Drift and Diffusion of Electrons in an Atomic Gas between Closely Spaced Electrodes. I Theory for Elastic Electron–Atom Collisions

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Abstract

The solution of the two-term approximation for the Boltzmann equation for electron drift and diffusion in an atomic gas between closely spaced electrodes has been obtained with the neglect of energy losses by elastic collisions. Although the omission of elastic collision energy losses (equivalent to using an infinite atomic mass) may appear to be a drastic oversimplification it does approximate the real physical situation where the cathode and anode are separated by a distance less than the thickness of the so-called boundary layer. The theory may be used as a guide in developing future, more accurate analytic theories of electron swarms between closely spaced electrodes. The present work has some special advantages however not so far achieved with previous analytic work on electron transport in the boundary layer region. There is no restriction to special model momentum transfer cross sections and the results can be written in easily interpreted analytic forms. The effects of inelastic collisions, the subject of a second paper, can also be included. In an attempt to obtain a better physical understanding of boundary effects, the present paper studies the influence of electron reflections from the electrode surfaces. Expressions are obtained which eliminate the electrode sensitive terms and relate experimentally measurable quantities to the momentum transfer cross section alone.

1. Introduction

For a nonzero electric field E and regions sufficiently close to an absorbing electrode the energy gained from the field by electrons which collide elastically with gas atoms or molecules is not perfectly balanced by the energy exchanged in collisions. Consequently the average energy of the swarm increases near an absorbing electrode (Lucas 1966; Sakai *et al.* 1972; Lowke *et al.* 1977; Braglia and Lowke 1979; Robson 1981, 1982; Chantry 1982; Braglia 1982; K. Ness and R. E. Robson, unpublished data). This phenomenon is also contributed towards (or against) by 'diffusion heating' (or 'diffusion cooling') by which either the slowest (or fastest) electrons, depending on the energy dependence of the momentum transfer cross section, are more rapidly removed from the swarm by diffusion to the electrode (Parker 1965; Robson 1976). The width of this 'boundary layer' is of the order of $l_\varepsilon = (M/2m)^{\frac{1}{2}}l$, where l is the mean free path, M is the gas atom or molecule mass and m is the electron mass. Here l_ε is the characteristic drift length for significant energy losses by elastic collisions. We introduce a new analysis of electron fluxes restricted to the small distances inside the boundary layer such that energy losses due to elastic collisions are, in the first approximation, unimportant. This requires

$D^2 \ll l_\varepsilon^2$ where D is the length scale of electrode–electron source separations. We further require $l \ll D$ for the Boltzmann equation to be applicable. Our treatment of the transport problem is equivalent to the infinite atomic mass limit $m/M \to 0$, although the results can be used to estimate the importance of energy losses and show that l_ε for finite M/m is the characteristic length for elastic collision energy losses (see Section 6). In this paper elastic collisions alone are considered in order to establish the basic ideas. In Part II (p. 45) it is shown how the theory can be extended to include inelastic collisions.

The results obtained in this paper may be regarded as generalizations of those due to Hertz (1925) and Langmuir (1931) on the drift and diffusion of electrons between closely spaced electrodes such that energy losses can be neglected. The early work has been generalized by allowing for an arbitrary energy dependence of the momentum transfer cross section. Further, the usual boundary condition of zero electron number density is avoided (which eliminates unphysical results obtained at very low densities) and is replaced by a flux relation which depends on the electron reflection probability at the electrode. The electron fluxes to the electrodes from a source in the gas are found to be insensitive to the reflection probabilities. The electron number density extrapolates to zero inside the electrodes at a distance of the order of one mean free path so that the usual zero number density boundary condition gives virtually identical results. If one electrode (the cathode) is the source of electrons, however, the zero number density boundary condition is approximately valid only for the anode.

Theoretical expressions which eliminate entirely effects from the electrodes are derived. These expressions relate experimentally measurable quantities to the momentum transfer cross section alone and may be useful in obtaining this cross section at energies above some lower limit depending on the identity of the atoms, the electrode spacing and gas pressure. The absolute scale of the cross section however has to be set by more traditional swarm techniques because elimination of the unknown electrode properties leads to relations which only give the form of the energy dependent momentum transfer cross section. Expressions are also obtained which enable the electron energy distribution to be scanned by a retarding field between electrodes inside a boundary layer.

The present work is of relevance to recent efforts to obtain a better understanding of electron drift and diffusion in the gas–electrode boundary layer (Lowke *et al.* 1977; Braglia and Lowke 1979; Chantry 1982; Braglia 1982). The need for analytical solutions has recently been emphasized by Robson (1981). Instead of Robson's limitation to special cases of energy dependent momentum transfer cross sections the present work obtains analytical results under the alternative approximation of negligible energy exchange by elastic collisions. A straightforward comparison of this work with the results of other authors is not possible however, because these works do not consider the case of two absorbing electrodes separated by a distance less than l_ε with a source emitting electrons with an arbitrary initial energy distribution between them.

Our results are shown in Section 7 to have an elegant representation in terms of generalized electron source strengths and absorption coefficients. The former refer to the effective source strength at any point in the gas distant from the actual 'bare' electron source. The generalized electron absorption coefficient takes into account the effect of electron–atom collisions due to the gas in front of a 'bare' absorbing

electrode. In other words an electrode with gas in front of it can be regarded as another electrode itself with electron absorbing properties given by the generalized absorption coefficient. These generalized quantities are defined by coordinate transformations from the source and absorbing electrode surfaces. Related invariant quantities can also be constructed. As shown in Part II, these concepts can be used to extend the present work to take into account inelastic collisions. Our approach may serve as a guide in future developments of more accurate analytic theories which allow for energy exchange by elastic collisions.

2. Two-term Approximation for Electron Flux without Energy Losses

The Boltzmann equation for electron transport is given by (Huxley and Crompton 1974)

$$(\partial nf/\partial t) + \nabla_r \cdot (nf\boldsymbol{c}) + \nabla_c \cdot (nfe\boldsymbol{E}/m) + J(nf) = 0, \tag{1}$$

where $f = f(\boldsymbol{r}, \boldsymbol{c}, t)$ is the distribution function, n is the electron number density, $J(nf)$ is the collision integral, $\boldsymbol{c}$ is the electron velocity and $\boldsymbol{E}$ is the applied electric field. Low electron densities are assumed to avoid space charge effects (electron–electron collisions). When the electron density gradient has axial symmetry with respect to the electric field the distribution function can be written as the expansion

$$f(\boldsymbol{r}, \boldsymbol{c}, t) = f_0(\boldsymbol{r}, \boldsymbol{c}, t) + \sum_{k=1}^{\infty} f_k(\boldsymbol{r}, \boldsymbol{c}, t) \, \mathrm{P}_k(\cos\theta), \tag{2}$$

where $\boldsymbol{c}.\boldsymbol{E} = -cE\cos\theta$ and $\mathrm{P}_k(u)$ are the Legendre polynomials. For atomic gases, where the elastic collision cross section is much larger than the inelastic cross section, the two-term approximation $k \leqslant 1$ is known to be adequate and is used in this work, although very close to an absorbing boundary the assumptions of this approximation break down (Robson 1981, 1982). The restriction to atomic gases and, for most foreseeable applications, heavy atoms is due to the neglect of energy losses for energies up to at least several electron volts. Thus the multiplicity of rotational and vibrational energy losses of molecular gases is avoided although electronic transitions are taken into account in Part II.

