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Some Applications of Transformation to the Space-independent Frame in the Processes of Strong Radiation in Plasmas

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Abstract

Transformation of nonlinear plasma equations from a lab frame S to the space-independent frame S′ (both inertial) for an electromagnetic (EM) wave in an unbounded plasma reduces the nonlinear partial differential equations in S to ordinary nonlinear differential equations in S′. This relativistically correct transformation is used (1) to find the intensity induced precessional rotation of the polarization ellipse of vibration of an EM wave, (2) in the S-frame Lagrangian of the particles and field produced by them to derive the exact nonlinearly correct dispersion relation for a strong circularly polarized wave in a cold unmagnetized plasma, and (3) to rectify some much discussed differential equations obtained by Akhiezer and Polovin (1956) to study the evolution of longitudinal and transverse waves in a cold plasma.

1. Introduction

Winkles and Eldridge (1972) first found a special Lorentz transformation (LT) from the lab frame S to a wave frame S′ (both inertial) in which the four-vector of the space–time continuum transforms to a time-like vector for velocities less than the vacuum speed of light c. Thus, in place of the nonlinear partial differential equations in the S frame, one obtains for solution a set of ordinary nonlinear differential equations in the S′ frame. An analogous transformation to a space-like vector is also theoretically possible, but at velocities invariably greater than c. So, for relativistic reasons, this transformation is not used.

The method of solving the field equations after transforming them to the space-independent frame also has the advantage that some of the field variables become either constant or zero in the S′ frame; for instance, the number density and the scalar potential become constant and the oscillation of the magnetic field vanishes in the S′ frame. As a result, some of the nonlinear terms which appear in the S-frame calculation vanish in the S′ frame. For this reason, investigations of some nonlinear effects, particularly the self-action effects (e.g. self-focussing, self-steepening, self-phase modulation, self-precession etc.) in plasmas and other media are expected to be interesting and easier by this method.

Subsequent to the discovery of this useful LT, several investigations on expanding its scope and applicability have already been reported. Among these may be mentioned the work of Clemmow (1974, 1975, 1977), Chian and Clemmow (1975), Kennal and Pellat (1976), Shih (1978), Decoster (1978) and Clemmow and Harding (1980). In addition, Akhiezer and Polovin (1956), Wong (1963) and Wong and Lojko (1963)

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also used similar transformation relations for the study of nonlinear propagation of waves through plasmas, which are relatively simpler but relativistically incorrect. Hence, it is possible to find the relativistically correct derivation of the main equations and solutions reported by these authors.

Our motivation in the present paper is to broaden the base of the work started by Winkles and Eldridge (1972) so that problems involving nonlinearly evolved self-precession of a strong EM wave and other self-induced effects in plasmas and other media can be easily investigated with the help of this transformation technique. In the present paper, for a cold, unmagnetized and collisionally undamped electron plasma, we derive the expressions for the Lagrangian $\mathscr{L}'$ in S′, starting from the Lagrangian $\mathscr{L}$ in the S frame, for a strong circularly polarized wave and hence obtain the exact nonlinear dispersion relation. The intensity-induced shift for a wave parameter for these waves follows easily from this relation. Another significant aspect reported here is the derivation by this method of the intensity-induced precessional rotation of the polarization ellipse of vibration for an EM wave. In addition, the complementary effects of the nonlinearly induced wave number shifts in the same plasma are derived as a consequence of the birefringence of the nonlinearly correct left and right circularly polarized components of the wave. This self-induced rotation has wide possibilities for generalization and application. It was first derived analytically in a plasma by Arons and Max (1974) using the direct method of obtaining a secular free solution by a convenient process of successive approximation. Also, we determine here the relativistically correct differential equations, the nonrelativistic simplified versions of which were obtained by Akhiezer and Polovin (1956) and further extensively considered by Akhiezer *et al.* (1975) and others for the study of nonlinear evolution of longitudinal and transverse waves in a cold plasma.

2. Assumptions and Basic Equations

We consider a cold, homogeneous, stationary plasma subject to a strong radiation of intensity less than about $3\times10^{22}\,\mathrm{W\,cm^{-2}}$ resulting in a relativistic electron velocity; the ion motion, small in comparison with the electron motion at the high frequencies of these radiations, is neglected. The forces arising due to other sources (e.g. collision, gravitation, ponderomotive force etc.) are negligible. Thus, the plasma equations in the lab inertial frame S can be written as

$$\frac{\partial \boldsymbol{p}}{\partial t}+(\boldsymbol{v}.\nabla)\boldsymbol{p} = -e\boldsymbol{E}-ec^{-1}(\boldsymbol{v}\times\boldsymbol{H}), \tag{1a}$$

$$\frac{\partial N}{\partial t}+\nabla.(N\boldsymbol{v}) = 0, \qquad \nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t}, \tag{1b,c}$$

$$\nabla\times\boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}-\frac{4\pi Ne\boldsymbol{v}}{c}, \qquad \nabla.\boldsymbol{E} = 4\pi e(N_i-N), \tag{1d,e}$$

$$\nabla.\boldsymbol{H} = 0, \qquad \boldsymbol{p} = m_0\boldsymbol{v}/(1-v^2/c^2)^{\frac{1}{2}}, \tag{1f,g}$$

where N_i and N are the number densities of ions and electrons, m_0 and $-e$ are the rest mass and charge of an electron, and the other symbols have their usual meaning.

