A Charged Analogue of the Vaidya–Tikekar Solution

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Abstract
We present here an interior solution of the Einstein–Maxwell equations for a charged static fluid sphere. The physical 3-space \( t = \text{constant} \) of the solution is spheroidal. The solution is interpreted as an exact relativistic model for a charged superdense star.

1. Introduction

Many exact interior solutions of the Einstein–Maxwell equations corresponding to a static charged sphere are available in the literature. The sphere of uniform density has been discussed by Kyle and Martin (1967) and by Mehra and Bohra (1979). The sphere of charged dust has been investigated by Bonnor and Wickramasuriya (1975), Raychaudhuri (1975) and Tikekar (1984). In all these cases, except Tikekar (1984), the physical 3-space \( t = \text{constant} \) is spherical. Patel and Pandya (1986) have obtained a Reissner–Nordstrom interior solution in which the physical 3-space \( t = \text{constant} \) is spheroidal. In all the above-mentioned solutions, the space–times are not conformally flat. Chang (1983) has obtained some conformally flat interior solutions of the Einstein–Maxwell equations for a charged static sphere, while Haji-Boutros and Sfeila (1986) have presented a generation technique to derive new exact solutions for a charged fluid sphere. They have also listed those papers which deal with different aspects of interior Reissner–Nordstrom solutions; for the sake of brevity, we do not repeat these references here.

Furthermore, it has been generally suggested that the collapse of a spherically symmetric distribution of matter to a point singularity can be avoided if the matter is accompanied by charge. The gravitational attraction is then balanced by the electrostatic repulsion and by the pressure gradient. This shows the importance of the interior Reissner–Nordstrom solutions.

It is the purpose of the present paper to obtain an interior solution for a charged perfect fluid sphere with the following properties:

(i) the space–time describing the geometry of the solution is not conformally flat;

(ii) the associated physical 3-space \( t = \text{constant} \) is spheroidal.
2. Metric Form and the Field Equations

Vaidya and Tikekar (1982) have discussed the space-times, with physical 3-space spheroidal, in some detail. They expressed the line element of such space-times in the form

\[ ds^2 = \exp(\gamma) \, dt^2 - \left(1 - \frac{K r^2}{R^2}\right) \left(1 - \frac{r^2}{R^2}\right)^{-1} \, dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]  

where \( \gamma \) is a function of \( r \) only. Here \( R \) and \( K \) are constants and \( K < 1 \). The line element (1) is regular at all points where \( r^2 < R^2 \). We denote the coordinates by \( x^1 = r, \ x^2 = \theta, \ x^3 = \phi \) and \( x^4 = t \). When \( K = 0 \), the physical 3-space \( t = \) constant becomes spherical.

The combined Einstein–Maxwell field equations are (using the geometric units \( c = G = 1 \))

\[ R^i_k - \frac{1}{2} R \delta^i_k = -8\pi T^i_k, \]  

where the energy–momentum tensor splits into two parts:

\[ T^i_k = M^i_k + E^i_k. \]  

For a charged perfect fluid we have

\[ M^i_k = (\rho + \rho) v^i v_k - p \delta^i_k, \]  

\[ E^i_k = \frac{1}{4\pi} \left( - F^{ik} F_{kn} + \frac{1}{4} \delta^i_k F_{mn} F^{mn} \right), \]  

where \( E^i_k \) is the electromagnetic energy tensor and \( v^i \) is the 4-vector velocity. The electromagnetic field tensor \( F_{ik} \) satisfies Maxwell’s equations

\[ F_{ik,j} + F_{kj,i} + F_{ji,k} = 0, \]  

\[ \frac{\partial}{\partial x^k} \{(-g)^{i/2} F^{ik}\} = 4\pi (-g)^{i/2} J^i, \]  

\( J^i \) being the 4-current vector. The fluid has been assumed to have null conductivity, so that

\[ J^i = \sigma v^i, \]  

where \( \sigma \) denotes the charge density and, since the field is static, we have

\[ v^i = (0, 0, 0, \exp(-\frac{1}{2} \gamma)). \]  

