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The Large Number Hypothesis and Einstein's Theory of Gravitation

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Abstract

In an attempt to reconcile the large number hypothesis (LNH) with Einstein's theory of gravitation, a tentative generalization of Einstein's field equations with time-dependent cosmological and gravitational constants is proposed. A cosmological model consistent with the LNH is deduced. The coupling formula of the cosmological constant with matter is found, and as a consequence, the time-dependent formulae of the cosmological constant and the mean matter density of the Universe at the present epoch are then found. Einstein's theory of gravitation, whether with a zero or nonzero cosmological constant, becomes a limiting case of the new generalized field equations after the early epoch.

1. Introduction

Over the past 50 years, there have been numerous suggestions that the gravitational constant G is time dependent. In particular, in 1937, Dirac (1938) proposed a theory of this kind which has as its basis the numerology uncovered by Weyl, Eddington, as well as Dirac himself. Dirac noticed that the ratio of the electrical to gravitational force beween a proton (mass m_p) and an electron (mass m_e) at a distance r apart, given by $Gm_p\, m_e/r^2$, is a large dimensionless number of the order 10^{40}. Similarly, the age of the Universe t, expressed in terms of a unit provided by atomic constants, say $e^2/m_e\, c^2$, is roughly of the same size. Dirac then went on to suggest that

$$Gm_p\, m_r/r^2 \sim t\,. \tag{1}$$

Assuming that (1) holds at all times, both in the past and future, and that the atomic parameters do not vary with time,* equation (1) tells us that

$$G \propto t^{-1}\,. \tag{2}$$

Dirac put this semi-quantitative argument on a formal footing with his large number hypothesis (LNH) which states that:

* Gamow (1967) once proposed keeping G constant and varying the fine structure constant $e^2/\hbar c$, but his attempt was unsuccessful.

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> Any two of the very large numbers occurring in Nature are connected by a simple mathematical relation in which the coefficients are of the order of unity.

A time-dependent G then follows as a natural consequence of the LNH. However, Einstein's theory of gravitation, with its great success in explaining local gravitational phenomena, requires G to be a genuine constant independent of any coordinates, in contradiction with the LNH.

Since Dirac's early work, several attempts have been made to formulate a generalized field theory of gravitation in which G is a scalar function of coordinates and Einstein's theory of gravitation appears as a special case of the new theory. Jordan (1949, 1959) attempted this by starting from a five-dimensional theory of relativity, where G is generalized to be a scalar function of the age of the Universe and the covariant divergence of the energy–momentum tensor is nonzero. The latest attempt was by Dirac (1975, 1979) himself who proposed the 'two metrics' theory, where two unit systems are set up, namely the Einstein and the atomic units. For the Einstein units, general relativity holds and G is a universal constant, whereas for the atomic units, G varies inversely with the age of the Universe. Hence, the success of Einstein's theory of gravitation was not disturbed.

In the present paper, another attempt is made to make the LNH compatible with Einstein's theory of gravitation, without needing the 'two metrics' theory. The original form of the Einstein field equations, with a nonzero cosmological constant λ, is retained except that λ and G receive a time-dependent form in the field equations.

2. Scale Factor of Expansion

From observations of a redshift in the spectra of the spiral nebulae, we believe that the Universe is in a state of expansion in the sense that the nebulae are uniformly moving away from each other. If we assume that the Universe is spatially homogeneous and isotropic, as proved by Robertson (1933) and Walker (1936), the general form of the metric of the Universe, irrespective of the model assumed, is

$$ds^2 = dt^2 - \frac{R^2(t)\{(dx^1)^2+(dx^2)^2+(dx^3)^2\}}{(1+\frac{1}{4}kr^2)^2}, \tag{3}$$

where the velocity of light c is taken to be unity, $r^2 = (x^1)^2+(x^2)^2+(x^3)^2$ and k is a real constant. Further, $R(t)$ is a function of t only, where t is the proper time on any clock carried by a particle with fixed coordinates (x^1, x^2, x^3). The function $R(t)$ gives the time dependence of the distance between two points in space at a particular epoch and therefore governs the rate of expansion of the Universe. We call $R(t)$ the scale factor of expansion and derive the form of its time dependence from the LNH. The derivation is entirely due to Dirac (1938, 1979); for the sake of completeness, it is repeated here.