It is useful to define $g_0 = nf_0$ and $g_1 = nf_1$ and to introduce the mean free path $l(\varepsilon)$ related to the momentum transfer cross section $q_m(\varepsilon)$ by $l(\varepsilon) = \{N q_m(\varepsilon)\}^{-1}$, where N is the number density of the gas atoms. Then the scalar and vector equations in the case of infinite plane parallel electrodes become [Huxley and Crompton (1974), equations (2.22) and (2.24) modified to the energy variable ε]

$$3(\tfrac{1}{2}m)^{\frac{1}{2}} \frac{\partial \varepsilon^{\frac{1}{2}} g_0}{\partial t} + \frac{\partial \varepsilon g_1}{\partial z} + eE \frac{\partial \varepsilon g_1}{\partial \varepsilon} = 0, \tag{3a}$$

$$(\tfrac{1}{2}m)^{\frac{1}{2}} \frac{\partial \varepsilon^{-\frac{1}{2}} g_1}{\partial t} + \frac{\partial g_0}{\partial z} + eE \frac{\partial g_0}{\partial \varepsilon} + \frac{1}{l(\varepsilon)} g_1(\varepsilon) = 0. \tag{3b}$$

The electron number density is given by

$$n(z, t) = \int f(z, \boldsymbol{c}, t) \, \mathrm{d}\boldsymbol{c} = 2\pi (2/m)^{3/2} \int_0^{\infty} \varepsilon^{\frac{1}{2}} g_0(z, \varepsilon, t) \, \mathrm{d}\varepsilon, \tag{4a}$$

and the electron flux is

$$\Gamma(z,t) = \int c\cos\theta\, f(z,\boldsymbol{c},t)\,\mathrm{d}\boldsymbol{c} = \tfrac{2}{3}\pi(2/m)^2\int_0^\infty \varepsilon\, g_1(z,\varepsilon,t)\,\mathrm{d}\varepsilon. \qquad (4\mathrm{b})$$

For the steady state case, equation (3a) shows that

$$\varepsilon\, g_1(z,\varepsilon) = F\left(\varepsilon - \varepsilon_0 - \int_{z_0}^z e\,E(z')\,\mathrm{d}z'\right).$$

The structure of the variable in F suggests the influence of simple energy conservation, which we bring out explicitly by writing for the case of all electrons produced at their source with the same energy

$$\varepsilon\, g_1(z,\varepsilon) = \gamma\delta\left(\varepsilon - \varepsilon_0 - e\int_{z_0}^z E(z')\,\mathrm{d}z'\right), \qquad (5)$$

where ε_0 is the electron energy at any z_0. As space charge effects are negligible we can drop the explicit z dependence of E here. In equation (5) γ is a constant independent of z and is related to Γ by

$$\Gamma = (8\pi/3m^2)\gamma. \qquad (6)$$

Let $\varepsilon(z) = \varepsilon_0 + eE(z-z_0)$ be the electron energy at z. Substituting equation (5) into (3b) we find that $g_0(z,\varepsilon)$ must be written in the form

$$g_0(z,\varepsilon) = h(z,\varepsilon)\,\delta(\varepsilon - \varepsilon(z)), \qquad (7)$$

which again represents an infinitely sharp electron energy distribution at all points in the gas. By equation (4a) we have

$$n(z) = 2\pi(2/m)^{3/2}\varepsilon(z)^{\frac{1}{2}}h(z,\varepsilon(z)). \qquad (8)$$

A relation between Γ and $n(z)$ is obtained by substituting equations (5) and (7) into (3b), multiplying by $8\pi\varepsilon\,l(\varepsilon)/3m^2$ and integrating over all energies:

$$\Gamma = -\frac{8\pi eE}{3m^2}\int_0^\infty \varepsilon\, l(\varepsilon)\frac{\partial g_0(z,\varepsilon)}{\partial\varepsilon}\,\mathrm{d}\varepsilon - \frac{8\pi}{3m^2}\frac{\partial}{\partial z}\int_0^\infty \varepsilon\, l(\varepsilon)\, g_0(z,\varepsilon)\,\mathrm{d}\varepsilon$$

$$= \frac{8\pi eE}{3m^2}h(z,\varepsilon)\left(l(\varepsilon) + \varepsilon\frac{\mathrm{d}l(\varepsilon)}{\mathrm{d}\varepsilon}\right) - \frac{8\pi}{3m^2}\frac{\mathrm{d}}{\mathrm{d}z}\Big(\varepsilon\, l(\varepsilon)\, h(z,\varepsilon)\Big),$$

where ε must now be understood as $\varepsilon(z)$. By using the identities

$$\mathrm{d}\varepsilon/\mathrm{d}z = eE, \qquad \mathrm{d}(\varepsilon/c)/\mathrm{d}z = \tfrac{1}{2}m\,\mathrm{d}c/\mathrm{d}z = eE/2c,$$

and reexpressing $h(z,\varepsilon)$ in terms of $n(z)$ it is found that $\mathrm{d}l/\mathrm{d}\varepsilon$ is eliminated. The final formula is

$$\Gamma = (eEl/3mc)n(z) - \tfrac{1}{3}cl\,\mathrm{d}n(z)/\mathrm{d}z, \qquad (9)$$

where c and l are to be understood as functions of z. The first term on the right-hand side represents the contribution of electron drift, whereas the second is due to electron diffusion with the diffusion constant $\tfrac{1}{3}cl$. Equation (9) is essentially

equivalent to that derived by Hertz (1925) and used by Langmuir (1931) for electron fluxes between close electrodes. The only difference now is that equation (9) has been established for arbitrary energy dependent $l(\varepsilon)$ rather than constant l.
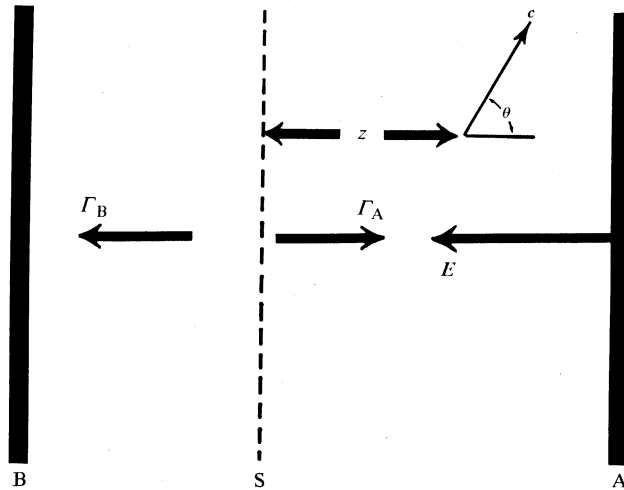

Fig. 1. Monoenergetic electrons are emitted isotropically from an infinite plane source S between parallel collecting electrodes A and B. An electric field E is directed perpendicularly out of A. The electron flux to A depends on the electron source strength and the relative absorption coefficients of A and B at the source. The effective source strength and absorption coefficients can be defined at any value of z (see Section 7).

3. Electron Flux between Infinite Plane Parallel Electrodes

Let an electron source S of strength S electrons per second per unit area be located in the gas a distance d from electrode B and distance $L-d$ from electrode A (see Fig. 1). The electrons are considered to be emitted isotropically with energy ε_S. The electron flux to either electrode A or B is nonzero only if E is such that the electron energy $\varepsilon_A = \varepsilon_S + eE(L-d)$ or $\varepsilon_B = \varepsilon_S - eEd$ at the electrode is equal to or greater than zero. We take E as positive if $\varepsilon_A > \varepsilon_S$.

