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Bound on the Number of Flavours

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Abstract

If one imposes the permutation symmetry S_n (n is the number of lepton flavours) reducibly on the different families ($e, \mu, \tau, ...$), it follows that at least two leptons have the same mass if $n > 6$. If equal lepton masses are excluded, this implies a bound on the number of flavours.

The question of why there are different families of leptons (and how many), with practically identical properties, is one of the most fundamental and long-standing problems in elementary particle physics.

Since all known leptons appear to have the same properties, except for their highly non-degenerate masses, it is natural to assume that, prior to spontaneous symmetry breaking, there is a permutation symmetry corresponding to the direct interchange of the family labels $e, \mu, \tau, ...$ To implement this scenario and at the same time still generate distinct lepton masses it proves essential to extend the Higgs fields to a non-singlet representation of the permutation group. The simplest choice of one Higgs doublet for each family is assumed. The Lagrangian of the model is taken to be the usual Weinberg–Salam–Glashow–Ward $SU(2) \times U(1)$ model (Weinberg 1967) with n families of leptons, quarks and Higgs bosons, supplemented by an S_n permutation symmetry. The present model (Christos 1979, unpublished results; 1980, Oxford preprint 54/80) is a natural extension of the $n = 2$ and 3 models considered earlier (Derman and Jones 1977; Derman 1978, 1979).

The most general quartic Higgs potential takes the form [where $\phi_i \equiv (\phi_i^+, \phi_i^0)$]

$$
\begin{aligned}
V(\phi_1, \phi_2, ..., \phi_n) = {} & \sum_{i=1}^{n} [-\mu^2(\phi_i^\dagger \phi_i) + \lambda(\phi_i^\dagger \phi_i)^2] \\
& + \sum_{i<j}^{n} [\tfrac{1}{2}\alpha\{(\phi_i^\dagger \phi_j) + \text{h.c.}\} + \beta(\phi_i^\dagger \phi_i)(\phi_j^\dagger \phi_j) + \gamma\,|\phi_i^\dagger \phi_j|^2 \\
& \qquad + \tfrac{1}{2}\delta\{(\phi_i^\dagger \phi_j)^2 + \text{h.c.}\}] \\
& + \tfrac{1}{2}\varepsilon \sum_{i \neq j}^{n} [(\phi_i^\dagger \phi_i)(\phi_i^\dagger \phi_j) + \text{h.c.}]
\end{aligned}
$$

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$$+\sum_{\substack{i\neq j\neq k\\ j<k}}^{n}\left[\tfrac{1}{2}\omega\{(\phi_i^\dagger\phi_j)(\phi_k^\dagger\phi_i)+\text{h.c.}\}+\tfrac{1}{2}\sigma\{(\phi_i^\dagger\phi_i)(\phi_k^\dagger\phi_j)+\text{h.c.}\}\right.$$
$$\left.+\tfrac{1}{2}\theta\{(\phi_i^\dagger\phi_j)(\phi_i^\dagger\phi_k)+\text{h.c.}\}\right]$$
$$+\xi\sum_{i\neq j\neq k\neq l}^{n}\left[(\phi_i^\dagger\phi_j)(\phi_k^\dagger\phi_l)\right]. \tag{1}$$

The last term in (1) is absent in the $n = 3$ case.

For simplicity we take the minimum of the potential (1) to occur at

$$\langle\phi_i^+\rangle = 0, \qquad \langle\phi_i^0\rangle = \rho_i/\sqrt{\,} = \text{real}. \tag{2}$$

This corresponds to the assumption of no spontaneous violation of both charge conjugation and time reversal invariances. With this choice the potential takes the form

$$V(\rho_1,\rho_2,\ldots,\rho_n) = \sum_{i=1}^{n}\left(-\tfrac{1}{2}\mu^2\rho_i^2+\lambda\rho_i^4\right)$$
$$+\sum_{i<j}^{n}\left\{\tfrac{1}{2}\alpha\rho_i\rho_j+\tfrac{1}{4}(\beta+\gamma+\delta)\rho_i^2\rho_j^2\right\}+\tfrac{1}{4}\varepsilon\sum_{i\neq j}^{n}\rho_i^3\rho_j$$
$$+\tfrac{1}{4}(\omega+\sigma+\theta)\sum_{\substack{i\neq j\neq k\\ j<k}}^{n}\rho_i^2\rho_j\rho_k+\tfrac{1}{4}\xi\sum_{i\neq j\neq k\neq l}^{n}\rho_i\rho_j\rho_k\rho_l. \tag{3}$$