For not very strong relativistic effects, if we expand $\boldsymbol{p}$ in equation (1g) in powers of v^2/c^2 then only the first two terms need be retained because these are sufficient for effects correct up to third order; thus the effective part, for this purpose, of equation (1a) is

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{v} + \frac{\partial}{\partial t}\left(\frac{v^2 \boldsymbol{v}}{2c^2}\right) = -\frac{e\boldsymbol{E}}{m_0} - \frac{e}{m_0 c}(\boldsymbol{v}\times\boldsymbol{H}). \tag{2}$$

The vector and scalar potentials A and ϕ in the Lorentz gauge are given by

$$\boldsymbol{H} = \nabla\times\boldsymbol{A}, \qquad \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi, \qquad \nabla.\boldsymbol{A} = -\frac{1}{c}\frac{\partial\phi}{\partial t}. \tag{3a, b, c}$$

Following Landau and Lifshitz (1975, p. 209), the Lagrangian $\mathscr{L}$ of the particles and the fields produced by them is

$$\mathscr{L} = -\sum_i m_i N_{i0} c^2 (1 - v_i^2/c^2)^{\frac{1}{2}} + \rho\phi + \boldsymbol{A}.\boldsymbol{j} + (E^2 - H^2)/8\pi, \tag{4}$$

where ρ and $\boldsymbol{j}$ are the charge density and electric current vector respectively and N_{i0} is the equilibrium state number density of the ith species of particles:

$$\rho = \sum_i e_i N_i, \qquad \boldsymbol{j} = \sum_i e_i N_i \boldsymbol{v}_i. \tag{5a, b}$$

For the problem under consideration we have

$$\mathscr{L} = -m_0 N_0 c^2 (1 - v^2/c^2)^{\frac{1}{2}} - eN\phi - eN\boldsymbol{v}.\boldsymbol{A} + (E^2 - H^2)/8\pi. \tag{6}$$

The corresponding Hamiltonian $\mathscr{H}$ is given by

$$\mathscr{H} = (\boldsymbol{v}.\partial\mathscr{L}/\partial\boldsymbol{v}) - \mathscr{L} = m_0 N_0 \gamma c^2 + eN\phi - (E^2 - H^2)/8\pi, \tag{7}$$

where $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$.

3. Space-independent Frame

The LT from the S frame to the S′ frame which is moving with relative velocity V_0 parallel to the z-axis is given by

$$x = x', \quad y = y', \quad z = \gamma_0(z' + V_0 t'), \quad t = \gamma_0(t' + V_0 z'/c^2), \tag{8}$$

where

$$\gamma_0^2 = (1 - \beta_0^2)^{-1}; \qquad \beta_0 = V_0/c. \tag{9}$$

Following Winkles and Eldridge (1972) we assume

$$V_0 = c^2/V = kc^2/\omega, \tag{10}$$

$V\,(=\omega/k)$ being the phase velocity of the wave. For transverse waves $V > c$, and so $V_0 < c$. Hence, the velocity of S′ relative to S is physically attainable, and therefore the wave phase reduces to

$$\omega t - kz = \omega(1 - \beta_0^2)^{\frac{1}{2}} t' = \omega' T, \tag{11}$$

where

$$t' \equiv T, \qquad \omega' \equiv \omega(1 - \beta_0^2)^{\frac{1}{2}} = \omega/\gamma_0. \tag{12a, b}$$

In place of equation (11), Akhiezer *et al.* (1975) considered the simple linear transformation relation

$$\omega t - kz = \omega T \tag{13}$$

for an investigation of nonlinear effects; Boyd and Sanderson (1969, p. 27) took the special value $V = c$ for similar studies. The relativistically correct improvement of the equations derived by Akhiezer *et al.* (1975) is given by equation (35) in Section 4. Based on equation (11) the partial derivatives with respect to z and t are changed to ordinary derivatives with respect to T. Some useful relations are

$$\frac{\partial z}{\partial t} = V - \frac{V}{\gamma_0}\frac{\partial T}{\partial t}, \qquad \frac{\partial}{\partial t} = \gamma_0 \frac{\partial}{\partial T}, \qquad \frac{\partial}{\partial z} = -\frac{\gamma_0}{V}\frac{\partial}{\partial T}. \tag{14}$$