Equation (9) can be rewritten in the form

$$\Gamma = \frac{eE\,l(c)}{3m}\left(\frac{n(c)}{c} - \frac{dn(c)}{dc}\right) \tag{10}$$

and can be solved to give $n(c)$ by the following procedure. Let n_S and c_S be the electron number density and speed at the source and let us introduce $p(c)$ by writing

$$n(c) = (cn_S/c_S)p(c). \tag{11}$$

Equation (11) substituted into (10) and specialized to Γ_A gives (returning to the variable ε)

$$\Gamma_A = -\frac{2eE\,l(\varepsilon)n_S}{3mc_S}\varepsilon\frac{dp(\varepsilon)}{d\varepsilon},$$

from which we find

$$p(\varepsilon) = 1 - \frac{3\Gamma_A\, mc_S}{2eEn_S} \int_{\varepsilon_S}^{\varepsilon} \frac{d\varepsilon'}{\varepsilon'\, l(\varepsilon')}. \tag{12}$$

We have $p(\varepsilon) \leqslant 1$ always because $1 - p(\varepsilon)$ is unchanged by a change in the sign of E. To progress further, the influence of the electrodes needs to be introduced at least as an auxiliary procedure for eliminating n_S, so as to relate Γ_A (and Γ_B) to S.

It is shown later that provided the mean free path is much smaller than d and $L - d$, the electrodes and the assumed boundary conditions do not actually have much bearing on the fluxes Γ_A and Γ_B. Electron–atom collisions dominate. Because of this and the fact that we are more interested anyway in developing relations that eliminate the influence of the electrodes altogether, we choose a convenient model to represent the complex events that may occur at the electrode surfaces. We do not however choose the usual boundary conditions $n_A = 0$ and $n_B = 0$ used by many authors [and by Langmuir (1931) for closely spaced electrodes], because they produce unphysical results at sufficiently low densities (namely $\Gamma_A + \Gamma_B$ larger than S). Our new formula is qualitatively correct at very low densities and compares favourably with the corresponding exactly solved flux in the zero density case.

Equation (4b) for Γ involves an integration over all θ between 0 and π. However, near the surface of an absorbing electrode the reverse flux for $\frac{1}{2}\pi \leqslant \theta \leqslant \pi$ originates only from reflected electrons. We model the surface effects at A by defining an incident flux Γ_{AI} for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ and a reflected flux Γ_{AR} for $\frac{1}{2}\pi < \theta < \pi$, such that

$$\Gamma_{AR} = r_A \Gamma_{AI},$$

where r_A is the reflection probability. The net flux Γ_A is then related to Γ_{AI} by

$$\Gamma_A = (1 - r_A)\Gamma_{AI}, \tag{13}$$

where Γ_{AI} is evaluated from equation (4b) but restricted to $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. This gives

$$\Gamma_{AI} = 2\pi n_A \int_0^\infty \int_0^{\frac{1}{2}\pi} c^3 \{f_0(L-d,c) + \cos\theta\, f_1(L-d,c)\}\cos\theta\, d\theta\, dc$$

$$= \tfrac{1}{4} n_A c_A + \tfrac{1}{2}\Gamma_A. \tag{14}$$

The first term on the RHS of equation (14) is the 'random flux' at A. Combining equation (14) with (13) leads to

$$\Gamma_A = \tfrac{1}{4}\alpha_A n_A c_A, \tag{15a}$$

where

$$\alpha_A = 2(1 - r_A)/(1 + r_A) \tag{15b}$$

is an effective absorption coefficient similar to that introduced by Hurst and Liley (1965) in their discussion of the Townsend–Huxley experiment (see also Huxley and Crompton 1974, pp. 381–6). Note that our theory is restricted to monoenergetic electrons so that no energy losses in electron–electrode collisions are considered.

The reverse flux at A corresponds to a distribution function of the form

$$f(L-d, c) = f_0(L-d, c) + \cos\theta\, f_1'(L-d, c); \qquad \tfrac{1}{2}\pi < \theta < \pi,$$

which leads to

$$\Gamma_{AR} = \tfrac{1}{4}n_A c_A - \tfrac{1}{2}\Gamma'_A, \tag{16}$$

where the primed variables f' and Γ'_A refer to $\tfrac{1}{2}\pi \leqslant \theta \leqslant \pi$. But because $\Gamma_A = \Gamma_{AI} - \Gamma_{AR}$ we see from equations (14) and (16) that $\Gamma'_A = \Gamma_A$. In particular this is satisfied by $f_1 = f'_1$ so that in effect the two-term approximation is being used right up to the electrode surfaces. This feature is retained in our later definition of generalized absorption coefficients $\alpha_A(z)$ where z is an arbitrary distance from the source S, i.e. $-d \leqslant z \leqslant L-d$. Our model of boundary effects is equivalent to Robson's (1981) equation (25) where the half-range expansions are equivalent to each other and the full two-term approximation.

By combining equations (11) and (15a) we find

$$\Gamma_A = \alpha_A(c_A^2 n_S/4c_S)p_A, \tag{17}$$

and Γ_A can now be eliminated from equation (12) to give

$$\frac{1}{p_A} = 1 + \frac{3\alpha_A \varepsilon_A}{4eE}\int_{\varepsilon_S}^{\varepsilon_A} \frac{d\varepsilon'}{\varepsilon' l(\varepsilon')}. \tag{18}$$

Analogous expressions exist for Γ_B and p_B. To eliminate n_S we use the continuity equation $S = \Gamma_A + \Gamma_B$ which gives

$$\frac{n_S}{2mc_S} = \frac{S}{\alpha_A \varepsilon_A p_A + \alpha_B \varepsilon_B p_B}.$$

Finally we obtain

$$\Gamma_A = \frac{\alpha_A \varepsilon_A p_A}{\alpha_A \varepsilon_A p_A + \alpha_B \varepsilon_B p_B}S, \qquad \Gamma_B = \frac{\alpha_B \varepsilon_B p_B}{\alpha_A \varepsilon_A p_A}\Gamma_A. \tag{19a, b}$$

4. Discussion of Electrode Effects

Due to the electron–gas atom collisions $p_A \ll 1$ and $p_B \ll 1$. To see this consider the limit $E \to 0$. Then equation (18) becomes

$$\frac{1}{p_A} = 1 + \frac{3\alpha_A \varepsilon_A}{4}\frac{L-d}{l(\varepsilon_A)},$$

and because $l(\varepsilon_A) \ll L-d$, p_A is dominated by electron–atom collisions and is very small if α_A is not too small. Written out fully equation (19a) is

$$\frac{\Gamma_A}{S} = \left\{1 + \left(\frac{1}{\alpha_A \varepsilon_A} + \frac{3}{4eE}\int_{\varepsilon_S}^{\varepsilon_A}\frac{d\varepsilon'}{\varepsilon' l(\varepsilon')}\right)\bigg/\left(\frac{1}{\alpha_B \varepsilon_B} + \frac{3}{4eE}\int_{\varepsilon_B}^{\varepsilon_S}\frac{d\varepsilon'}{\varepsilon' l(\varepsilon')}\right)\right\}^{-1}. \tag{20}$$

For $d \gg l$ and $L-d \gg l$ the terms in (20) that depend on l are usually the most important. In other words, electron–atom collisions rather than the boundary condition dominate the fluxes, provided that r_A and r_B are not too close to unity.