Since (2) is assumed to be the minimum of the potential, it follows that

$$V_i \equiv \partial V(\rho)/\partial\rho_i = 0 \qquad (i = 1,\ldots,n). \tag{4}$$

These are third-order polynomial equations in the ρ. The difference, $V_k(\rho)-V_l(\rho)$ $(k\neq l)$, can then be written as $(\rho_k-\rho_l)F_{kl}(\rho)$, where $F_{kl}(\rho)$ is a second-order polynomial in the ρ and is symmetric under the interchange of k and l, and under the S_{n-2} permutation operations on the ρ, bar ρ_k and ρ_l. Similarly, the difference $F_{kl}(\rho)-F_{pl}(\rho)$ can be written as $(\rho_k-\rho_p)G_{klp}(\rho)$, where $G_{klp}(\rho)$ is linear in the ρ and is symmetric under the interchange of k, l and p, and under the S_{n-3} permutations on the ρ, bar ρ_k, ρ_l and ρ_p. If $\rho_k\neq\rho_l\neq\rho_p\neq\rho_k$ it follows from (4) that $F_{kl}(\rho) = 0 = G_{klp}(\rho)$. The specific form of $G_{klp}(\rho)$ is given by

$$G_{klp}(\rho) = \tfrac{1}{4}\chi(\rho_k+\rho_l+\rho_p)+\tfrac{1}{4}\psi\sum_{m\neq k,l,p}\rho_m, \tag{5}$$

where $\chi = 4\lambda-2\gamma-2\beta-2\delta-\varepsilon+\omega+\sigma+\theta$ and $\psi = 3\varepsilon-3\omega-3\sigma-3\theta+12\xi$. Suppose now that four of the ρ_i are distinct, say ρ_1, ρ_2, ρ_3 and ρ_4. From the difference $G_{123}(\rho)-G_{124}(\rho) = \tfrac{1}{4}(\chi-\psi)(\rho_3-\rho_4) = 0$, it follows that $\chi = \psi$. This is viewed as an unnatural constraint (not implied or protected by any symmetry) among the coupling constants in (1). Excluding such unnatural constraints, implies that the ρ_i can take on at most three distinct values.

This means that the possible vacua of the S_4, S_5 and S_6 models are characterized by ($\rho \neq \rho' \neq \rho'' \neq \rho$)

$[1]^4$	$\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho;$		
$[2]^4$	$\rho_1 = \rho_2 = \rho_3 = \rho,$	$\rho_4 = \rho';$	
$[3]^4$	$\rho_1 = \rho_2 = \rho,$	$\rho_3 = \rho_4 = \rho';$	
$[4]^4$	$\rho_1 = \rho_2 = \rho,$	$\rho_3 = \rho',$	$\rho_4 = \rho'';$
$[1]^5$	$\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_5 = \rho;$		
$[2]^5$	$\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho,$	$\rho_5 = \rho';$	
$[3]^5$	$\rho_1 = \rho_2 = \rho_3 = \rho,$	$\rho_4 = \rho_5 = \rho';$	
$[4]^5$	$\rho_1 = \rho_2 = \rho_3 = \rho,$	$\rho_4 = \rho',$	$\rho_5 = \rho'';$
$[5]^5$	$\rho_1 = \rho_2 = \rho,$	$\rho_3 = \rho_4 = \rho',$	$\rho_5 = \rho'';$
$[1]^6$	$\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_5 = \rho_6 = \rho;$		
$[2]^6$	$\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_5 = \rho,$	$\rho_6 = \rho';$	
$[3]^6$	$\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho,$	$\rho_5 = \rho_6 = \rho';$	
$[4]^6$	$\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho,$	$\rho_5 = \rho',$	$\rho_6 = \rho'';$
$[5]^6$	$\rho_1 = \rho_2 = \rho_3 = \rho,$	$\rho_4 = \rho_5 = \rho_6 = \rho';$	
$[6]^6$	$\rho_1 = \rho_2 = \rho_3 = \rho,$	$\rho_4 = \rho_5 = \rho',$	$\rho_6 = \rho'';$
$[7]^6$	$\rho_1 = \rho_2 = \rho,$	$\rho_3 = \rho_4 = \rho',$	$\rho_5 = \rho_6 = \rho''.$

The residual (i.e. unbroken) vacuum permutation symmetries of the above vacua can be read off by inspection; for example, the vacua $[2]^4$, $[3]^5$ and $[7]^6$ have respectively an S_3, $S_3 \times S_2$ and $S_2 \times S_2 \times S_2$ permutation symmetry.

