

Additional Rigidly Rotating Solutions in the String Model of Hadrons

C. J. Burden^A and L. J. Tassie^B

^A Department of Physics, Weizmann Institute of Science,
P.O. Box 26, Rehovot 76100, Israel.

^B Department of Theoretical Physics, Research School of Physical Sciences,
Australian National University, G.P.O. Box 4, Canberra, A.C.T. 2601.

Abstract

Solutions to the relativistic string equation are found which correspond to rigid body rotation about the z -axis with azimuthal velocity greater than the velocity of light. If the solutions lie entirely in the x - y plane they are rotating epicycloids, complimentary to the hypocycloid solutions found previously. The use of a general solution to the string equation in terms of two arbitrary world-lines with null tangents provides an alternative derivation of the rigidly rotating solutions.

1. Introduction

Previously we determined the set of solutions to the string equation corresponding to rigid body rotation about the z -axis (Burden and Tassie 1982*b*; hereafter referred to as Paper I). The solutions fell into two categories which we labelled tachyonic and tardyonic, the names referring to the component of velocity in the $\hat{\theta}$ direction about the z -axis. On physical grounds, however, it is more relevant to consider the component of velocity of the string normal to the string itself (see e.g. Goddard *et al.* 1973), and with this in mind we shall see here that use of the term 'tachyonic' above is misleading. The 'tachyonic' solutions referred to in Paper I do in fact have the velocity normal to the string everywhere less than or equal to the velocity of light, and can lead to physically acceptable string glueballs.

We examine the extra solutions here in detail, with particular emphasis on the planar solutions, which turn out to be rotating epicycloids. We also show that all the planar rigidly rotating string solutions can be found more directly as specific examples of a general solution to the string equation.

2. Additional Solutions

In Paper I we considered solutions to the relativistic string equation of the form $X^\mu(\tau, r) = (\tau, X(\tau, r))$ where, in polar coordinates,

$$X(\tau, r) = (r, \theta(r) + \omega\tau, z(r)), \quad (1)$$

with $\theta(r)$ and $z(r)$ the initial azimuthal and axial coordinates of the string and ω constant. (We set $c = 1$.) The form (1) clearly corresponds to rigid body rotation

about the z -axis. Setting $\phi = d\theta/dr$ and $\zeta = dz/dr$, we saw that the string equation leads to differential equations whose solutions are

$$\zeta = \frac{A\lambda r}{\{\lambda^2 r^2(1-A^2-\omega^2 r^2)-A^2(1-\omega^2 r^2)\}^{\frac{1}{2}}}, \quad (2)$$

$$\phi = \frac{A(1-\omega^2 r^2)}{r\{\lambda^2 r^2(1-A^2-\omega^2 r^2)-A^2(1-\omega^2 r^2)\}^{\frac{1}{2}}}, \quad (3)$$

where A and λ are real constants of integration. The restriction that ζ and ϕ be real implies that the term in braces in the denominators of equations (2) and (3) be positive over some range of r . This gives the constraint

$$|A\omega - \lambda| \geq \lambda A, \quad (4)$$

that is, the allowable values of λ and A lie in the shaded part of the graph of Fig. 1.

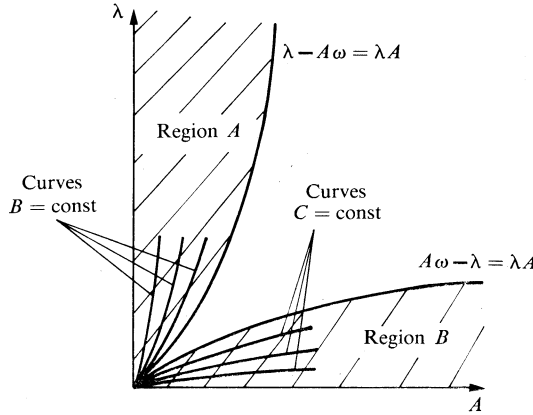


Fig. 1. Parameter space of rigidly rotating solutions to the string equation. The regions A and B were labelled 'tardyonic' and 'tachyonic' respectively in Paper I, referring to the θ component of velocity.