Let ρ_0 be the average matter density of the Universe. Without continuous creation of matter*, mass is conserved and hence $\rho_0 R^3(t)$ is constant, that is

$$\rho_0 \propto R^{-3}(t). \tag{4}$$

* Dirac (1938) originally proposed this idea to account for the time dependence of the number of protons in the Universe, but abandoned it in his subsequent 1979 paper.

We now take the general law of expansion $R \propto t^n$, and fix our attention on a particular galaxy whose velocity of recession is $\frac{1}{2}c$. The velocity of recession of that galaxy is then given by

$$\mathrm{d}R(t)/\mathrm{d}t = nR(t)/t = \tfrac{1}{2},$$

where c is taken to be unity, so that the distance of the galaxy from us is presently $t/2n$. Therefore, the total mass within this distance is proportional to ρt^3.

We consider the mass of that part of the Universe that is receding from us with velocity less than $\frac{1}{2}c$ which is, in terms of the proton mass, roughly 10^{78}, with a suitable factor allowed for invisible matter. The LNH then requires this large number to be proportional to t^2 and, as a result, we have $\rho_0\, t^3 \propto t^2$, which gives

$$\rho_0 \propto t^{-1}. \tag{5}$$

From (4) and (5), we also get

$$R(t) \propto t^{\frac{1}{3}}, \tag{6}$$

which is the time-dependence form of the scale factor required.

3. The Model of the Universe

In the previous section, we determined the time-dependence form of the scale factor $R(t)$. Equation (3) is then determined up to a real constant k which represents the curvature of the three-dimensional space corresponding to a particular epoch. As worked out by Dirac (1979), the value of k can be determined by considering the cases $k > 0$, $k < 0$ and $k = 0$ separately. For $k > 0$, the three-dimensional space at a particular epoch is finite and hence has a finite total mass which, expressed in proton units, gives a constant large number, this being not allowed by the LNH.

For $k < 0$, the three-dimensional space is hyperbolic and hence infinite. By picking an arbitrary point in the three-dimensional space and considering another point in the immediate neighbourhood, the distance between these two points $\mathrm{d}s$ must have the same time-dependence form as that of the scale factor of expansion, i.e. $\mathrm{d}s \propto t^{\frac{1}{3}}$. The radius of curvature of the three-dimensional space $\mathscr{R}$ is determined by $\mathrm{d}s$, and therefore $\mathscr{R} \propto t^{\frac{1}{3}}$. We consider a sphere with radius $\mathscr{R}$, where the total mass within the sphere is proportional to $\rho_0 \mathscr{R}^3$. From (5), we get $\rho_0 \propto t^{-1}$. Thus, $\rho_0 \mathscr{R}^3$ gives a constant large number independent of time, in contradiction to the LNH.

We are thus left with $k = 0$ as the only case consistent with the LNH. Equation (3) then becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\{(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2\}, \tag{7}$$

where $R(t) \propto t^{\frac{1}{3}}$, and this is the metric compatible with the LNH and the cosmological principle.

Incidentally, we note that (7) is the same as the metric in the Einstein–De Sitter (1932) model. However, the model represented by (7) differs from the Einstein–De Sitter in that (i) instead of $R(t) \propto t^{\frac{2}{3}}$, we have $R(t) \propto t^{\frac{1}{3}}$ and (ii) the cosmological pressure is not assumed to be zero. As a mattter of fact, the pressure can be calculated easily since we know the exact time-dependence form of $R(t)$.

4. Generalized Field Equations with Time-dependent λ and G

The most general form of the Ricci tensor with zero covariant divergence, as proved by Cartan (1922), is of the form

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R+\lambda g^{\mu\nu},$$

where $R^{\mu\nu} = R_{a}^{a\ \mu\nu}$ is the contracted Riemann–Christoffel tensor, λ is a real constant and R is the curvature scalar. Hence, the general form of Einstein's field equations is