From equation (1.4-5) of Hughes and Young (1966) we have

$$(1-\beta^2)^{\frac{1}{2}} = (1-\beta'^2)^{\frac{1}{2}}/\beta_0(1+\beta_0\beta'_z), \tag{15}$$

where $\boldsymbol{\beta} = \boldsymbol{v}/c$ and $\boldsymbol{\beta}' = \boldsymbol{v}'/c$. Transformation of $\boldsymbol{v}$, $\boldsymbol{E}$ and $\boldsymbol{H}$ from the S frame to the S′ frame gives

$$(v_x, v_y) = \frac{(v'_x, v'_y)}{\gamma_0(1+\beta_0\beta'_z)}, \qquad v_z = \frac{v'_z + V_0}{1+\beta_0\beta'_z}, \tag{16a, b}$$

$$E_x = \gamma_0(E'_x + \beta_0 H'_y), \qquad E_y = \gamma_0(E'_y - \beta_0 H'_x), \qquad E_z = E'_z, \tag{17a}$$

$$H_x = \gamma_0(H'_x - \beta_0 E'_y), \qquad H_y = \gamma_0(H'_y + \beta_0 E'_x), \qquad H_z = H'_z = 0. \tag{17b}$$

Following Section 1.7 of Hughes and Young (1966) and using equation (15) we find that mass is transformed as

$$m = m'\gamma_0(1+\beta_0\beta'_z), \tag{18}$$

and momentum components as

$$p_x = p'_x, \qquad p_y = p'_y, \qquad p_z = \gamma_0(p'_z + m_0 V_0 \gamma'). \tag{19}$$

The potentials A and ϕ of equations (3) form a four-vector, so their transformation to the S′ frame gives

$$A_x = A'_x, \qquad A_y = A'_y, \qquad A_z = \gamma_0(A'_z + \beta_0\phi'), \tag{20a}$$

$$\phi = \gamma_0(\phi' + \beta_0 A'_z). \tag{20b}$$

The Lorentz gauge condition of (3c) when transformed to the S′ frame gives

$$-\frac{\gamma_0^2}{V}\frac{\partial}{\partial T}(A'_z + \beta_0\phi') = -\frac{\gamma_0^2}{c}\frac{\partial}{\partial T}(\phi' + \beta_0 A'_z). \tag{21}$$

Since the terms containing A'_z cancel from both sides and since a constant potential is ignored here, we have

$$\phi' = 0. \tag{22}$$

Also, the z component of (3b) on transformation to the S′ frame becomes

$$E'_z = -\frac{\gamma_0^2}{c}\frac{\partial}{\partial T}(A'_z+\beta_0\phi') - \frac{\gamma_0^2}{V}\frac{\partial}{\partial T}(A'_z+\beta_0\phi'). \tag{23}$$

From (22) this equation reduces to

$$E'_z = -\frac{\gamma_0^2}{cV}(V+c)\frac{\partial A'_z}{\partial T}. \tag{24}$$

4. Transformation of Field Equations to Space-independent Frame

With the help of the transformation relations of Section 3, the Maxwell equations (1c) and (1d) give

$$(H'_x, H'_y, H'_z) = 0 \tag{25}$$

(because there is no applied d.c. magnetic field) and

$$(\dot{E}'_x, \dot{E}'_y) = \frac{4\pi e N_0(v'_x, v'_y)}{\gamma_0(1+\beta_0\beta'_z)}, \qquad \dot{E}'_z = \frac{4\pi e N_0(v'_z+V_0)}{\gamma_0(1+\beta_0\beta'_z)}, \tag{26a, b}$$

where N_0 is constant and a derivative with respect to T is denoted by a dot.

From equation (16b) we see that $v_z = 0$ when $v'_z = -V_0$. Thus, we write

$$v'_z = -V_0+\delta v'_z, \tag{27}$$

and expand $1/(1+\beta_0\beta'_z)$ in powers of β'_z. Then equations (16) give

$$(v_x, v_y) = \gamma_0(v'_x, v'_y)(1-\gamma_0^2\beta_0\,\delta\beta'_z+\gamma_0^4\beta_0^2\,\delta\beta'^2_z - \ldots), \tag{28a}$$

$$v_z = \gamma_0^2\,\delta v'_z(1-\gamma_0^2\beta_0\,\delta\beta'_z+\gamma_0^4\beta_0^2\,\delta\beta'^2_z - \ldots), \tag{28b}$$

and equations (26) reduce to

$$(\dot{E}'_x, \dot{E}'_y) = (\gamma_0 m_0\,\omega_p^2/e)(v'_x, v'_y)(1-\gamma_0^2\beta_0\,\delta\beta'_z), \tag{29a}$$

$$\dot{E}'_z = (\gamma_0 m_0\,\omega_p^2/e)\delta v'_z(1-\gamma_0^2\beta_0\,\delta\beta'_z), \tag{29b}$$

where the electron plasma frequency $\omega_p = (4\pi N_0 e^2/m_0)^{\frac{1}{2}}$.

