, x_3, p_3, U_+, U_-, V_+, V_-, α_3 and β are more fundamental than $\boldsymbol{x}$, $\boldsymbol{p}$, $\boldsymbol{\alpha}$ and β in this problem. In particular, U_+, U_-, V_+ and V_- replace the a and $a^\dagger$ appearing in the corresponding problem for a scalar particle. One could define energy eigenvectors by applying these shift operators to suitably defined 'vacuum' states with energy $\pm mc^2$, but we shall not pursue this matter here.

6. Centre of Mass and Relative Variables for an Electron in the Magnetic Field

From the form of the solutions (92) and (96a) for $\boldsymbol{x}(t)$, it is clear how to distinguish a c.m. or mean position $\boldsymbol{x}_A(t)$ and a relative position $\boldsymbol{Q}^{\mathrm{rel}}(t)$ for the electron in the field. We take

$$x_{A\pm}(t) = \tfrac{1}{2}\{x_\pm(0) \mp \mathrm{i}\,\theta^{-1}p_\pm(0)\} + (1/4c\theta)(H+H_\pm)U_\pm(t), \tag{97a}$$

$$x_{A3}(t) = x_3(0) + \tfrac{1}{2}\mathrm{i}\hbar c\{H^{-1}\alpha_3(0) - cp_3(0)H^{-2}\} + cp_3(0)H^{-1}t, \tag{97b}$$

$$Q^{\mathrm{rel}}_\pm(t) = (1/4c\theta)(H-H_\pm)V_\pm(t), \tag{98a}$$

$$Q^{\mathrm{rel}}_3(t) = -\tfrac{1}{2}\mathrm{i}\hbar c H^{-1}\{\alpha_3(t) - cp_3(0)H^{-1}\}. \tag{98b}$$

Then, as for the free particle, the position of the charge is given by

$$\boldsymbol{x} = \boldsymbol{x}_A + \boldsymbol{Q}^{\mathrm{rel}}, \tag{99}$$

and $\boldsymbol{x}_A$ behaves similarly to the position operator of a scalar particle (cf. equations 53a and 61a), while $\boldsymbol{Q}^{\mathrm{rel}}$ is highly oscillatory, even for vanishingly weak fields. It can be shown that as $|B| \to 0$, these operators reduce to those of (15) and (17).

Since the canonical momentum $\boldsymbol{p}$ is not defined in a gauge-invariant way, we do not attempt to identify a corresponding c.m. canonical momentum. Instead, we look at the kinetic momentum, which according to (92), (93) and (96b) has the form

$$\begin{aligned}\pi_\pm(t) &= p_\pm(t) \mp \mathrm{i}\,\theta x_\pm(t)\\ &= \mp(\mathrm{i}/2c)(H+H_\pm)U_\pm(t) \mp (\mathrm{i}/2c)(H-H_\pm)V_\pm\,, \end{aligned}\tag{100a}$$

$$\pi_3(t) = p_3(t) = \pi_3(0)\,. \tag{100b}$$

We see that, in contrast to the case of the free electron where $\boldsymbol{\pi}(t) = \boldsymbol{p}(t) = \boldsymbol{\pi}(0)$, the components $\pi_\pm$ contain Zitterbewegung terms involving $V_\pm$, as well as terms involving the even operators $U_\pm$. These Zitterbewegung terms vanish as $|B| \to 0$. For the kinetic momentum $\boldsymbol{\pi}_A$ of the c.m., we take the even part of $\boldsymbol{\pi}$, giving

$$\pi_{A\pm}(t) = \mp(\mathrm{i}/2c)(H+H_\pm)U_\pm(t)\,, \qquad \pi_{A3}(t) = p_3(t) = \pi_{A3}(0)\,. \tag{101a, b}$$

Turning to the relative momentum, we see that the simplest identification consistent with our choice for the free particle is, as in (18),

$$\boldsymbol{P}^{\mathrm{rel}} = (\hbar/\lambda c)\dot{\boldsymbol{Q}}^{\mathrm{rel}}\,. \tag{102}$$

Then from (90b) and (98) we get

$$P^{\mathrm{rel}}_\pm = \mp(\mathrm{i}\hbar/\lambda)V_\pm(t)\,, \qquad P^{\mathrm{rel}}_3 = (\hbar/\lambda)\{\alpha_3(t) - cp_3(0)H^{-1}\}\,. \tag{103a, b}$$