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R+\lambda g^{\mu\nu} = -8\pi G T^{\mu\nu}, \tag{8}$$

where $T^{\mu\nu}$ is the energy–momentum tensor, c is equal to unity and λ is identified as the cosmological constant. Clearly, equation (8) is not compatible with the LNH. For, if G is a function of time, then

$$(-8\pi G T^{\mu\nu})_{:\nu} = -8(G T^{\mu\nu}{}_{:\nu} + T^{\mu 0} G_{,0}),$$

where the colon before a suffix always denotes a covariant derivative, while a comma always denotes an ordinary partial derivative. The number following the comma means differentiation with respect to the corresponding coordinate, where

$$x^0 = t,\quad x^1 = x,\quad x^2 = y,\quad x^3 = z.$$

As a consequence of the principles of conservation of energy and momentum, we have

$$T^{\mu\nu}{}_{:\nu} = 0,$$

and therefore $(-8\pi G T^{\mu\nu})_{:\nu} = -8\pi G T^{\mu 0} G_{,0}$, which is not equal to zero in general. However, it is always true that

$$(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R+\lambda g^{\mu\nu})_{:\nu} = 0,$$

and so (8) is not valid in general if G is a function of time.

To generalize (8) to make the LNH compatible with Einstein's theory of gravitation, we postulate that both G and λ are scalar functions of time. To be compatible with the LNH, we further postulate that G is inversely proportional to the age of the Universe. The covariant divergence of (8), instead of zero, is then

$$g^{\mu 0}\lambda_{,0} = -8\pi T^{\mu 0} G_{,0}. \tag{9}$$

From (7) we have

$$g_{00} = 1,\qquad g_{11} = g_{22} = g_{33} = -\beta t^{\frac{2}{3}}, \tag{10a}$$

where β is a constant of proportionality, and

$$g^{\mu\nu} = 0 \qquad \text{for} \quad \mu \neq \nu. \tag{10b}$$

It follows that

$$g^{00} = 1, \qquad g^{11} = g^{22} = g^{33} = -\beta^{-1} t^{-\frac{2}{3}}; \qquad g^{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu. \tag{11}$$

To determine $T^{\mu 0}$ we need to know the energy–momentum tensor of the Universe. If we assume that the cosmological principle is valid, then on a large scale the Universe is isotropic and homogeneous, and local irregularities are smoothed out. The energy–momentum tensor of the Universe with c set to unity will be of the form

$$T^{\mu\nu} = \rho_0\, v^\mu v^\nu + p(v^\mu v^\nu - g^{\mu\nu}),$$

where ρ_0 is the proper matter density of the Universe, p is the cosmological pressure and v is the velocity of matter. Furthermore, by the principle of relativity, physical laws are invariant under transformation of a frame of reference. To make the calculations simpler, we choose a co-moving coordinate system in which matter is at rest. Hence, we have $T^{00} = \rho_0$ and $T^{\mu 0} = 0$ for $\mu \neq 0$. Thus, when $\mu \neq 0$, (9) is equal to zero, being identical to the result with constant λ and G.

For $\mu = 0$, we have $g^{00} = 1$ and $T^{00} = \rho_0$, and (9) then becomes $\lambda_{,0} = -8\pi\rho_0\, G_{,0}$. From (2) and (5), we have

$$G \propto t^{-1} \quad \text{and} \quad \rho_0 \propto t^{-1}$$

or, equivalently, $G = \beta_1 t^{-1}$ and $\rho_0 = \beta_2 t^{-1}$, where β_1 and β_2 are constants of proportionality. Thus, we have $G_{,0} = -\beta_1 t^{-2}$ and, as a result,

$$\lambda_{,0} = 8\pi\beta_1\beta_2 t^{-3}.$$

In turn, this implies

$$\lambda = -4\pi\beta_1\beta_2 t^{-2} + C, \tag{12}$$

where C is a constant of integration. We can see that, apart from a constant of integration, λ has an inverse-square form for its time dependence.

5. Exact Time-dependence Form of λ

From now on, we refer to equation (8) as the generalized field equation with time-dependent λ and G. In (12), λ is indeterminate up to a constant of integration. To find a more exact time-dependence form of λ, we write the Ricci tensor in the form

$$R^{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}.$$

Using (10) and (11), we find for the non-vanishing Christoffel symbols

$$\Gamma^{0}_{11} = \Gamma^{0}_{22} = \Gamma^{0}_{33} = \tfrac{1}{3}\beta t^{-\frac{1}{3}},$$

$$\Gamma^{1}_{01} = \Gamma^{1}_{10} = \Gamma^{2}_{02} = \Gamma^{2}_{20} = \Gamma^{3}_{03} = \Gamma^{3}_{30} = \tfrac{1}{3} t^{-1}.$$