If N and N' are the electron number densities in the S and S′ frames respectively, then from Section 1.6 of Hughes and Young (1966) we obtain

$$N_0 = N(1-v^2/c^2)^{\frac{1}{2}} = \text{const.}, \tag{30a}$$

$$m_0 N' = m' N_0, \qquad m_0 N = m N_0, \tag{30b}$$

$$N/N' = m/m' = (\beta_0/\gamma_0^2)(1+\gamma_0^2\beta_0\,\delta\beta'_z), \tag{30c}$$

$$N'_0 = (\gamma_0^2/\beta_0)N_0, \tag{30d}$$

where N'_0 is the unperturbed state value of N'. Equation (30a) shows that in the S frame the number density is not constant and nor is charge neutrality ($N_e = N_i$) necessarily ensured, although the total charge of the plasma is constant in all inertial frames of reference. Transforming now the equation of continuity (1b) to the S′ frame with the help of equations (14), (16) and (30c) we find that

$$N' = N_0' = (\gamma_0^2/\beta_0)N_0 = \text{const.}, \tag{31}$$

and hence

$$\delta N' = 0. \tag{32}$$

With (25) equations (17) give

$$(E_x, E_y) = \gamma_0(E_x', E_y'), \qquad E_z = E_z', \tag{33a, b}$$

$$(H_x, H_y) = -\gamma_0\beta_0(E_y', -E_x'). \tag{33c}$$

From equations (25) and (33) the components of the equation of motion (1a), when transformed to the S′ frame, become

$$(\dot{p}_x', \dot{p}_y') = -e(E_x', E_y'), \tag{34a}$$

$$\begin{aligned}\dot{p}_z' = {}& \frac{eE_z'}{1+\beta_0\gamma_0(\gamma_0-1)\delta\beta_z'}\left(\frac{p_z'}{m_0\gamma_0^2\gamma' V_0}-1\right)\\ &+\frac{e(p_x' E_x' + p_y' E_y')}{1+\beta_0\gamma_0(\gamma_0-1)\delta\beta_z'}\\ &\times\frac{\gamma_0^2 V(V-V_0)-2V_0V+V_0^2+\delta v_z'\{V(\gamma_0^2-\gamma_0+1)-V_0\}}{m_0\gamma' c_2(V-V_0+\delta v_z')}.\end{aligned} \tag{34b}$$

Differentiating (34a) with respect to T and using (29a) we find that

$$\ddot{\boldsymbol{q}}_\perp' = -\frac{\gamma_0\omega_p^2\boldsymbol{q}_\perp'}{(1+q'^2)^{\frac{1}{2}}+\beta_0\gamma_0^2\delta q_z'}, \tag{35}$$

where $\boldsymbol{q}' = \boldsymbol{p}'/m_0 c$ and $\boldsymbol{q}_\perp' = (q_x', q_y')$.

Equation (35) is more general than the first and second equations of (8.1.2.16) in Akhiezer *et al.* (1975). Equation (34b) with the help of (29) and (34a) can be similarly reduced to a form which is more general than the third of these equations of Akhiezer *et al.* (1975) and which is relativistically correct. Therefore, the investigations reported by these authors on the nonlinear evolution of longitudinal and transverse waves using their equations (8.1.2.16) can be generalized and made relativistically correct, even if only approximately, with the help of (34) and (35).

Linearized Equations and Their Solution

In the linearized approximations (26a) and (28a) the transverse components give

$$(\dot{E}_x', \dot{E}_y') = (\gamma_0 m_0\omega_p^2/e)(v_x', v_y'), \qquad (\dot{v}_x', \dot{v}_y') = -(e/m_0\gamma_0)(E_x', E_y'). \tag{36a, b}$$