Summarizing these identifications, in terms of $\boldsymbol{x}$, $\boldsymbol{p}$, $\boldsymbol{Q}$, $\boldsymbol{P}$ and β, we have the c.m. variables

$$\begin{aligned} x_{A\pm}(t) &= \{\tfrac{3}{4} + H(4H_\pm)^{-1}\}x_\pm(t) \pm \mathrm{i}\,\theta^{-1}\{\tfrac{1}{4} - H(4H_\pm)^{-1}\}p_\pm(t)\\ &\quad -(\hbar c/\lambda)(H_\pm)^{-1}\beta(t)Q_\pm(t)\\ &= x_{A\pm}(0) - \tfrac{1}{4}\{1 + H(H_\pm)^{-1}\}[1 - \exp\{\mathrm{i}\hbar^{-1}(H-H_\pm)t\}]\\ &\quad \times\{x_\pm(0) \pm \mathrm{i}\,\theta^{-1}p_\pm(0) \pm (1/c\theta\lambda)(H-H_\pm)\beta(0)Q_\pm(0)\}\,, \end{aligned}\tag{104a}$$

$$\begin{aligned} x_{A3}(t) &= x_3(t) - \{(\hbar c/\lambda)Q_3(t)\beta(t)H^{-1} - \tfrac{1}{2}\mathrm{i}\hbar c^2 p_3(0)H^{-2}\}\\ &= x_{A3}(0) + cp_3(0)H^{-1}t\,, \end{aligned}\tag{104b}$$

$$\begin{aligned} \pi_{A\pm}(t) &= \tfrac{1}{2}(H+H_\pm)\{p_\pm(t) \mp \mathrm{i}\,\theta x_\pm(t) - (\lambda/2c\hbar)(H-H_\pm)P_\pm(t)\}H^{-1}\\ &= \exp\{\mathrm{i}\hbar^{-1}(H-H_\pm)t\}\,\pi_{A\pm}(0)\,, \end{aligned}\tag{105a}$$

$$\pi_{A3}(t) = p_3(0)\,, \tag{105b}$$

and the relative variables

$$\begin{aligned} Q^{\mathrm{rel}}_\pm(t) &= (\hbar c/\lambda)(H_\pm)^{-1}\beta(t)Q_\pm(t) + \tfrac{1}{4}\{1 - H(H_\pm)^{-1}\}\{x_\pm(t) \pm \mathrm{i}\,\theta^{-1}p_\pm(t)\}\\ &= \exp\{\mathrm{i}\hbar^{-1}(H+H_\pm)t\}\,Q^{\mathrm{rel}}_\pm(0)\,, \end{aligned}\tag{106a}$$

$$Q_3^{\mathrm{rel}}(t) = (\hbar c/\lambda) Q_3(t)\beta(t)H^{-1} - \tfrac{1}{2}\mathrm{i}\hbar c^2 p_3(0)H^{-2}$$

$$= \exp(2\mathrm{i}\hbar^{-1}Ht)\, Q_3^{\mathrm{rel}}(0), \tag{106b}$$

$$P_{\pm}^{\mathrm{rel}}(t) = \tfrac{1}{2}\{1 + H(H_{\pm})^{-1}\}P_{\pm}(t) - (\hbar c/\lambda)(H_{\pm})^{-1}\{p_{\pm}(t) \mp \mathrm{i}\,\theta x_{\pm}(t)\}$$

$$= \exp\{\mathrm{i}\hbar^{-1}(H + H_{\pm})t\}\, P_{\pm}^{\mathrm{rel}}(0), \tag{107a}$$

$$P_3^{\mathrm{rel}}(t) = P_3(t) - (\hbar c/\lambda)H^{-1}p_3(0)$$

$$= \exp(2\mathrm{i}\hbar^{-1}Ht)\, P_3^{\mathrm{rel}}(0). \tag{107b}$$

The c.m. variables describe a motion of the same general form as that for a scalar particle in the same field. In comparing the relative motion with that for a free electron, we see that the major effect is to replace the exponent $2\mathrm{i}\hbar^{-1}Ht$ in the time dependence of the components $Q_{\pm}^{\mathrm{rel}}$ and $P_{\pm}^{\mathrm{rel}}$ for the free particle by the exponent

$$\mathrm{i}\hbar^{-1}(H + H_{\pm})t = \mathrm{i}\hbar^{-1}\{2Ht + (H_{\pm} - H)t\}$$

$$\approx 2\mathrm{i}\hbar^{-1}Ht \pm 2\mathrm{i}\,\theta c^2 H^{-1}t, \tag{108}$$

for weak fields (cf. equation 54). At low energies where $H^2 \approx m^2c^4$, the effect is, roughly speaking, to modify the Zitterbewegung frequency ω_Z $(=2mc^2/\hbar)$ by plus or minus the cyclotron frequency $2\theta/m$ $(=eB/m)$. However, the precise interpretation of operator expressions like (106) and (107) can properly be determined only by consideration of the time dependence of expectation values.

7. Further Remarks and Applications

The method we used here can in principle be applied to other types of external fields. The Heisenberg equations define the quantum or operator analogue of a classical dynamical system. For most external fields, for example an external Coulomb or constant electric field, the system will be nonlinear. It would be very interesting to see if the remarkable properties of nonlinear dynamical systems, such as limit cycles, the onset of stochasticity, strange attractors etc. also occur in the quantum case for the electron. From the point of view of quantum electrodynamics, it would be important to study the Zitterbewegung in the presence of the self-field of the electron in order to calculate the radiative effects.

As for applications, our solutions for $x(t)$ can be used together with a radiation formula to calculate transition probabilities in external magnetic fields, in analogy with the calculation of the Einstein A coefficients by this method (Barut 1979).

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