The points in region A correspond to solutions for which $\omega r < 1$. These solutions were examined in detail in Paper I. In region B we have $\omega r > 1$, such solutions being labelled 'tachyonic' in Paper I. However, the component of velocity normal to the string is less than the speed of light provided $(X_\tau X_r)^2 > X_\tau^2 X_r^2$ (see e.g. Goddard *et al.* 1973). Subscripts are used to denote partial derivatives, for example $X_\tau = \partial X / \partial \tau$. In our notation, this condition is equivalent to

$$r^4 \omega^2 \phi^2 > -(1 - r^2 \omega^2)(1 + r^2 \phi^2 + \zeta^2). \quad (5)$$

Direct substitution shows that condition (5) is satisfied for the solutions (2) and (3) for all points in regions A and B .

The integrals of (2) and (3), namely

$$z - \text{const} = -\frac{A}{2\omega} \arccos\left(\frac{r_1^2 + r_2^2 - 2r^2}{r_2^2 - r_1^2}\right), \quad (6)$$

$$\theta - \text{const} = \frac{1}{2} \arcsin\left(\frac{r_1^2 + r_2^2}{r_2^2 - r_1^2} - \frac{2r_1^2 r_2^2}{r^2(r_2^2 - r_1^2)}\right) + \frac{A\omega}{2\lambda} \arcsin\left(\frac{r_1^2 + r_2^2 - 2r^2}{r_2^2 - r_1^2}\right), \quad (7)$$

where

$$r_{1,2}^2 = (2\omega^2\lambda^2)^{-1}(\{(1-A^2)\lambda^2 + A^2\omega^2\} \mp [\{(1-A^2)\lambda^2 + A^2\omega^2\}^2 - 4A^2\omega^2\lambda^2]^{\frac{1}{2}}) \quad (8)$$

(see equation 13 in Paper I), are valid in both regions A and B .

We showed in Paper I that if the point $(A, \lambda) = (0, 0)$ is approached along the lines $B = \text{const}$, where $B = A(\lambda + \omega)/\lambda$, planar solutions are obtained, each the shape of a hypocycloid.* Consider now the solutions obtained by approaching the point $(A, \lambda) = (0, 0)$ along the curves

$$A(\omega - \lambda) = C\lambda, \quad C > 1 \quad (9)$$

shown in Fig. 1. Writing the solutions (2) and (3) in terms of the parameters λ and C and taking the limit $\lambda \rightarrow 0$ gives the solutions

$$\frac{dz}{dr} = \zeta = 0, \quad \frac{d\theta}{dr} = \phi = -\frac{C(\omega^2 r^2 - 1)^{\frac{1}{2}}}{r(C^2 - \omega^2 r^2)^{\frac{1}{2}}}. \quad (10a, b)$$

Once again we have curves lying in the plane $z = \text{const}$. Integrating equation (10b) gives

$$\theta - \text{const} = \frac{1}{2}(C-1)\arccos\left(\frac{C-\omega^2 r^2}{\omega r(C-1)}\right) - \frac{1}{2}(C+1)\arccos\left(\frac{C+\omega^2 r^2}{\omega r(C+1)}\right), \quad (11)$$

the equation of an epicycloid. Points on the curve lie at positions satisfying $1 < \omega r < C$, and the cusps move at the speed of light.

We shall refer to the hypocycloid solutions as type A solutions and the epicycloid solutions as type B solutions.

Two other solutions are worthy of note:

(a) If $A\omega - \lambda = \lambda A$, then $r_1 = r_2 = (1+A)^{\frac{1}{2}}/\omega$, and the string is the shape of a helix making an angle of

$$\arctan\left(\frac{1}{r} \frac{dz}{d\theta}\right) = -\arctan(1+A)^{-\frac{1}{2}} \quad (12)$$

with a plane arranged perpendicular to the z -axis. A similar case is given† in Paper I for the case $\lambda - A\omega = \lambda A$.

