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Stability of Modified Korteweg–de Vries Waves

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Abstract

A nonlinear wave theory is developed on the basis of the Infeld–Rowlands method to study the stability of modified Korteweg–de Vries waves. A general stability criterion is derived in order to show that unstable waves exist.

1. Introduction

Let us consider the modified Korteweg–de Vries (mKdV) equation written in the form

$$u_t + 6\alpha u^2 u_x + u_{xxx} = 0. \tag{1}$$

This equation was derived via Shen's method (Shen and Zhong 1981) for water waves in a two-layer fluid filling a channel with varying cross section (Murawski 1986). We now look for a solution of the form

$$u = B(\xi \equiv x - ct). \tag{2}$$

For the Zakharov–Kuznetsov equation c may be assumed to be zero because of Galilean invariance of this equation (Infeld 1985), but in our case we have to put $c \neq 0$.

Upon integration of (1) we get

$$2\alpha B^3 - cB + B_{\xi\xi} = a, \tag{3}$$

where a is a constant. Multiplying this equation by B and integrating, we get

$$B_\xi^2 = cB^2 - \alpha B^4 + 2aB + l, \tag{4}$$

where l is an integration constant. This equation must have double roots, so that $\alpha = \pm 1$ requires $c^3 > \frac{27}{2} a^2$ and $c^3 < -\frac{27}{2} a^2$ respectively. From this equation we can find solutions of (1) without analytical calculations. Fig. 1 presents phase diagrams for the case of $\alpha < 0$ and $a = 0$. The range of periodic waves (curve b is one such wave) is limited by a linear wave (curve a) and a shock wave (curve c). Fig. 2 shows phase diagrams for the case $a \neq 0$. Periodic waves (curve b) are bounded by linear

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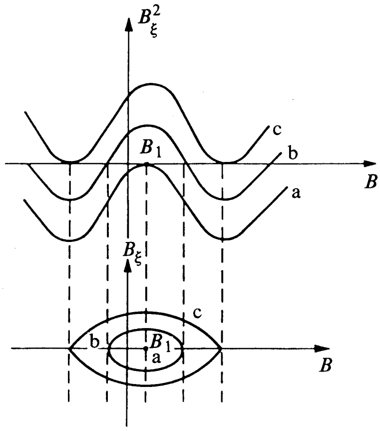

Fig. 1. Phase diagrams for equation (4) for the case $\alpha = -1$ and $a = 0$: curve a, linear wave limit; curve b, cnoidal wave; curve c, shock wave.

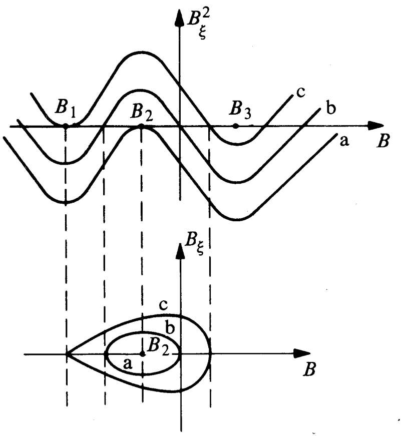

Fig. 2. As for Fig. 1, but $a \neq 0$ and curve c is for solitons.

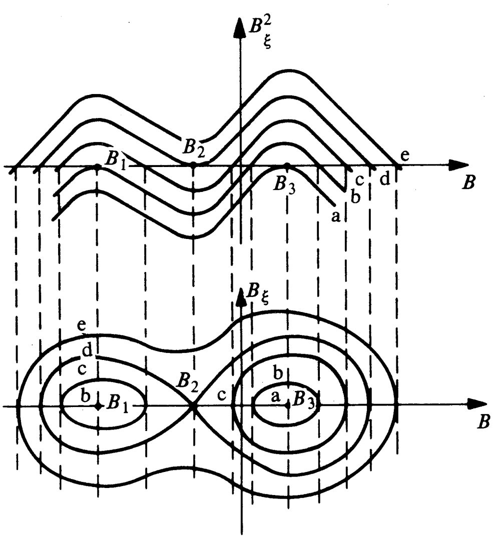

Fig. 3. The case of $\alpha = 1$:
curve a, linear wave limit;
curve b, cnoidal wave on the right and linear wave on the left;
curve c, cnoidal waves;
curve d, solitons;
curve e, cnoidal wave.