The fermion (lepton) masses are generated by the Yukawa interaction and (2). The most general S_n invariant Yukawa Lagrangian is given by

$$\mathscr{L}_Y = \sum_{i=1}^{n} a\bar{l}_i \phi_i r_i + \sum_{i \neq j}^{n} (b\bar{l}_i \phi_i r_j + c\bar{l}_i \phi_j r_i + d\bar{l}_j \phi_i r_i) + \sum_{i \neq j \neq k}^{n} e\bar{l}_i \phi_j r_k, \tag{6}$$

where the $l_i = (\nu_i, e_i^-)_L$ are the left-handed lepton doublets and $r_i = (e_i^-)_R$ are the right-handed lepton singlets.

As an explicit example, let us consider the $[2]^4$ vacuum of the S_4 theory. This vacuum has a residual S_3 permutation symmetry. In this case, the mass matrix takes the form

$$\mathbf{M} = \left[\begin{array}{ccc|c} a' & b' & b' & c' \\ b' & a' & b' & c' \\ b' & b' & a' & c' \\ \hline d' & d' & d' & e' \end{array}\right], \qquad \mathbf{MM}^\dagger = \left[\begin{array}{ccc|c} \alpha' & \beta' & \beta' & \gamma' \\ \beta' & \alpha' & \beta' & \gamma' \\ \beta' & \beta' & \alpha' & \gamma' \\ \hline \gamma' & \gamma' & \gamma' & \varepsilon' \end{array}\right],$$

where a', b', α' etc. are some functions of a, b, c, d, e and the ρ. The S_3 vacuum invariance is clearly visible in the submatrix. The fermion masses are obtained by diagonalizing $\mathbf{M}$ by a biunitary left–right transformation, or equivalently $\mathbf{MM}^\dagger$ (or $\mathbf{M}^\dagger\mathbf{M}$) by a unitary transformation. It is easy to see that $\alpha' - \beta'$ is an eigenvalue of $\mathbf{MM}^\dagger$ with two eigenvectors.

This result is quite general. As long as there is a residual S_3 vacuum symmetry (or anything larger) there will always be at least two degenerate eigenvectors. If one is to exclude the possibility that any two leptons have the same mass (based on the present trend $m_e \ll m_\mu \ll m_\tau$), all such vacua can be ruled out. This means that the only acceptable vacua for the S_4, S_5 and S_6 models are $[3]^4$, $[4]^4$, $[5]^5$ and $[7]^6$. Since only three distinct values of the ρ_i are possible, it is easy to see that S_n models with $n > 6$ always have at least a residual S_3 permutation symmetry, and consequently lead to degenerate lepton masses. If we exclude this possibility, this means that the hypothesis of imposing a permutation symmetry on the lepton families naturally leads to a bound on the number of lepton flavours, $n \leqslant 6$. It would be interesting to see if this feature also occurs in other horizontal unification schemes.

A direct consequence of models which mix families is that there are decays in which the ordinary (additive) lepton number conservation laws are violated. In the permutation models considered above, there does however remain a multiplicative lepton number conservation law (Feinberg and Weinberg 1961). This is because all of the acceptable vacua of these models seem to at least contain a residual S_2 permutation symmetry. Consequently, every state of the theory has associated with it a 'lepton parity' (± 1). In models like $[3]^4$ and $[5]^5$ there are two parities, each of which is separately conserved in any process. For $[7]^6$ there are three. The details of these conservation laws and the decays that may be possible depend on the model (i.e. the value of n and the vacuum structure) and the 'lepton parity' assignments of the individual leptons (for further details see Derman 1979; Derman and Tsao 1979).

The application of the permutation symmetry to the quark sector leads to a similar set of results, and in particular to the bound on the number of quark flavours, $n^* \leqslant 12$. This agrees with anomaly cancellation arguments which require that $n^* = 2n$.

Similar considerations have also lead Derman and Tsao (1979) to a bound on the number of flavours. However, these authors apparently overlooked the vacuum structure $[7]^6$ of the S_6 model and consequently obtained the wrong bound ($n \leqslant 5$).

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