Counterpart of the Kasner Model
in Brans–Dicke Theory

V. B. Johri\textsuperscript{A} and G. K. Goswami\textsuperscript{B}

\textsuperscript{A} Department of Mathematics, Indian Institute of Technology,
Madras 600036, India.
\textsuperscript{B} B.S.P. Middle School, Sector-VII, Bhilai 490001, India.

Abstract
We present a simple and elegant generalization of the Kasner model in Brans–Dicke (BD) theory by solving the BD field equations corresponding to the Bianchi type I metric.

1. Introduction
In this paper we obtain vacuum solutions of the Brans–Dicke (1961) field equation corresponding to the spatially homogeneous and anisotropic Bianchi type I metric. It is shown in Section 3 that our solution is a generalization of the well-known Kasner (1921) model in BD theory. Some of the properties of model are given in Section 4.

2. BD Field Equations
We consider the Bianchi type I metric
\[ ds^2 = \frac{dt^2}{A^2} - \frac{dx^2}{B^2} - \frac{dy^2}{C^2} - dz^2, \]
where \( A, B \) and \( C \) are functions of time only. The BD field equations for vacuum space \((T_{ij} = 0)\) are
\[ G_{ij} = -(\omega/\phi^2)(\phi_i \phi_j - \frac{1}{2} g_{ij} \phi_k \phi^k) - \phi^{-1}(\phi_{ij} - g_{ij} \phi^k \phi_k), \]
\[ \phi^k {}_{;k} = 0, \]
where the symbols have their usual meaning. The BD field equations corresponding to the metric (1) are
\[ \frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} = -\frac{\omega \phi_4^2}{2 \phi^2} + \frac{C_4 \phi_4}{C \phi}, \]
\[ \frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} = -\frac{\omega \phi_4^2}{2 \phi^2} + \frac{A_4 \phi_4}{A \phi}, \]
\[ \frac{C_{44}}{C} + \frac{A_{44}}{A} + \frac{C_4 A_4}{CA} = -\frac{\omega \phi_4^2}{2 \phi^2} + \frac{B_4 \phi_4}{B \phi}. \]
\[ \frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{C_4 A_4}{CA} = \frac{\omega \phi_4^2}{2\phi^2} - \frac{(ABC)_4 \phi_4}{ABC \phi}, \]  (4d)

\[ \frac{\phi_4}{\phi} + \frac{(ABC)_4 \phi_4}{ABC \phi} = 0. \]  (4e)

3. The Solution

Equations (4a)-(4c) yield

\[ \frac{s_{44}}{s_4} + \frac{\phi_4}{\phi} = 0, \]  (5)

where

\[ s = ABC. \]  (6)

Equations (4e) and (5) give the solution

\[ \phi = (at + b)^{p_1}, \quad s = s_0(at + b)^{1-p_1}, \]  (7a, b)

where \( a, b, s_0 \) and \( p_1 \) are arbitrary constants.

Now equations (4a)-(4c) along with equations (7) ultimately give the solution

\[ A = A_0(at + b)^{p_1}, \quad B = B_0(at + b)^{p_3}, \quad C = C_0(at + b)^{p_4}, \]  (8a, b, c)

where the arbitrary constants \( p_2, p_3, p_4 \) and \( A_0, B_0, C_0 \) satisfy

\[ p_2 + p_3 + p_4 = 1 - p_1 \quad \text{or} \quad \sum_{i=1}^{4} p_i = 1, \]  (9a)

\[ A_0 B_0 C_0 = s_0. \]  (9b)

One more restriction on the \( p_i \) may be imposed with the help of equation (4d), which along with (9a), gives

\[ (\omega + 1)p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1. \]  (10)

Thus, we get the following metric for an anisotropic empty BD universe:

\[ ds^2 = dt^2 - A_0^2(at + b)^{2p_1} dx^2 - B_0^2(at + b)^{2p_3} dy^2 - C_0^2(at + b)^{2p_4} dz^2. \]

This metric can be transformed through a proper choice of coordinates to the form

\[ ds^2 = dT^2 - T^{2p_1} dx^2 - T^{2p_3} dy^2 - T^{2p_4} dz^2, \]  (11)

with the scalar field \( \phi = \phi_0 T^{1-p_1}, \phi_0 \) being constant.

4. Some Physical Properties

(1) The metric (11) is the generalization of the well-known Kasner (1921) metric in BD theory. This can be seen in the following way. As we know that BD theory goes over to relativistic theory as \( \omega \to \infty \), equation (10) shows us immediately that in this limit

\[ p_1 = 0, \]
which gives $\phi = \phi_0$, and then the constants $p_i$ satisfy

$$\sum_{i=2}^{4} p_i = 1 \quad \text{and} \quad \sum_{i=2}^{4} p_i^2 = 1.$$  

Thus, in the limit $\omega \to \infty$, metric (11) is converted into the Kasner metric.

(2) The volume element in the BD model is

$$(-g)^{\frac{1}{2}} = T^{1-p},$$

which shows the expansion of the universe with time.

(3) The expansion is anisotropic, occurring at the rates $p_2/t$, $p_3/t$ and $p_4/t$ along the $x$, $y$ and $z$ axes respectively.

References


Manuscript received 27 September, accepted 9 December 1982