For $d \to 0$ equation (20) becomes

$$\frac{\Gamma_A}{S} = \left(1 + \frac{\alpha_B \varepsilon_B}{\alpha_A \varepsilon_A} + \frac{3\alpha_B \varepsilon_B}{4eE}\int_{\varepsilon_B}^{\varepsilon_A}\frac{d\varepsilon'}{\varepsilon' l(\varepsilon')}\right)^{-1}. \tag{21a}$$

Because the electron source is considered to emit electrons isotropically, S here cannot be regarded as the flux S_{ej} of electrons ejected from a cathode B. To get S_{ej} from S we note that half of the electrons in S are incident on B of which the fraction r_B is reflected. Thus the actual flux that appears from a cathode is given by

$$S_{ej} = \tfrac{1}{2}(1 + r_B)S,$$

and equation (21a) can be rewritten as

$$\frac{\Gamma_A}{S_{ej}} = \left[1 + \frac{r_A(1 - r_B)}{2(1 - r_A)} \left\{ 1 + \frac{\varepsilon_B - \varepsilon_A}{\varepsilon_A} \left(1 + \frac{1}{r_A} \right) \right\} + \frac{3(1 - r_B)\varepsilon_B}{4eE} \int_{\varepsilon_B}^{\varepsilon_A} \frac{d\varepsilon'}{\varepsilon' l(\varepsilon')} \right]^{-1}. \quad (21b)$$

Equation (21b) generalizes Langmuir's (1931) result as follows. Firstly a general energy dependent $l(\varepsilon)$ has been allowed for. Secondly, Langmuir solved equation (10) using the boundary condition $n_A = 0$. This has the effect of eliminating the first two terms on the RHS of (21b) with the consequence that at sufficiently low densities Γ_A exceeds S_{ej}. This is not a particularly serious omission because equation (10) would not be valid at such low densities anyway. However, small corrections for low density saturation effects are probably necessary under the gas pressure conditions of negligible energy losses by elastic collisions. The approximate validity of $n_A = 0$ is discussed further below. Thirdly, equations (21) show clearly that Γ_A is not sensitive to the boundary condition at A provided that $1 - r_A$ is not too small. This provides some further justification for the usual boundary condition $n_A = 0$, deduced assuming perfectly absorbing electrodes (McDaniel 1964), even though in practice electrodes can be quite reflective of low energy electrons (see Section 9). The physical significance of this is that it does not matter much whether electrons are initially reflected from A or whether they suffer changes of energy in doing so because the chance of their getting back to the cathode is practically zero anyway. An electron initially reflected by the anode is multiply reflected between the gas and the anode until eventually absorbed.

In the limit $N \to 0$, Γ_A tends to the saturation value

$$(\Gamma_A)_0 = S_{ej} \bigg/ \left[1 + \frac{r_A(1 - r_B)}{2(1 - r_A)} \left\{ 1 + \frac{\varepsilon_B - \varepsilon_A}{\varepsilon_A} \left(1 + \frac{1}{r_A} \right) \right\} \right]. \quad (22a)$$

The Boltzmann equation analysis fails as l becomes comparable in size with L so at best the low density limit of equation (21a) is only qualitatively correct. The exact flux $(\Gamma_A)_0$ is calculated in the Appendix for the model of isotropic elastic scattering from the electrodes. When electrode B is the cathode we find for $\varepsilon_A \geqslant \varepsilon_B$

$$(\Gamma_A)_0 = S_{ej} \bigg/ \left[1 + \frac{1 - r_B}{1 - r_A} r_A \left\{ 1 - \left(1 - \frac{\varepsilon_B}{\varepsilon_A} \right)^{\frac{1}{2}} \right\} \right], \quad (22b)$$

and for $\varepsilon_B \geqslant \varepsilon_A$

$$(\Gamma_A)_0 = S_{ej} \bigg/ \left\{ 1 + \frac{1 - r_B}{1 - r_A} r_A \left(1 + \frac{1}{r_A} \frac{(1 - \varepsilon_A/\varepsilon_B)^{\frac{1}{2}}}{1 - (1 - \varepsilon_A/\varepsilon_B)^{\frac{1}{2}}} \right) \right\}. \quad (22c)$$

The qualitative similarity with respect to r_A and r_B of equation (22a) with (22b) and (22c) lends confidence to the belief that our model of the boundary effects is not

drastically wrong and that equations (19) and (21) have acceptable forms for electrode effects on the fluxes.

It is of interest to calculate the distance ξ_A into A for which $n(z)$ extrapolates to zero. This can be found from the definition

$$\xi_A = -n_A \Big/ \left(\frac{dn(z)}{dz}\right)_A .$$

By equations (9) and (15a) we have

$$\left(\frac{dn(z)}{dz}\right)_A = \frac{eEn_A}{2\varepsilon_A}\left(1 - \frac{3\alpha_A\varepsilon_A}{2eE\,l(\varepsilon_A)}\right),$$

which then gives

$$\xi_A = \{\lambda_A/(\tfrac{3}{4}\alpha_A\lambda_A - 1)\}l(\varepsilon_A), \tag{23a}$$

where

$$\lambda_A = 2\varepsilon_A/eE\,l(\varepsilon_A). \tag{23b}$$

If $\alpha_A \neq 0$ and provided that $\varepsilon_A \gg \tfrac{1}{2}eE\,l(\varepsilon_A)$, which is certainly needed for the Boltzmann equation analysis to apply, we find

$$\xi_A \approx \tfrac{4}{3}l(\varepsilon_A)/\alpha_A .$$

That $n(z)$ extrapolates to zero about a mean free path inside the anode agrees with Lowke *et al.* (1977) who used the boundary condition $f_0 = -f_1$. In our approach the ratio of the drift and diffusion contributions to Γ_A is equal to $eE\xi_A/2\varepsilon_A \ll 1$ so that the flux between closely spaced electrodes is diffusion dominated. The same applies to any boundary layer (Huxley and Crompton 1974, pp. 121–4; Lowke *et al.* 1977). These overall results also apply of course to electrode B. In conclusion, $n_A \approx 0$ and $n_B \approx 0$ are acceptable boundary conditions when l is much smaller than the source–electrode separations $L-d$ and d. For $d = 0$, $n_S = n_B$ cannot be approximated by zero however, because $\Gamma_B \approx S$ and n_B is given by the B electrode version of equation (15a).