(b) When $\lambda = 0$, we have $\phi = (\omega^2 r^2 - 1)^{\frac{1}{2}}/r$, independent of A . This integrates to give

$$\theta = (\omega^2 r^2 - 1)^{\frac{1}{2}} - \arctan(\omega^2 r^2 - 1)^{\frac{1}{2}}, \quad (13)$$

the polar equation of an involute of a circle. The normal velocity is given by

$$v_{\perp} = \omega r(1 + r^2 \phi^2)^{-1} = 1, \quad (14)$$

indicating that this solution has infinite energy per unit length.

* A hypocycloid is the locus of a point on the circumference of a cylinder which rolls without slipping on the interior of a larger cylinder. If the small cylinder rolls instead on the exterior of the larger cylinder, an epicycloid is obtained.

† There are two mistakes in Paper I for this case, namely subsection (iii) of Section 2. The correct equations are

$$r_1 = r_2 = (1-A)^{\frac{1}{2}}\omega^{-1} \quad \text{and} \quad \arctan(r^{-1}dz/d\theta) = \arctan(\zeta/r\phi) = \arctan(1-A)^{-\frac{1}{2}}.$$

3. Dynamics

We next consider the classical string hadron model of Kikkawa *et al.* (1979) and Bars (1976*a*, 1976*b*) which is set out in detail in Paper I and applied to the solutions in the region *A* of Fig. 1.

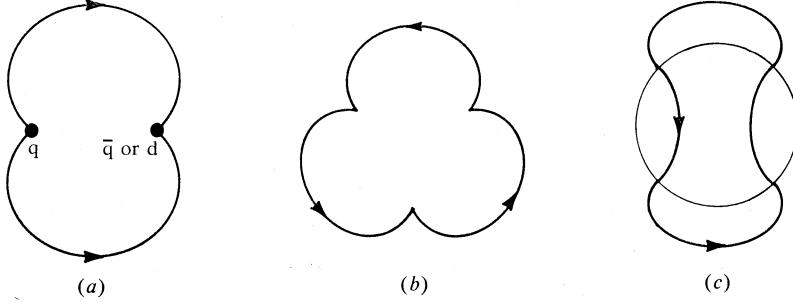


Fig. 2. Hadrons composed of rigidly rotating string segments: (a) meson or quark-diquark baryon with tachyonic quarks; (b) glueball (three-cusped epicycloid); (c) glueballs constructed from type *A* and type *B* solutions joining across the interface $\omega r = 1$ are forbidden.

If any quarks are attached directly to strings of the type in region *B* (including the planar strings of the previous section) to form rigidly rotating hadrons, the quarks must be tachyonic. Mesons and quark-diquark baryons of the type shown in Fig. 2*a* would themselves be tardyonic, while the quarks would be confined tachyons.

More realistic possible structures are the glueballs made from type *B* planar solutions (epicycloids) which close on themselves (see Fig. 2*b*). We shall calculate the Chew-Frautschi plot slopes for these glueballs and see how they compare with the hypocycloid glueballs examined in Burden and Tassie (1982*a*) and Paper I. By similar calculations to those given in Paper I, the energy and angular momentum densities of a segment of the string described by equation (11) are respectively

$$\mathcal{E} = \frac{1}{2\pi\alpha'} \frac{\omega r(C^2 - 1)}{(C^2 - \omega^2 r^2)^{\frac{1}{2}}(\omega^2 r^2 - 1)^{\frac{1}{2}}}, \quad (15a)$$

$$\mathcal{J}_x = \mathcal{J}_y = 0, \quad \mathcal{J}_z = \frac{1}{2\pi\alpha'} \frac{r(C^2 - \omega^2 r^2)^{\frac{1}{2}}}{(\omega^2 r^2 - 1)^{\frac{1}{2}}}. \quad (15b, c)$$

These integrate to give, for a segment of string with $r_1 < r < r_2$,

$$E(r_1 \rightarrow r_2) = \frac{1 - C^2}{4\pi\alpha'\omega} \arcsin\left(\frac{1 + C^2 - 2\omega^2 r^2}{C^2 - 1}\right) \Big|_{r_1}^{r_2}, \quad (16)$$

$$J_z(r_1 \rightarrow r_2) = \frac{1}{4\pi\alpha'\omega^2} \left\{ \frac{1}{2}(1 - C^2) \arcsin\left(\frac{1 + C^2 - 2\omega^2 r^2}{C^2 - 1}\right) + (C^2 - \omega^2 r^2)^{\frac{1}{2}}(\omega^2 r^2 - 1)^{\frac{1}{2}} \right\} \Big|_{r_1}^{r_2}. \quad (17)$$