When considered together these equations yield solutions for simple harmonic oscillation with a frequency ω' given by

$$\omega'^2 = \omega_p^2. \tag{37}$$

When transformed to the S frame with the help of (11), (12) and (37) this equation leads to the standard dispersion relation of the linearized theory:

$$\omega^2 - k^2c^2 = \omega_p^2. \tag{38}$$

Nonlinear Equations for Secular Free Solution

For convenience we adopt rotating (complex) coordinates, using the substitutions

$$v_\pm = v_x' \pm \mathrm{i}\, v_y', \qquad E_\pm = E_x' \pm \mathrm{i}\, E_y', \tag{39a, b}$$

and find that the nonlinear field equations correct up to third order reduce to

$$\dot v_\pm \pm \frac{e}{m_0 \gamma_0} E_\pm = \frac{\gamma_0^2}{V}\left(\dot v_\pm \, \delta v_z' + \frac{\partial}{\partial T}(v_\pm \, \delta v_z')\right)$$
$$+\gamma_0^2 \frac{\partial}{\partial T}\left(\frac{v_\pm^2 \, v_\mp}{2c^2}\right) + \frac{e\beta_0}{m_0 c}\gamma_0 \, E_\pm \, \delta v_z', \tag{40}$$

$$\dot E_\pm = (m_0 \, \omega_\mathrm{p}^2 \, \gamma_0 / e) v_\pm \, , \tag{41}$$

$$\delta \dot v_z' = -\frac{eE_z'}{m_0 \gamma_0^3} - \frac{e\beta_0}{2m_0 \, c\gamma_0}(E_+ \, v_- + E_- \, v_+), \tag{42a}$$

$$\dot E_z' = (m_0 \, \omega_\mathrm{p}^2 \, \gamma_0 / e)\delta v_z' . \tag{42b}$$

From (40) and (41) we obtain

$$\frac{e}{m_0}(\ddot E_\pm + \omega_\mathrm{p}^2 \, E_\pm) = \frac{\gamma_0^3 \, \omega_\mathrm{p}^2}{V}(\dot v_\pm \, \delta v_z' + v_\pm \, \delta \dot v_z')$$
$$-\gamma_0^3 \, \omega_\mathrm{p}^2 \frac{\partial}{\partial T}\left(\frac{v_\pm^2 \, v_\mp}{2c^2}\right) + \frac{\gamma_0^2 \, \beta_0 \, e\omega_\mathrm{p}^2}{m_0 \, c} E_\pm \, \delta v_z', \tag{43}$$

while equations (42) give

$$\delta \ddot v_z' + \frac{\omega_\mathrm{p}^2}{\gamma_0}\delta v_z' = -\frac{e\beta_0}{2m_0 \, c\gamma_0}\frac{\partial}{\partial T}(E_+ \, v_- + E_- \, v_+). \tag{44}$$

We now write

$$E_\pm = \tfrac{1}{2}\{(a \pm b)\exp(\mathrm{i}\,\theta_0) + (a \mp b)\exp(-\mathrm{i}\,\theta_0)\} = a\cos\theta_0 \pm \mathrm{i}\, b\sin\theta_0 , \tag{45}$$

where a and b are the amplitudes of the EM wave along the principal directions of its polarization ellipse and $\theta_0 = \omega_\mathrm{p} T$. Then equations (36b) and (44) give

$$v_\pm = (e/m_0 \, \gamma_0 \, \omega_\mathrm{p})(-a\sin\theta_0 \pm \mathrm{i}\, b\cos\theta_0), \tag{46}$$

$$\delta v_z' = -e^2(a^2 - b^2)\omega^2 \cos 2\theta_0 / m_0^2 \, V\gamma_0^2 \, \omega_\mathrm{p}^2(3\omega^2 + k^2c^2), \tag{47}$$

where we have used (9) and (10) to find an expression for $4\gamma_0^2 - 1$ in the denominator of (47). Now we evaluate all terms on the RHS of (43) which give the first harmonic field effect and obtain

$$\ddot E_\pm + \omega_\mathrm{p}^2 \, E_\pm = -\frac{e^2(a^2 - b^2)\omega^2}{2m_0^2 \, V^2(3\omega^2 + k^2c^2)}(a\cos\theta_0 \mp \mathrm{i}\, b\sin\theta_0)$$
$$+\frac{e^2}{8m_0^2 \, c^2}\{(3a^2 + b^2)a\cos\theta_0 \pm \mathrm{i}(a^2 + 3b^2)b\sin\theta_0\}. \tag{48}$$

Nonlinearly Induced Birefringence for Precessional Rotation and Wave Number Shift

It has already been mentioned that the left and right circularly polarized components of a wave propagate at different rates due to plasma nonlinearities and thus give rise to the effect of birefringence. When these components propagate at the same rate, the polarization ellipse does not experience any precessional rotation about the direction of propagation. But, as argued by Maker *et al.* (1964), when the two rates differ, the two wave components undergo a different phase retardation, which results in a rotation of the polarization ellipse through an angle φ, equal to half the difference in the phase delay of the two components of the wave. The spatial rate of the precessional rotation can therefore be evaluated from the relation

$$\varphi = \tfrac{1}{2}(k_- - k_+)z, \tag{49}$$

where $k_\pm$ are the wave numbers of the two wave components.