5. Electron Flux for More General Electrode Geometries

Consider a small surface area da_S at the source S defined at an equipotential surface between electrodes A and B (see Fig. 2). Let ds be a line element anywhere along the electric field lines. Then at any two points s_1 and s_2 the elements of electric flux $E(s_1).da_1$ and $E(s_2).da_2$ are equal for the same flux tube. Let da_A and da_B be the surface elements at A and B corresponding to da_S and let da be the surface element anywhere between A and B. Equation (9) is now generalized for a current element at s given by

$$dJ(s) = \{eE.da\,l\,n(s)\}/3mc - \tfrac{1}{3}cl\,da.dn(s)/ds ,$$

where ε (and c) are parametrically dependent on s by

$$\varepsilon(s) = \varepsilon(s_0) + \int_{s_0}^{s} eE(s').ds' .$$

Because $d\boldsymbol{a}$ and $d\boldsymbol{s}$ are perpendicular we can drop the vector notation hereafter. The expression for $d\boldsymbol{J}(\boldsymbol{s})$ is actually independent of $\boldsymbol{s}$ due to continuity of the electron flow along the flux tubes. End effects, due to the finite size of any practical apparatus, are ignored so that electron fluxes due to density gradients perpendicular to the electric field lines are omitted.
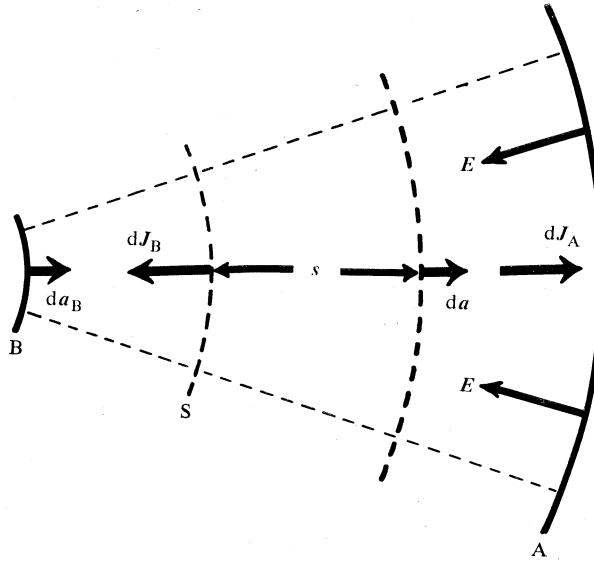

Fig. 2. Electron flux and electric field configuration for a non-planar source and collecting electrode geometry. The solution of the drift and diffusion problem closely follows that for plane geometry because, by assuming no transverse electron density gradients, the current is constant along each electric flux tube.

The analysis leading to equations (19) goes through again virtually unchanged. One simply notes that $E(s)\,da$ is a constant along the flux tube. We then have

$$dJ_A = dJ_S \left/ \left(1 + \frac{\alpha_B\,\varepsilon_B\,p_B}{\alpha_A\,\varepsilon_A\,p_A}\frac{da_B}{da_A}\right)\right., \qquad dJ_B = \frac{\alpha_B\,\varepsilon_B\,p_B}{\alpha_A\,\varepsilon_A\,p_A}\frac{da_B}{da_A}dJ_A, \qquad (24a,b)$$

where

$$\frac{1}{p_A} = 1 + \frac{3\alpha_A\,\varepsilon_A}{4eE(s)}\frac{da_A}{da}\int_{\varepsilon_S}^{\varepsilon_A}\frac{d\varepsilon'}{\varepsilon'\,l(\varepsilon')}, \qquad (24c)$$

and similarly for p_B.

Following Langmuir (1931) we can write from the definition of the capacitance C_{AB}

$$eE(s)\,da = \frac{\varepsilon_A - \varepsilon_B}{\kappa}\left(\frac{dC_{AB}}{da_A}\right)da_A, \qquad (25)$$

where κ is the permittivity of the medium between A and B. It is readily checked that if we set $\alpha_B = 1$ and $l = $ constant, let B become the source and ignore the electrode corrections to the flux, then dJ_A is precisely the formula given by Langmuir.

6. Energy Losses to Elastic Collisions

We establish general conditions for negligible energy losses due to elastic collisions with an atomic gas between electrodes of arbitrary geometry. Let dX be the average number of collisions per electron for the drift distance ds. We have

$$dX = c\,ds/l(\varepsilon)\,c_d(s), \tag{26}$$

where $c_d(s)$ is the diffusion modified electron drift velocity related to dJ by

$$dJ/da = n(s)\,c_d(s). \tag{27}$$

Equation (26) follows from $c\,dt/l(\varepsilon)$ collisions in the time interval $dt = ds/c_d(s)$ that it takes the electrons to flow the distance ds. It has been assumed that the collision rate is dominated by the local random flux. This corresponds to $c \gg c_d(s)$. In equation (26), $c_d(s)$ can be eliminated using (27), while $n(s)$ can be eliminated using (11). In equation (27), dJ/da can be evaluated from (24a) and (24b). Considering only the flux to A we have

$$dX = \frac{4\varepsilon}{\alpha_A\,\varepsilon_A\,p_A}\frac{da}{da_A}\,p(\varepsilon)\frac{ds}{l(\varepsilon)}. \tag{28}$$

Equation (12) can be rewritten in the form

$$p(\varepsilon) = p_A + \tfrac{3}{2}\,dJ_A\,\frac{mc_S}{n_S}\frac{1}{eE.da}\int_\varepsilon^{\varepsilon_A}\frac{d\varepsilon'}{\varepsilon'l(\varepsilon')}$$

$$= p_A\left(1 + \frac{3\alpha_A\,\varepsilon_A}{4eE(s)}\frac{da_A}{da}\int_\varepsilon^{\varepsilon_A}\frac{d\varepsilon'}{\varepsilon'l(\varepsilon')}\right), \tag{29}$$

where dJ_A has been eliminated using the obvious generalization of equation (17). Equation (29) can then be used to eliminate p_A from (28). The final expression is obtained by eliminating ds by $d\varepsilon = eE(s)\,ds$, and $E(s)$ can be eliminated by using the invariance of the electric flux $E(s)\,da$. Our final expression is

$$dX(\varepsilon) = \frac{4}{\alpha_A\,\varepsilon_A\,eE_A}\left(\frac{da}{da_A}\right)^2\frac{\varepsilon\,d\varepsilon}{l(\varepsilon)} + \frac{3}{(eE_A)^2}\left(\frac{da}{da_A}\right)^2\frac{\varepsilon\,d\varepsilon}{l(\varepsilon)}\int_\varepsilon^{\varepsilon_A}\frac{d\varepsilon'}{\varepsilon'l(\varepsilon')}, \tag{30}$$

where da/da_A is a function of s and hence is also a function of ε.

The second term on the RHS of equation (30) is second order in l^{-1} and dominates dX. It has been previously derived for a constant mean free path model and infinite plane parallel electrodes by Hertz (1925). To see this, we denote the second order contribution to the number of collisions between S and A by $X_2(S, A)$, put $da = da_A$ for plane parallel electrodes and set $l =$ constant. We find

$$X_2(S, A) = \frac{3}{(eE_A)^2 l^2}\int_{\varepsilon_S}^{\varepsilon_A}\varepsilon\ln(\varepsilon_A/\varepsilon)\,d\varepsilon$$

$$= \frac{3}{2}\left(\frac{L-d}{l}\right)^2\frac{\varepsilon_S^2}{(\varepsilon_A-\varepsilon_S)^2}\ln(\varepsilon_S/\varepsilon_A) + \frac{3}{4}\left(\frac{L-d}{l}\right)^2\frac{\varepsilon_A+\varepsilon_S}{\varepsilon_A-\varepsilon_S},$$

where eE has been replaced by $(\varepsilon_A - \varepsilon_S)/(L-d)$. Now we define $\varepsilon_S = x_1 eE$, $\varepsilon_A = aeE$ and $L-d = a-x_1$, such that

$$X_2(S, A) = \frac{3(a^2 - x_1^2)}{4l^2} - \frac{3x_1^2}{2l^2}\ln(a/x_1),$$

which agrees with Hertz. The first order contribution sensitive to the electrode boundary condition has not been previously observed. Although it should be qualitatively correct it is perhaps not quantitatively accurate because, near say electrode A, $c_d(A) = \frac{1}{4}\alpha_A c_A$, which is not necessarily very much smaller than c_A as assumed in the derivation of dX. The assumptions of the derivation are valid however at distances of a few mean free paths from the electrode surface.