After some algebraic manipulation, we find

$$R_0^0 = R^{00} = R_{00} = -\tfrac{2}{3}t^{-2}, \qquad R_1^1 = R_2^2 = R_3^3 = 0,$$

and hence

$$R = R_\mu^\mu = R_0^0 = -\tfrac{2}{3}t^{-2}.$$

If we put $\mu = \nu = 0$, then equation (8) becomes

$$\tfrac{1}{2}R^{00} + \lambda = -8\pi G T^{00} = -\tfrac{1}{2}R^{00} - 8\pi G T^{00},$$

and hence

$$\lambda = \tfrac{1}{3}t^{-2} - 8\pi G \rho_0. \tag{13}$$

We can see that λ and G are not independent, but coupled with the age and matter of the Universe via equation (13). Using (10), we can further deduce from $G = \beta_1 t^{-1}$ and $\rho_0 = \beta_2 t^{-1}$ that

$$\lambda = (\tfrac{1}{3} - 8\pi\beta_1\beta_2)t^{-2}. \tag{14}$$

By comparing (12) and (14), we note that the constant of integration is zero and that

$$\tfrac{1}{3} - 8\pi\beta_1\beta_2 = -4\pi\beta_1\beta_2.$$

We then obtain

$$\beta_1\beta_2 = (12\pi)^{-1}. \tag{15}$$

Substituting (15) into (14) we find that

$$\lambda = -\tfrac{1}{3}t^{-2}.$$

Thus, λ depends on the inverse square of time and is always negative. It will approach zero when the age of the Universe tends to infinity; otherwise, λ is nonzero as a consequence of the LNH.

At the present epoch, we have $t \approx 6\times10^{17}$ s, giving a present value of λ of about $9\times10^{-37}\ \mathrm{s}^{-2}$. The limit for λ set by experiment is (Ohanian 1976) $|\lambda| < 2\times10^{-35}\ \mathrm{s}^{-2}$, and so our theory is not impossible.

6. Average Matter Density of the Universe

From (15), we have $\beta_1\beta_2 = (12\pi)^{-1}$. Since $\beta_1 = Gt$ and since G is known accurately to be $6{\cdot}67\times10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$, then $\beta_1 = 4\times10^{7}\ \mathrm{m^3\,kg^{-1}\,s^{-1}}$, and this implies

$$\beta_2 = (12\pi\beta_1)^{-1} = 7\times10^{-10}\ \mathrm{m^{-3}\,kg\,s}.$$

As a result, the mean matter density of the Universe is given by

$$\rho_0 = \beta_2 t^{-1} = 1\times10^{-27}\ \mathrm{kg\,m^{-3}} = 1\times10^{-30}\ \mathrm{g\,cm^{-3}}.$$

The experimental value of ρ_0 varies from 10^{-29} to 10^{-31} g cm^{-3}, depending on the method of determination used. Our predicted value is within the range, and so the theory is not inconsistent with observation.

7. Einstein's Field Equations as an Approximation

With a time varying G and λ, (8) is satisfied by the LNH. From our calculations, we know that at the present epoch λ is of the order of 10^{-38} s^{-2} or 10^{-54} m^{-2}. It follows that the order of $\lambda^{-\frac{1}{2}}$ is around 10^{27} m. For a local region in the Universe with dimensions small compared with $\lambda^{-\frac{1}{2}}$, for instance the Solar System, λ has a negligible influence and can be approximated to zero in most calculations at the present epoch and in the future, or even in the not too distant past. In addition, $|G_{,0}/G| = t^{-1}$, and thus we have $|G_{,0}/G| \sim 10^{-10}$ yr^{-1} at present. Even over a period of 10^5 yr, the fractional change of G with time is only about 10^{-5}. Therefore, under most circumstances, G can be regarded as a genuine constant after a time which is distant enough from the Big Bang. The implication of a time varying G will take effect only in the early epoch or when the overall history and evolution of the Universe is considered. Hence, with the approximation of zero λ and constant G, we can immediately go over to Einstein's field equations of the form

$$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = -8\pi G T^{\mu\nu},$$

and the success of Einstein's theory of gravitation in explaining local gravitational phenomena is preserved.

We note also that $|\lambda_{,0}/\lambda| = t^{-1}$. For similar reasons as in the case of G, λ can be treated as a genuine constant after the early epoch. With this approximation of constant G and λ, we can arrive again at (8) with time-independent G and λ.

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