Since there is only a radial electric field, the only surviving component of \( F_{ik} \) is \( F_{14} \).
The Maxwell equations (6) lead us to write

$$F_{14} = \frac{\exp(\frac{1}{2} \gamma)}{r^2} \left( \frac{1 - K r^2/R^2}{1 - r^2/R^2} \right)^{\frac{1}{2}} \int_0^\gamma 4\pi \sigma \gamma^2 \left( \frac{1 - K r^2/R^2}{1 - r^2/R^2} \right)^{-\frac{1}{2}} \, dr,$$

and also

$$-F_{14} F^{14} = E^2(r),$$

where $E(r)$ can be interpreted as the field intensity. The results (9) and (10) imply that

$$4\pi \sigma = \frac{1}{r^2} \left( \frac{d}{dr} (r^2 E) \right) \left( 1 - \frac{r^2}{R^2} \right)^{\frac{1}{2}} \left( 1 - \frac{K r^2}{R^2} \right)^{-\frac{1}{2}},$$

and, subsequently,

$$Q(r) = 4\pi \int_0^\gamma \left( 1 - \frac{K r^2}{R^2} \right)^{\frac{1}{2}} \left( 1 - \frac{r^2}{R^2} \right)^{-\frac{1}{2}} \sigma r^2 \, dr,$$

which represents the total charge contained within the sphere of radius $r$.

The Einstein–Maxwell equations (2), along with (3), (4) and (5), give

$$-8\pi p + E^2 = \left\{ \frac{1 - K}{R^2} - \left( \frac{1}{r} \left( 1 - \frac{r^2}{R^2} \right) \right) \left( 1 - \frac{K r^2}{R^2} \right)^{-1} \right\},$$

$$8\pi \rho + E^2 = \frac{3(1 - K)}{R^2} \left( 1 - \frac{K r^2}{3 R^2} \right) \left( 1 - \frac{K r^2}{R^2} \right)^{-2},$$

$$-8\pi p - E^2 = -\left( \frac{\gamma'}{2} + \frac{\gamma^2}{4} + \frac{\gamma'}{2 r} \right) \left( 1 - \frac{r^2}{R^2} \right) \left( 1 - \frac{K r^2}{R^2} \right)^{-1}$$

$$+ \left( \frac{1 - K}{R^2} \right) \frac{r}{2} \left( \frac{\gamma'}{2} + \frac{1}{r} \right) \left( 1 - \frac{K r^2}{R^2} \right)^{-2}.$$
where $\beta$ is a constant. Clearly $E^2$ is positive. Equating the expressions for the pressure $p$ from (13) and (15), we get the differential equation

$$(1 - K + Kz^2) \frac{d^2 F}{dz^2} - Kz \frac{dF}{dz} + K(K - 1) F = 0,$$  \hspace{1cm} (17)

where $z^2 = 1 - r^2/R^2$ and $F = \exp(\frac{1}{2} \gamma) - 2\beta^2/K(K - 1)$. Equation (17) has been solved by Vaidya and Tikekar (1982) in the uncharged case (i.e. $\beta = 0$) for $K = -2$. Following the same method, the closed-form solution of (17) for $K = -2$ in the charged case can be expressed as

$$\exp(\frac{1}{2} \gamma) = \frac{1}{3} \beta^2 + Az(1 - \frac{4}{3} z^2) + B(1 - \frac{2}{3} z^2)^{\frac{3}{2}},$$  \hspace{1cm} (18)

where $A$ and $B$ are integration constants. The closed-form solutions of (17) have also been obtained for $K = -7$ and $-14$ etc., but are not reported here.