To estimate the energy loss due to elastic collisions we weight $dX(\varepsilon)$ by $2m\varepsilon/M$ and integrate over all ε between ε_S and ε_A. Thus we have

$$\Delta\varepsilon(S, A) = \frac{8m}{M}\frac{1}{\alpha_A \varepsilon_A eE_A}\int_{\varepsilon_S}^{\varepsilon_A}\frac{\varepsilon^2}{l(\varepsilon)}\left(\frac{da}{da_A}\right)^2 d\varepsilon$$

$$+ \frac{6m}{M}\frac{1}{(eE_A)^2}\int_{\varepsilon_S}^{\varepsilon_A}\frac{\varepsilon^2}{l(\varepsilon)}\left(\frac{da}{da_A}\right)^2 d\varepsilon \int_{\varepsilon}^{\varepsilon_A}\frac{d\varepsilon'}{\varepsilon' l(\varepsilon')}. \tag{31}$$

An accelerating field would partially replenish the energy lost in collisions so that the greatest relative effect would occur for zero and retarding electric fields. Consider $E \to 0$ and $\varepsilon_A \to \varepsilon_S$; in this limit

$$\frac{\Delta\varepsilon(S, A)}{\varepsilon_S} \to \frac{1}{\alpha_A}\frac{8m}{M}\frac{L-d}{l(\varepsilon_S)} + \frac{3m}{M}\left(\frac{L-d}{l(\varepsilon_S)}\right)^2. \tag{32}$$

As expected equation (32) shows that $\Delta\varepsilon \ll \varepsilon_S$ if $l_\varepsilon^2 \gg (L-d)^2$. Equation (32) also shows that the neglect of collisional energy losses in the theory is inadmissable if α_A is small. Highly reflective electrodes build up the number of electron–atom collisions leading to a significant accumulated energy loss by elastic collisions.

7. Generalized Absorption Coefficients and Source Strengths

The basic ideas are illustrated here for infinite plane parallel electrodes and are easily extended to arbitrary geometry by the method of Section 5. Let z be the distance from S to any point in the gas between B and A. Thus $-d \leqslant z \leqslant L-d$ defines the range of z. We assert that the structure of the theory of electron fluxes between closely spaced electrodes is independent of the locations of the 'electrodes' and the 'source'. Rather than regard electrode A at a separation $L-d$ from S, we can consider A to be at any distance $0 \leqslant z \leqslant L-d$ but with a generalized absorption coefficient $\alpha_A(z)$ (see Fig. 1). Similarly the bare electrode B can be replaced by a generalized electrode at $-d \leqslant z \leqslant 0$ with the absorption coefficient $\alpha_B(z)$. In addition, rather than regard the source S as at a distance $L-d$ in front of A, we can define $S_A(z)$ as the source strength at a distance of $L-d-z$ from A, for $0 \leqslant z \leqslant L-d$. We now derive expressions for $\alpha_A(z)$, $\alpha_B(z)$, $S_A(z)$ and $S_B(z)$. In Part II it is shown that these ideas are fundamental to solving the problem of electron fluxes between close electrodes taking into account inelastic collisions.

Equations (19) are appropriate to defining Γ_A and Γ_B in terms of $\alpha_A(z)$, $\alpha_B(z)$, $S_A(z)$ and $S_B(z)$ at $z = 0$. Let $\varepsilon(0)\alpha_A(0) = \varepsilon_A \alpha_A p_A$, $\varepsilon(0)\alpha_B(0) = \varepsilon_B \alpha_B p_B$ and $S_A(0) = S_B(0) = S$. Then equations (19) become

$$\frac{\Gamma_A}{S_A(0)} = \frac{\alpha_A(0)}{\alpha_A(0)+\alpha_B(0)}, \qquad \frac{\Gamma_B}{S_B(0)} = \frac{\alpha_B(0)}{\alpha_A(0)+\alpha_B(0)}, \tag{33a, b}$$

where

$$\frac{1}{\alpha_A(0)} = \frac{\varepsilon(0)}{\alpha_A \varepsilon_A} + \frac{3\varepsilon(0)}{4eE} \int_{\varepsilon(0)}^{\varepsilon_A} \frac{d\varepsilon'}{\varepsilon' l(\varepsilon')}, \tag{34a}$$

$$\frac{1}{\alpha_B(0)} = \frac{\varepsilon(0)}{\alpha_B \varepsilon_B} + \frac{3\varepsilon(0)}{4eE} \int_{\varepsilon_B}^{\varepsilon(0)} \frac{d\varepsilon'}{\varepsilon' l(\varepsilon')}. \tag{34b}$$

Equation (33a) for $\Gamma_A/S_A(0)$ is essentially the probability of an electron originating at $z = 0$ migrating to electrode A rather than electrode B. This probability is in direct proportion to the absorption probability $\alpha_A(z)$ at $z = 0$. It is easily checked from equation (17) that

$$\Gamma_A = \tfrac{1}{4}\alpha_A(0)\,n(0)\,c(0), \tag{35}$$

where $n(0)$ is equivalent to n_S. Equation (35) is just a logical extension of (15a) so that in effect we are treating the electrode A as if it is right at the source at $z = 0$. Equation (34a) tells us how to modify the absorption coefficient to take into account electron–atom collisions.

The extension to any z is straightforward. The logical generalizations of equations (34) are

$$\frac{1}{\alpha_A(z)} = \frac{\varepsilon(z)}{\alpha_A \varepsilon_A} + \frac{3\varepsilon(z)}{4eE} \int_{\varepsilon(z)}^{\varepsilon_A} \frac{d\varepsilon'}{\varepsilon' l(\varepsilon')}, \tag{36a}$$

$$\frac{1}{\alpha_B(z)} = \frac{\varepsilon(z)}{\alpha_B \varepsilon_B} + \frac{3\varepsilon(z)}{4eE} \int_{\varepsilon_B}^{\varepsilon(z)} \frac{d\varepsilon'}{\varepsilon' l(\varepsilon')}. \tag{36b}$$

The connection between $\alpha_A(z)$ and α_A can be extended to one between $\alpha_A(z)$ and $\alpha_A(z')$, for any z', by replacing α_A by $\alpha_A(z')$ and ε_A by $\varepsilon(z')$ in equation (36a), and similarly for $\alpha_B(z)$. The generalization of equations (33) serves essentially as a definition of $S_A(z)$ and $S_B(z)$. For $0 \leqslant z \leqslant L-d$ for instance we have

$$\frac{\Gamma_A}{S_A(z)} = \frac{\alpha_A(z)}{\alpha_A(z)+\alpha_B(z)}, \tag{37a}$$

and the 'back flux' to B for positive z is given by

$$\frac{\Gamma_B(z)}{S_A(z)} = \frac{\alpha_B(z)}{\alpha_A(z)+\alpha_B(z)}. \tag{37b}$$

Note that $\Gamma_B(z)$ is z-dependent for positive z unlike the case of negative z, and it represents the flux of electrons that have reached at least as far as z and have been finally repulsed by the gas to eventually flow to B. The ratio $\Gamma_B(z)/S_A(z)$ is essentially

the probability that an electron is repulsed by electrode A and by the gas in the interval $z \leqslant z' \leqslant L-h$ and flows to electrode B. For $z = L-d$ this probability is seen to be very small, consistent with our discussion of equation (21b). Using equations (11), (12), (17) and (35) one readily verifies that $z = 0$ can be replaced in equation (35) by any z, for $0 \leqslant z \leqslant L-d$.