To find $k_\pm$ we must use in the LHS of (48) the nonlinearly correct expression for $E_\pm$ in place of that of (45). To determine the appropriate nonlinear complex expressions for the first harmonic fields, following the plane polarized wave evaluation of the Faraday rotation angle by Krall and Trivelpiece (1975, p. 185), the more general elliptically polarized wave is written as

$$E_\pm = \tfrac{1}{2}\{(a \pm b)\exp(\mathrm{i}\,\theta_\pm) + (a \mp b)\exp(-\mathrm{i}\,\theta_\mp)\}, \tag{50}$$

where

$$\theta_\pm = \omega'_\pm T = \theta_0 + \delta\theta_\pm = (\omega_\mathrm{p} + \delta\omega_\pm)T, \qquad \dot{\theta}_\pm = \omega'_\pm\,. \tag{51a, b}$$

The frequencies $\omega'_\pm$ are determined from the relation (12b). This form of $E_\pm$ enables us to find the nonlinearly developed birefringence effect of an elliptically polarized wave.

Equation (48) now gives

$$\omega'^2_\pm - \omega_\mathrm{p}^2 = \{k^2c^2/2(4-X)\}(\alpha \mp \eta)^2 - \tfrac{1}{8}\omega^2\{3(\alpha^2+\eta^2) \mp 2\alpha\eta\}, \tag{52}$$

where

$$(\alpha, \eta) = e(a, b)/m_0\,\omega_\mathrm{p}\,c, \qquad X = \omega_\mathrm{p}^2/\omega^2, \tag{53a, b}$$

and α and η are the dimensionless field amplitudes of the elliptically polarized wave. Putting the value of $\omega'_\pm$ from equation (12b) in (52) we find the nonlinear dispersion relation correct up to third order for the left and right circularly polarized components.

Next we write $k_\pm = k + \delta k_\pm$ and $\omega_\pm = \omega$, and consider the spatial evolution problem. Retaining only first order terms in $\delta k_\pm$ we obtain

$$\delta k/k = \tfrac{1}{2}k^{-1}(\delta k_+ + \delta k_-) = \frac{X(\alpha^2+\eta^2)}{4(1-X)}\left(\frac{3}{4} - \frac{1-X}{4-X}\right), \tag{54a}$$

$$\tfrac{1}{2}k^{-1}(\delta k_+ - \delta k_-) = \frac{X\alpha\eta}{2(1-X)}\left(\frac{1-X}{4-X} - \frac{1}{4}\right), \tag{54b}$$

δk being the average of the two wave-number shifts. These two equations are identical, apart from notation, to the results (24) and (25) of Arons and Max (1974). Equation (54b) gives the precessional rotation of the polarization ellipse of transverse vibration of the wave as a function of the field intensity. As shown in (49) this is a consequence of the birefringence effect because solving (52) with the help of

(12b) shows that ω/k_+ and ω/k_- have different values. Chakraborty (1978, Ch. 12) has applied several other methods to find this effect [see also the forthcoming review by Chakraborty *et al.* (1983)].

For radiation generated by an Nd–glass laser we have typically $\lambda = 1{\cdot}06\,\mu$m, $\omega = 1{\cdot}78\times10^{15}\,\mathrm{s}^{-1}$, $N_0 = 5\times10^{20}\,\mathrm{cm}^{-3}$ and a laser power $W = 10^{16}\,\mathrm{W\,cm^{-2}}$. Therefore, taking $\alpha^2 = \eta^2 \approx 0{\cdot}005$ we find that $\delta k = 1{\cdot}05\times10^2\,\mathrm{cm}^{-1}$, and as the wave travels through a length of $5{\cdot}5\times10^{-3}$ cm its polarization ellipse rotates through an angle of 1°. The power of $10^{16}\,\mathrm{W\,cm^{-2}}$ is less than the threshold power ($\approx 10^{20}\,\mathrm{W\,cm^{-2}}$) necessary to generate self-focussing and other distortions due to the nonlinearly growing inhomogeneity in a dense plasma.