The matter density and the fluid pressure are found to be

$$8\pi \rho = \frac{3}{R^2(3 - 2z^2)^2} \left[ 5 - 2z^2 - \frac{\beta^2(1 - z^2)}{\beta^2 + 3Az(1 - \frac{4}{3} z^2) + 3B(1 - \frac{2}{3} z^2)^{\frac{3}{2}}} \right],$$  \hspace{1cm} (19)

$$8\pi p R^2(3 - 2z^2)\left[ \frac{1}{3} \beta^2 + Az(1 - \frac{4}{3} z^2) + B(1 - \frac{2}{3} z^2)^{\frac{3}{2}} \right]$$

$$= \frac{\beta^2(1 - z^2)}{3 - 2z^2} - \beta^2 + B(6z^2 - 3)(1 - \frac{3}{2} z^2)^{\frac{1}{2}} + Az(4z^2 - 5).$$  \hspace{1cm} (20)

The density $\rho$ and the pressure $p$ at the centre $(r = 0)$ attain the values

$$8\pi \rho_0 = \frac{9}{R^2},$$  \hspace{1cm} (21)

$$8\pi p_0 = \frac{9}{R^2} \left( \frac{B\sqrt{3} - A - \beta^2}{B\sqrt{3} + 5A + 3\beta^2} \right).$$  \hspace{1cm} (22)

The charge density $\sigma$ can be determined from (11) in the form

$$8\pi \sigma R^2 \left( 1 + \frac{2r^2}{R^2} \right)^{\frac{3}{2}} \exp(\frac{3}{2} \gamma) = \beta \left( \frac{4}{9} \left[ 30 + \frac{11r^2}{R^2} - \frac{22r^4}{R^4} + \frac{8r^6}{R^6} \right] \right)$$

$$+ \frac{2B}{\sqrt{3}} \left( 1 - \frac{r^2}{R^2} \right)^{\frac{1}{2}} \left( 1 + \frac{2r^2}{R^2} \right)^{\frac{3}{2}} \left( 1 - \frac{r^2}{3R^2} \right) + \frac{3}{2} \beta^2 \left( 3 + \frac{2r^2}{R^2} \right) \left( 1 - \frac{r^2}{R^2} \right)^{\frac{3}{2}},$$  \hspace{1cm} (23)

where $\exp(\gamma)$ is given by (18). It is easy to see that $E^2 = 0$ at the centre $r = 0$.

We consider a situation where the spherical charged perfect fluid distribution extends to a finite radius $a < R$. The interior metric (1) with $\exp(\gamma)$ given by (18)
should then match with the external Reissner–Nordstrom metric

\[ ds^2 = \left( 1 - \frac{2m}{r} + \frac{q^2}{r^2} \right) dt^2 - \left( 1 - \frac{2m}{r} + \frac{q^2}{r^2} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]

across the boundary \( r = a \). Here \( m \) and \( q \), respectively, denote the total mass and the total charge of the sphere. The relations (12) and \( Q(a) = q \) give us

\[ \beta^2 a^6 \int R^4 \left( 1 + \frac{2a^2}{R^2} \right)^2 \left\{ \frac{\beta^2}{3} + A \left( 1 - \frac{a^2}{R^2} \right) \left( 5 \frac{4a^2}{R^2} \right) \right\} \]

\[ + \frac{B}{3\sqrt{3}} \left( 1 + \frac{2a^2}{R^2} \right)^{3/2} \right\} = q^2. \]

The appropriate boundary conditions are

\[ \exp \{ \gamma(a) \} = \left( 1 - \frac{a^2}{R^2} \right) \left( 1 + \frac{2a^2}{R^2} \right)^{-1} = 1 - \frac{2m}{a} + \frac{q^2}{a^2}, \]

and the pressure at \( r = a \) vanishes. These conditions determine the constants \( A, B \) and \( m \) to be

\[ A = \frac{3}{2} \left( 1 - \frac{2a^2}{R^2} \right) \left( 1 + \frac{2a^2}{R^2} \right)^{-\frac{1}{2}} - \frac{\beta^2}{6} \left( 4 - \frac{5a^2}{R^2} \right) \left( 1 - \frac{a^2}{R^2} \right)^{-\frac{1}{2}}, \]

\[ B = \frac{\sqrt{3}}{2} \left( 1 + \frac{4a^2}{R^2} \right) \left( 1 - \frac{a^2}{R^2} \right)^{\frac{3}{2}} \left( 1 + \frac{2a^2}{R^2} \right)^{-1} \]

\[ + \frac{\beta^2}{6\sqrt{3}} \left( 2 - \frac{9a^2}{R^2} - \frac{20a^4}{R^4} \right) \left( 1 + \frac{a^2}{R^2} \right)^{-\frac{3}{2}}, \]

\[ \frac{m}{a} = \frac{3a^2}{2R^2} \left( 1 + \frac{2a^2}{R^2} \right)^{-1} + \frac{\beta^2 a^4}{2R^4} \left( 1 - \frac{a^2}{R^2} \right)^{-\frac{1}{2}} \left( 1 + \frac{2a^2}{R^2} \right)^{-\frac{1}{2}}. \]

From (28) it is evident that the mass parameter \( m \) is positive.