A very useful property which follows from equations (36) is the identity

$$\frac{\varepsilon(z)\,\alpha_A(z)\,\alpha_B(z)}{\alpha_A(z)+\alpha_B(z)} = \frac{\varepsilon(z')\,\alpha_A(z')\,\alpha_B(z')}{\alpha_A(z')+\alpha_B(z')}, \tag{38}$$

for any z and z'. This invariant under coordinate transformation can be used to construct other invariants, e.g.

$$\frac{S(0)}{\varepsilon(0)\,\alpha_B(0)} = \frac{S_A(z)}{\varepsilon(z)\,\alpha_B(z)}, \qquad \frac{S(0)}{\varepsilon(0)\,\alpha_A(0)} = \frac{S_B(z)}{\varepsilon(z)\,\alpha_A(z)}. \tag{39a, b}$$

Equation (39a) is obtained by eliminating Γ_A using equations (33a) and (37a).

8. Relations between Momentum Transfer Cross Section and Experimentally Measurable Quantities Alone

The simplest method of separating off unknown electrode sensitive contributions to Γ_A and Γ_B is to ensure that d and $L-d$ are sufficiently large that electron–atom collisions dominate. However, because these fluxes only involve the ratio $\alpha_A(0)/\alpha_B(0)$, the price one pays is that only information on the dimensionless form of $q_m(\varepsilon)$ is obtained. From equation (19b) we have

$$\frac{\Gamma_A}{\Gamma_B} \approx \left(\int_{\varepsilon_B}^{\varepsilon_S} q_m(\varepsilon')/\varepsilon' \, d\varepsilon' \bigg/ \int_{\varepsilon_S}^{\varepsilon_A} q_m(\varepsilon')/\varepsilon' \, d\varepsilon' \right)$$

$$\times \left[1 - \tfrac{4}{3}eE\left\{ \left(\alpha_A \varepsilon_A \int_{\varepsilon_S}^{\varepsilon_A} \frac{d\varepsilon'}{\varepsilon' l(\varepsilon')} \right)^{-1} \right. \right.$$

$$\left. \left. - \left(\alpha_B \varepsilon_B \int_{\varepsilon_B}^{\varepsilon_S} \frac{d\varepsilon'}{\varepsilon' l(\varepsilon')} \right)^{-1} \right\} + O(N^{-2}) \right]. \tag{40}$$

The leading factor of interest can be separated off because it is density independent. The range of energies covered is controlled by E. For $E \to 0$ the leading term approaches $d/(L-d)$. A similar expression is obtainable if S is an actual fine mesh cathode with an absorption coefficient α_S. In that case accelerating or retarding fields are simultaneously possible on opposite sides of S.

Another approach which avoids the need to observe pressure change effects is obtained when B is the electron source ($d = 0$). From equation (21a) we have

$$\left(\frac{\partial(S/\Gamma_A)}{\partial L} \right)_{\varepsilon_A} = \frac{3\alpha_B \varepsilon_B}{4(\varepsilon_A-\varepsilon_B)} \int_{\varepsilon_B}^{\varepsilon_A} \frac{d\varepsilon'}{\varepsilon' l(\varepsilon')}, \tag{41a}$$

$$\frac{\partial}{\partial \varepsilon_A} \left(\frac{\partial(S/\Gamma_A)}{\partial L} \right)_{\varepsilon_A} = \frac{3\alpha_B \varepsilon_B}{4(\varepsilon_A-\varepsilon_B)} \left(\frac{1}{\varepsilon_A\, l(\varepsilon_A)} - \frac{1}{\varepsilon_A-\varepsilon_B} \int_{\varepsilon_B}^{\varepsilon_A} \frac{d\varepsilon'}{\varepsilon' l(\varepsilon')} \right). \tag{41b}$$

By combining equations (41a) and (41b) α_B and S can be eliminated. This gives the required relation

$$\frac{\partial\{\ln(\partial\Gamma_A^{-1}/\partial L)_{\varepsilon_A}\}}{\partial\varepsilon_A} = \varepsilon_A^{-1} q_m(\varepsilon_A) \bigg/ \left(\int_{\varepsilon_B}^{\varepsilon_A} q_m(\varepsilon')/\varepsilon' \, d\varepsilon'\right) - \frac{1}{\varepsilon_A - \varepsilon_B}. \tag{42}$$

A third expression which eliminates the electrode sensitive terms is obtained by solving for the RHS integral in equation (42) and taking the partial derivative with respect to ε_B. This leads to

$$\frac{q_m(\varepsilon_B)}{q_m(\varepsilon_A)} = -\frac{\varepsilon_B}{\varepsilon_A} \frac{\partial G(\varepsilon_B, \varepsilon_A)}{\partial\varepsilon_B}, \tag{43a}$$

where

$$G(\varepsilon_B, \varepsilon_A) = \left(\frac{\partial\{\ln(\partial\Gamma_A^{-1}/\partial L)_{\varepsilon_A}\}}{\partial\varepsilon_A} + \frac{1}{\varepsilon_A - \varepsilon_B}\right)^{-1}. \tag{43b}$$

Alternatively the partial derivative with respect to ε_A can be taken, leading to

$$\frac{\partial[\ln\{\varepsilon_A^{-1} q_m(\varepsilon_A)\}]}{\partial\varepsilon_A} = \frac{1}{G(\varepsilon_B, \varepsilon_A)} - \frac{\partial\{\ln G(\varepsilon_B, \varepsilon_A)\}}{\partial\varepsilon_A}.$$

This can be integrated again to give

$$\frac{q_m(\varepsilon)}{q_m(\varepsilon_0)} = \frac{\varepsilon}{\varepsilon_0} \frac{G(\varepsilon_B, \varepsilon_0)}{G(\varepsilon_B, \varepsilon)} \exp\left(\int_{\varepsilon_0}^{\varepsilon} \frac{1}{G(\varepsilon_B, \varepsilon')} \, d\varepsilon'\right),$$

which simplifies further to

$$\frac{q_m(\varepsilon)}{q_m(\varepsilon_0)} = \frac{\varepsilon}{\varepsilon_0} \frac{G(\varepsilon_B, \varepsilon_0)}{G(\varepsilon_B, \varepsilon)} \frac{\varepsilon - \varepsilon_B}{\varepsilon_0 - \varepsilon_B} \frac{(\partial\Gamma_A^{-1}/\partial L)_{\varepsilon}}{(\partial\Gamma_A^{-1}/\partial L)_{\varepsilon_0}}. \tag{44}$$

The RHS is actually independent of ε_B and this can in principle be experimentally tested.

9. Electron Flux for a Distribution of Energies at the Source

In any practical situation with an electron source that can be used for the close electrode configuration (e.g. back illuminated photocathodes), the electrons are produced with some distribution of initial energies. Equation (37a) is then replaced by, for $0 \leqslant z \leqslant L - d$,

$$\Gamma_A(\varepsilon_m) = \int_{\varepsilon_m}^{\infty} \frac{\alpha_A(z, \varepsilon)}{\alpha_A(z, \varepsilon) + \alpha_B(z, \varepsilon)} \frac{dS_A(z, \varepsilon)}{d\varepsilon} \, d\varepsilon, \tag{45}$$

where $dS_A(z, \varepsilon)$ is the effective source strength in the energy range $\varepsilon, \varepsilon + d\varepsilon$. Here ε_m is nonzero only if the electric field is chosen to de-accelerate electrons approaching the anode so that $\varepsilon_m = eE(L - d)$.