5. Exact Dispersion Relation for Circularly Polarized Wave from S′ Frame Lagrangian

In equation (6) the Lagrangian $\mathscr{L}$ of the particles and the field produced by them is valid in the lab inertial frame S. When transformed to the S′ frame with the help of the relations in Section 3 it becomes

$$\begin{aligned}\mathscr{L}' = -N_0\Big\{&m_0 c^2\Big(1-\frac{\gamma_0^2}{(1+\gamma_0\beta_0\,\delta\beta_z')^2}(\beta_x'^2+\beta_y'^2+\gamma_0^2\,\delta\beta_z'^2)\Big)^{\frac{1}{2}}\\ &-\frac{e\gamma_0}{1+\gamma_0^2\beta_0\,\delta\beta_z'}(A_x'\beta_x'+A_y'\beta_y'+\gamma_0^2A_z'\,\delta\beta_z')-e\gamma_0\beta_0A_z'\Big\}\\ &+(E_x'^2+E_y'^2+E_z'^2)/8\pi\,.\end{aligned}\tag{55}$$

The equation of motion in the S′ frame is

$$(\partial/\partial T)(\gamma'\boldsymbol{v}') = -(e/m_0)\boldsymbol{E}' = (e/m_0)\dot{\boldsymbol{A}}',\tag{56}$$

and so

$$\gamma'\boldsymbol{v}' = -e\boldsymbol{A}'/m_0 c\,.\tag{57}$$

For a circularly polarized wave in the S′ frame we write

$$\boldsymbol{A}' = (-s\sin\omega'T,\, s\cos\omega'T,\, 0),\tag{58}$$

and find that

$$\delta\beta_z' = 0,\qquad A_z' = 0,\qquad E_z' = 0\,.\tag{59}$$

Then we get

$$\beta_\perp' = \frac{es}{\gamma_0(e^2s^2+m_0^2c^4)^{\frac{1}{2}}},\qquad \gamma' = \frac{\gamma_0(e^2s^2+m_0^2c^4)^{\frac{1}{2}}}{m_0c^2},\tag{60a, b}$$

$$\boldsymbol{v}' = -\frac{ec(-s\sin\omega'T,\, s\cos\omega'T,\, -V_0)}{\gamma_0(e^2s^2+m_0^2c^4)^{\frac{1}{2}}},\tag{60c}$$

$$A_x'\beta_x'+A_y'\beta_y' = -\frac{es^2}{(e^2s^2+m_0^2c^4)^{\frac{1}{2}}},\qquad E_x'^2+E_y'^2 = \omega'^2s^2/c^2\,.\tag{60d, e}$$

Therefore the Lagrangian becomes

$$\mathscr{L}' = -N_0(e^2s^2+m_0^2c^4)^{\frac{1}{2}}+s^2\omega'^2/8\pi c^2\,.\tag{61}$$

For a variation in s, the Euler–Lagrange equation gives

$$-\frac{N_0 e^2 s}{(e^2 s^2 + m_0^2 c^4)^{\frac{1}{2}}} + \frac{s\omega'^2}{4\pi c^2} = 0, \tag{62}$$

and so the exact value of ω' is given by

$$\omega'^2 = \omega_p^2/(1 + e^2 s^2/m_0^2 c^4)^{\frac{1}{2}}. \tag{63}$$

Transforming to the S frame with the help of (12b), we find the well known exact dispersion relation for a circularly polarized wave:

$$\omega^2 - k^2 c^2 = \omega_p^2/(1 + e^2 s^2/m_0^2 c^4)^{\frac{1}{2}}. \tag{64}$$

In place of equation (58) we can now put

$$\boldsymbol{E} \equiv (a\cos\theta, a\sin\theta, 0) \tag{65}$$

in the lab frame S, where $\theta = kz - \omega t$. Then since $\boldsymbol{E} = -c^{-1}\partial\boldsymbol{A}/\partial t$, we can replace s by $-ac/\omega$ and find that (64) becomes

$$\omega^2 - k^2 c^2 = \omega_p^2/(1+\alpha^2)^{\frac{1}{2}}, \tag{66}$$

where α is the dimensionless amplitude parameter $ea/m_0 c\omega$. Expanding the RHS in powers of α^2 and retaining only the first two terms of the binomial expansion we obtain the result (18) of Arons and Max (1974), if their notation is adopted. Replacing ω by $\omega + \delta\omega$ or k by $k + \delta k$ in the LHS of equation (66) and keeping terms correct to the first power of the nonlinear increment $\delta\omega$ or δk, we easily determine its intensity-dependent value.

It may be mentioned here that as the dispersion relation is exact, the right and left circularly polarized components have the same dispersion rates and so the effect of nonlinearly induced birefringence is absent for circular polarization.

6. Remarks

Intensity-dependent precessional rotation of strong EM waves in plasmas and other material media becomes significant under some physically possible conditions of phase matching or resonance (Arons and Max 1974; Katz *et al.* 1975; Lie and Wonnacott 1976; Chakraborty 1977; Chakraborty and Chandra 1977; Khan and Chakraborty 1979; Bhattacharyya and Chakraborty 1979). Induced magnetization (IM) and synchrotron radiation (SR) subsequently produced by the precessional rotation should provide useful information on the conditions found inside the material body. The IM and SR and the resultant radiation reaction leading to the damping of the accelerated charge motion are processes which change with time. So verification of the nonlinear rotation with the help of transformation to the space-independent time-like frame is expected to give rise to an elegant and concise method of deducing subsequent results.