Thus, the final form of the metric of our solution is

\[ ds^2 = \frac{1}{2} \left\{ \beta^2 + A \left( 1 - \frac{r^2}{R^2} \right)^{\frac{1}{2}} \left( 5 + \frac{4r^2}{R^2} \right) + \frac{B}{\sqrt{3}} \left( 1 + \frac{2r^2}{R^2} \right)^{\frac{3}{2}} \right\}^2 \]

\[ dt^2 - \frac{1 + 2r^2/R^2}{1 - r^2/R^2} \, dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]

where \( A \) and \( B \) are given by (26) and (27). When \( \beta = 0 \), the electromagnetic field disappears and we get the solution discussed by Vaidya and Tikekar (1982) in connection with the exact relativistic model for a superdense star. Thus, our solution
Table 1. Variation of parameters with $a/b$ for $(a) \beta^2 = 0.1$ and $(b) \beta^2 = 2.0$

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<th>$B$</th>
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$(b) \beta^2 = 2.0$

If $\rho_a$ is the value of $\rho$ at the boundary $r = a$ and $\lambda = \rho_a/\rho_0$, then a straightforward calculation yields

$$\lambda = \frac{1}{3} \left( 3 + \frac{2a^2}{R^2} \right) \left( 1 + \frac{2a^2}{R^2} \right)^{-1} - \frac{\beta^2 a^2}{9R^2} \left( 1 - \frac{a^2}{R^2} \right)^{-\frac{1}{2}} \left( 1 + \frac{2a^2}{R^2} \right)^{-\frac{1}{2}}.$$

(30)

It is not hard to see that $\lambda < 1$. Thus, the central density $\rho_0$ is greater than the density $\rho_a$ at the boundary. For a physically significant model we must have

$$\rho_0 > 0, \quad \rho_0 > 0, \quad \rho_0 - \rho_0 > 0, \quad \rho_0 - 3\rho_0 > 0.$$
If we set \( f(A, B) = (B\sqrt{3} - A - \beta^2)/(B\sqrt{3} + 5A + 3\beta^2) \), we see that the conditions (31) are equivalent to

\[
f(A, B) > 0, \quad 1 - f(A, B) > 0, \quad 1 - 3f(A, B) > 0.
\] (32)

If \( a/R \) is given, then \( A, B \) and \( \lambda \) can be determined from (26), (27) and (30) respectively for a fixed value of \( \beta \). For the uncharged case, Vaidya and Tikekar (1982) have noted that \( a/R \) must be less than 0.5567 for a physically significant model. For \( \beta^2 = 0.1 \) and \( \beta^2 = 2.0 \), and for \( 0.025 < a/R < 0.60 \), the values of \( \lambda, A, B, f(A, B), 1 - 3f(A, B) \) and \( 1 - f(A, B) \) are given in Table 1. From this table it is clear that, when \( a/R \) increases, \( \lambda \) decreases. From Table 1a it is clear that for \( \beta^2 = 0.1, 1 - 3f(A, B) \) becomes negative for \( a/R = 0.60 \). Thus, for \( \beta^2 = 0.1 \), the conditions (31) are satisfied for \( 0.025 < a/R < 0.575 \). From Table 1b it is evident that \( f(A, B) \) becomes negative for \( a/R = 0.425 \). Thus, for \( \beta^2 = 2.0 \), the conditions (31) are satisfied for \( 0.025 < a/R < 0.40 \). Though the numerical calculations have been carried out for the exact solution corresponding \( K = -2 \), the method is quite general and can be used for a whole series of models with \( K < 1 \).

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