Where electron–atom collisions dominate $\alpha_A(z, \varepsilon)$, equation (45) can be approximated by

$$\Gamma_A(\varepsilon_m) \approx \frac{4}{3(L-d-z)} \int_{\varepsilon_m}^{\infty} F_A(z, \varepsilon) \frac{\varepsilon_A - \varepsilon}{\varepsilon} \left(\int_{\varepsilon}^{\varepsilon_A} \frac{d\varepsilon'}{\varepsilon' l(\varepsilon')} \right)^{-1} d\varepsilon, \qquad (46a)$$

where

$$F_A(z, \varepsilon) = \{\alpha_A(z, \varepsilon) + \alpha_B(z, \varepsilon)\}^{-1} dS_A(z, \varepsilon)/d\varepsilon. \qquad (46b)$$

Equations (45) and (46a) may be used to obtain rough 'scans' of the energy distribution, assuming that the electron reflection probabilities at the electrodes are not too strongly energy dependent. An experiment of this type has been performed for electrons in Hg vapour (McMahon 1978, 1979, 1983) showing the essential correctness of this electron transport theory in the boundary layer. These observations also provide some information on the low energy electron–electrode reflection probability and suggest quite high reflectivities of 80–90% (or $\alpha \approx 0.1$). Such small absorption coefficients would have a substantial effect on the extrapolation distance ξ_A (see equation 23a). For most purposes where the electron drift lengths are much longer than l_ε, the effect of electron reflections at the electrodes on the drift length is not important. High electron reflection probabilities would be of significance however for the Cavalleri and Huxley–Townsend experiments where the theoretical analysis assumes perfectly absorbing electrodes in the boundary condition $n(z) = 0$ at all electrode surfaces. For closely spaced electrodes high reflectivities can make significant contributions to the electron flux and ideally one would like to separate these by experimental means (see Section 8).

10. Discussion

In its present form, this theory of electron drift and diffusion between closely spaced electrodes is only applicable to heavy atomic gases. It is desirable to extend the theory to nonzero m/M values where presumably for a monoenergetic electron source between closely spaced electrodes the δ functions of equations (5) and (7) are replaced by less strongly peaked functions with widths depending on m/M. It may be possible to go much further with an analytical theory, without previous restrictions (Robson 1981) to special collision models, by using the present work as a guide.

Another area in which the present work can be profitably exploited is in the study of the distribution function at distances of a few mean free paths or less from the boundary. There the neglect of energy losses by elastic collisions is especially accurate. At electrode A the two-term distribution function is proportional to $1 + \frac{3}{4}\alpha_A \cos\theta$ which assumes unphysical negative values for $\alpha_A \cos\theta < -\frac{4}{3}$. Such problems are solved either by going to a multi-term theory or Robson's (1981) mathematically equivalent half-range formulation. The four-term extension of the present work is readily obtained and shows that the two-term theory is accurate at distances greater than a few mean free paths from the boundary. The present 'full-range' two-term theory can also be easily extended by going to the half-range two-term theory. The results are almost identical to the present work except that the more accurate treatment of the distribution function within a few mean paths of the boundary gives a modified expression for the absorption coefficient. This is of no practical con-

sequence when the electron flux is dominated by electron–atom collisions and for most purposes the present full-range two-term treatment is adequate.

Acknowledgments

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Appendix

The two-term approximation leads to equation (22a) in the limit of zero gas density. Although this approximation is meaningless when l becomes comparable with L, our boundary condition approach does give a reasonable qualitative form for the flux. Here we derive equations (22b) and (22c) which apply to elastic isotropic reflections of the electrons from infinite plane parallel electrode surfaces.

Let the electrons be ejected from electrode B with the source strength S_{ej}. Consider firstly an accelerating field from B to A. The initial flux incident on A is therefore S_{ej} of which $r_A S_{ej}$ is reflected. Of the flux S_{ej}, let $\lambda_A S_{ej}$ be that flux reflected back to B to be either absorbed or reflected. To find λ_A one has to take into account the fact that some electrons are multiply reflected from A before developing a sufficiently large velocity component normal to the plane of the electrodes to overcome the A to B retarding field. We leave the calculation of λ_A to later. Of the flux $\lambda_A S_{ej}$ incident on B, $(1 - r_B)\lambda_A S_{ej}$ is absorbed and a second flux $r_B \lambda_A S_{ej}$ is incident on A. For each of the first, second, third etc. flux incident A, the fraction $1 - \lambda_A$ is absorbed. Then Γ_A is given by the infinite series

$$(\Gamma_A)_0 = (1 - \lambda_A)S_{ej}\{1 + r_B \lambda_A + (r_B \lambda_A)^2 + ...\}$$

$$= \{(1 - \lambda_A)/(1 - r_B \lambda_A)\}S_{ej}. \tag{A1}$$

The only electrons reflected from A that can make it back to B are those for which

$$\varepsilon_A \cos^2\theta > eEL = \varepsilon_A - \varepsilon_B,$$

where θ is the angle between the normal to A and the electron velocity. We define θ_m by $\cos^2\theta_m = 1 - \varepsilon_B/\varepsilon_A$. By Lambert's law, the fraction of electrons isotropically reflected into $0 \leqslant \theta \leqslant \theta_m$ is $1 - \cos\theta_m$. Thus, of the incident flux, the fraction of electrons that make it back to B without one or more rebounds from A is $r_A(1 - \cos\theta_m)$. The remaining fraction $r_A \cos\theta_m$ require rebounds and each rebound gives a further loss of reflected intensity by the fraction r_A. The first rebound contributes the fraction $r_A \cos\theta_m r_A(1 - \cos\theta_m)$ of the incident flux that gets back to B. In this way λ_A is seen to be the infinite sum

$$\lambda_A = r_A(1 - \cos\theta_m)\{1 + r_A \cos\theta_m + (r_A \cos\theta_m)^2 + ...\}$$

$$= r_A(1 - \cos\theta_m)/(1 - r_A \cos\theta_m). \tag{A2}$$

Combining equations (A1) and (A2), after a little algebra, equation (22b) is obtained.

When the electric field retards electrons emitted or reflected from B we merely need to modify the above derivation to take into account the fact that only some fraction $\gamma_B S_{ej}$ of the initially emitted electron flux makes up the flux first incident on A. Here γ_B is calculated from an infinite series for rebounds and is given by

$$\gamma_B = (1 - \cos\theta_m)\{1 + r_B \cos\theta_m + (r_B \cos\theta_m)^2 + ...\} = \lambda_B/r_B, \tag{A3}$$

where $\cos^2\theta_m = 1 - \varepsilon_A/\varepsilon_B$ and λ_B has the same form as equation (A2). The flux reflected from A and thus incident on B is $r_A \gamma_B S_{ej}$. The second flux incident on A is therefore $\gamma_B \lambda_B r_A S_{ej}$, of which the fraction $1 - r_A$ is absorbed. Adding up the obvious infinite series gives

$$(\Gamma_A)_0 = \frac{(1 - r_A)\gamma_B}{1 - r_A \lambda_B} S_{ej}. \tag{A4}$$

After some algebra equation (A4) finally gives (22c).