In a magnetized plasma the nonlinearly correct refractive index becomes inhomogeneous and so nonlinearly correct closed-form solutions are not obtained. For this reason Chakraborty *et al.* (unpublished) have used the WKB approximation to derive expressions for the nonlinearly correct evolution of the Faraday effect.

But if, from the beginning, equations are transformed relativistically to a space-independent frame, it is expected that difficulties arising from the nonlinearly developed refractive index can be bypassed mathematically.

An investigation of the Lienard–Wiechert potential and other results following from it with the help of the LT to the space-independent frame should be simple exercises. Another useful exercise would be the transformation of the inhomogeneous Klein–Gordon equation, which has been written in the four-vector covariant form by De Jager (1967). The method of an averaged Lagrangian, developed by Whitham (1967, 1974) and extended by Dysthe (1974), Sihi (1980) and others to nonlinear wave processes in plasmas, is another topic in which the application of the LT to a space-independent frame should be made in the study of modulational instability.

References

Akhiezer, A. I., Akhiezer, I. A., Polovin, R. V., Sitenko, A. G., and Stepanov, K. N. (1975). 'Plasma Electrodynamics', Vol. 2 (Pergamon: Oxford).

Akhiezer, A. I., and Polovin, R. V. (1956). *Sov. Phys. JETP* **3**, 696.

Arons, J., and Max, C. E. (1974). *Phys. Fluids* **17**, 1983.

Bhattacharyya, B., and Chakraborty, B. (1979). *Physica* C **97**, 257.

Boyd, T. J. M., and Sanderson, J. J. (1969). 'Plasma Dynamics' (Nelson: London).

Chakraborty, B. (1977). *Phys. Rev.* A **16**, 1297.

Chakraborty, B. (1978). 'Principles of Plasma Mechanics' (Wiley Eastern: New Delhi).

Chakraborty, B., and Chandra, S. K. (1977). *Phys. Fluids* **21**, 2336.

Chakraborty, B., Paul, S. N., Khan, M. R., and Bhattacharyya, B. (1983). Intensity-induced precessional rotation and related nonlinear effects of electromagnetic waves in plasmas. *Phys. Reports* (to be published).

Chian, A. C. L., and Clemmow, P. C. (1975). *J. Plasma Phys.* **14**, 505.

Clemmow, P. C. (1974). *J. Plasma Phys.* **12**, 297.

Clemmow, P. C. (1975). *J. Plasma Phys.* **13**, 213.

Clemmow, P. C. (1977). *J. Plasma Phys.* **17**, 301.

Clemmow, P. C., and Harding, R. D. (1980). *J. Plasma Phys.* **23**, 91.

Decoster, A. (1978). *Phys. Reports* **47**, 285.

De Jager, E. M. (1967). *SIAM* (*Soc. Ind. Appl. Math.*) *J. Appl. Math.* **15**, 944.

Dysthe, K. B. (1974). *J. Plasma Phys.* **11**, 63.

Hughes, W. F., and Young, F. J. (1966). 'The Electromagnetodynamics of Fluids' (John Wiley: New York).

Katz, I., Parks, D. E., and Rotenburg, M. (1975). *Phys. Fluids* **18**, 895.

Kennal, C. F., and Pellat, R. (1976). *J. Plasma Phys.* **15**, 962.

Khan, M., and Chakraborty, B. (1979). *J. Appl. Phys.* **50**, 7977.

Krall, N. A., and Trivelpiece, A. W. (1975). 'Principles of Plasma Physics' (McGraw–Hill: New York).

Landau, L. D., and Lifshitz, E. M. (1975). 'The Classical Theory of Fields', p. 209 (Pergamon: Oxford).

Lie, C. S., and Wonnacott, E. (1976). *Can. J. Phys.* **54**, 962.

Maker, P. D., Terhune, R. W., and Savage, C. M. (1964). *Phys. Rev.* **12**, 507.

Shih, L. Y. (1978). *J. Plasma Phys.* **20**, 289.

Sihi, S. (1980). Ph.D. Thesis, Kalyani University, West Bengal, India.

Whitham, G. B. (1967). *Proc. R. Soc. London* A **299**, 6.

Whitham, G. B. (1974). 'Linear and Nonlinear Waves, Ch. 4 (John Wiley: New York).

Winkles, H. S. C., and Eldridge, O. (1972). *Phys. Fluids* **15**, 1790.

Wong, H. S. C. (1963). *Phys. Fluids* **6**, 1115.

Wong, H. S. C., and Lojko, M. S. (1963). *Phys. Fluids* **6**, 145.

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