In (16) and (17), if r passes through its maximum value $r_{\max} = C/\omega$, the expressions must be evaluated in two pieces, namely $E(r_1 \rightarrow r_2) = E(r_1 \rightarrow r_{\max}) + E(r_2 \rightarrow r_{\max})$ and similarly for J_z .

For an epicycloid with N cusps we have

$$C = 1 + 2/N. \quad (18)$$

Using equations (16)–(18) we calculate the total energy and angular momentum of the glueball with N cusps to be

$$E = 2(1 + N^{-1})/\alpha'\omega, \quad J = (1 + N^{-1})/\alpha'\omega^2, \quad (19a, b)$$

giving the straight Chew–Frautschi plot

$$J = \{\alpha'/4(1 + N^{-1})\}E^2. \quad (20)$$

Comparing this with the previous result (see equation 37 of Paper I) for hypocycloidal glueballs, namely $J = \{\alpha'/4(1 - N^{-1})\}E^2$, we see that for a given angular momentum the hypocycloidal glueballs are energetically more favourable.

Finally in this section we investigate the possibility of type A and type B planar solutions joining across the interface $\omega r = 1$ to form, for example, glueballs such as that shown in Fig. 2c. For this to happen the string tension for the inner (type A) and outer (type B) solutions must match across the junction. From equations (27) in Paper I, we have for the absolute value of the $\hat{\theta}$ and time components of tension for the type A solutions, at $\omega r = 1$,

$$|\mathcal{T}_\theta| = |\mathcal{T}_0| = (1/2\pi\alpha')B; \quad 0 < B < 1, \quad (21)$$

the other components being zero. For the type B solutions a similar calculation gives, at $\omega r = 1$,

$$|\mathcal{T}_\theta| = |\mathcal{T}_0| = (1/2\pi\alpha')C; \quad C > 1, \quad (22)$$

and it is clear that the inner and outer solutions cannot match across the junction.

4. Alternative Derivation of the Planar Solutions from a General Solution

As an alternative to the method used in Paper I, the planar rigidly rotating solutions can be obtained as particular cases of a general solution.

For any world sheet $X^\mu(\sigma, \tau)$ with a time-like tangent at each point it is always possible (Goddard *et al.* 1973) to choose σ and τ to be an orthonormal set of coordinates satisfying

$$X_\tau X_\sigma = 0, \quad X_\tau^2 = -X_\sigma^2 > 0 \quad (23a, b)$$

[we use the metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$], and the string equation becomes

$$X_{\tau\tau}^\mu - X_{\sigma\sigma}^\mu = 0.$$

Defining a set of null coordinates ξ and η by

$$\xi = \frac{1}{2}(\tau + \sigma), \quad \eta = \frac{1}{2}(\tau - \sigma), \quad (24a, b)$$

equations (23) can be written as

$$X_{\xi}^2 = X_{\eta}^2 = 0. \quad (25)$$

With these coordinates the string equation becomes

$$X_{\xi\eta}^{\mu} = 0, \quad (26)$$

with the general solution

$$X^{\mu}(\xi, \eta) = r^{\mu}(\xi) + q^{\mu}(\eta), \quad (27)$$

where, from equation (25),

$$(r^{\mu'})^2 = (q^{\mu'})^2 = 0. \quad (28)$$

That is to say, every solution to the string equation can be written as the sum of two world-lines each of which has a null tangent at each point.

In particular we choose r^{μ} and q^{μ} to be the world-lines of points executing uniform circular motion at the speed of light in the x - y plane:

$$r^{\mu}(\xi) = (\xi, \Omega_1^{-1} \cos \Omega_1 \xi, \Omega_1^{-1} \sin \Omega_1 \xi), \quad (29a)$$

$$q^{\mu}(\eta) = (\eta, \Omega_2^{-1} \cos \Omega_2 \eta, \Omega_2^{-1} \sin \Omega_2 \eta). \quad (29b)$$

Then $X^{\mu}(\xi, \eta) = r^{\mu}(\xi) + q^{\mu}(\eta)$ is a solution to the string equation. In order to interpret this solution we consider the string's shape at $X^0 = 0$, which will consist of the locus of points of the form

$$X(\Delta, -\Delta) = r(\Delta) + q(-\Delta), \quad (30)$$

where $r^{\mu} = (\xi, r(\xi))$ and $q^{\mu} = (\eta, q(\eta))$.

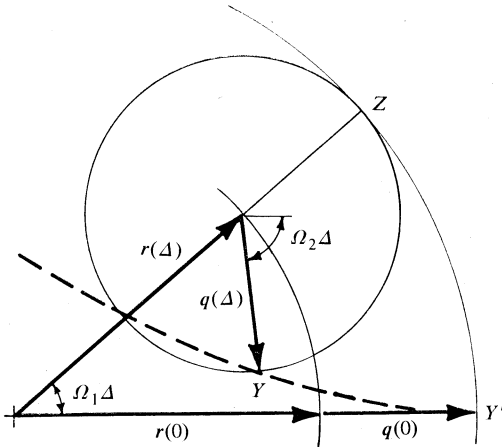


Fig. 3. Points $Y' = X(0, 0)$ and $Y = X(\Delta, -\Delta)$ lie on the same hypocycloid (dashed curve).

In Fig 3. we show the point $Y' = X(0, 0)$ and some arbitrary point $Y = X(\Delta, -\Delta)$, assuming that Ω_1 and Ω_2 have the same sign. It is straightforward to see that

$$\text{arc } ZY = (\Omega_1 \Delta + \Omega_2 \Delta) \Omega_2^{-1} = \Omega_1 \Delta (\Omega_1^{-1} + \Omega_2^{-1}) = \text{arc } ZY', \quad (31)$$

and so Y and Y' lie on the same hypocycloid (dashed curve). Choosing X^0 at any other fixed time will produce the same hypocycloid, rotated around the central point.

By considering the world-line of constant ξ passing through Y' in Fig. 3 we see that the cusps move at the speed of light, and so have the properties of the planar solutions found in Paper I. If Ω_1 and Ω_2 have opposite sign, a similar argument gives the rotating epicycloids of Section 2. The quantities B , C and ω which parametrize the solutions are related to Ω_1 and Ω_2 by

$$B = C = (\Omega_2 - \Omega_1)/(\Omega_2 + \Omega_1), \quad \omega^{-1} = \Omega_1^{-1} + \Omega_2^{-1}. \quad (32a, b)$$

For the singular case $\Omega_1 = -\Omega_2 = \Omega$, the solution (27) is a breathing circle whose radius $R(t)$ is given by

$$R(t) = 2\Omega^{-1} \sin \frac{1}{2}\Omega t. \quad (33)$$

This solution has been found previously (Vilenkin 1981).

5. Conclusions

We have examined certain rigidly rotating solutions to the relativistic string equation, namely those whose azimuthal velocity is greater than the speed of light. Of particular interest are the planar solutions, which turn out to be epicycloids whose cusps move at the speed of light (cf. the rotating hypocycloids in Paper I). These solutions suggest the existence of a set of string glueballs further to those conjectured in Paper I. The Chew–Frautschi plot slopes of the new epicycloid glueballs are always greater than those for the hypocycloid glueballs, so we expect the new glueballs to be less stable.

There seem to be no other simple rigidly rotating hadrons which can be constructed from the new solutions, unless quarks are tachyons, in which case mesons and baryons such as those in Fig. 2a are allowed.

In Section 4 we used the result that every classical solution to the string equation is the sum of two world-lines, each with null-vector tangents. By choosing the two world-lines to be those corresponding to coplanar uniform circular motion we reconstructed the planar solutions to the string equation. This method of solution serves to exhibit the massless nature of the string: the rigidly rotating solutions are in fact a collection of massless points, each executing uniform circular motion at the speed of light.

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