waves (curve a) and solitons (curve c). The case $\alpha > 0$ is treated in Fig. 3. We can distinguish three regions assigned by $l_{min} < l_{av} < l_{max}$ which correspond to double roots of equation (4). In the region $l_{min} < l < l_{av}$ there are periodic waves. For $l = l_{min}$ a linear wave appears and for $l = l_{av}$ we have periodic and other linear waves. In the region $l_{av} < l < l_{max}$ there are two periodic waves corresponding to the same parameter l. For $l = l_{max}$ they are both limited by solitons. The region $l > l_{max}$ contains only periodic waves.

Nonlinear wavetrain instabilities of the mKdV equation have been considered by Fornberg and Whitham (1978) in numerical and theoretical studies of the KdV equation and its generalisations. Here we shall discuss the stability of the whole set of mKdV waves and thus generalise the problem using the Infeld–Rowlands method (Infeld 1981, 1985; Infeld *et al.* 1978, 1985; Infeld and Rowlands 1979*a*, 1979*b*, 1980). This method includes nonlinear terms and may be regarded as an extension of linear methods (see e.g. Lashmore-Davies and Stenflo 1979; Bateman 1980; Storer 1983).

The paper is arranged as follows. The next section considers the stability of the mKdV waves in the limit of long wavelength perturbations, while Section 3 presents numerical results which reveal the fact that unstable waves exist.

2. Infeld–Rowlands Method

Suppose a nonlinear wave given by (4) is perturbed by a long wavelength linear wave with small amplitude δu. We then have

$$u = B(\xi) + \delta u(\xi)\exp\{i(k\xi + wt)\}\,, \tag{5}$$

where we have the stretched coordinates

$$\xi = x - ct\,, \qquad t = t\,, \tag{6a, b}$$

and $\delta u(\xi)$ is periodic with the same period λ. We assume k is small and expand as

$$w = w_1\, k + w_2\, k^2 + \ldots, \qquad \delta u = \delta u_0 + k\delta u_1 + \ldots. \tag{7, 8}$$

In the frame moving with speed c we may write

$$u_t - c u_\xi + 6\alpha u^2 u_\xi + u_{\xi\xi\xi} = 0\,. \tag{9}$$

Introducing (5) into (9), we find (neglecting terms quadratic in δu and proportional to k^3)

$$(L\delta u)_\xi = -\mathrm{i}\,\tilde{w}\delta u - 6\mathrm{i}\,k\alpha B^2\delta u - 3\mathrm{i}\,k\delta u_{\xi\xi} + 3k^2\delta u_\xi\,, \tag{10}$$

where the following notation is used:

$$L \equiv \partial_\xi^2 + 6\alpha B^2 - c\,, \qquad w \equiv \tilde{w} - kc\,. \tag{11, 12}$$

In zeroth order of k, after substitution of (7) and (8) into (10), we have

$$L\delta u_0 = K\,, \tag{13}$$

where K is an integration constant. The homogeneous equation corresponding to

(13) is solved by

$$\delta u_{01} = B_\xi , \qquad \delta u_{02} = B_\xi \int \frac{\mathrm{d}\xi}{B_\xi^2} . \tag{14, 15}$$

So, the general solution of equation (13) may be written as

$$\delta u_0 = B_\xi + G\Psi + K\Phi , \tag{16}$$

where

$$\Psi \equiv B_\xi \int \frac{\mathrm{d}\xi}{B_\xi^2} = \beta\xi B_\xi + Q_0(\xi) , \qquad \Phi \equiv B_\xi \int \frac{B\,\mathrm{d}\xi}{B_\xi^2} = \gamma\xi B_\xi + Q_1(\xi) . \tag{17a, b}$$

The functions Q_0 and Q_1 are both periodic with the same period as B, i.e. λ. The second and third terms of the RHS of (16) are both secular. Removal of these secular expressions determines G as

$$G = -K\gamma/\beta , \tag{18}$$

and consequently

$$\delta u_0 = B_\xi + K\left(Q_1 - \frac{\gamma}{\beta} Q_0 \right) . \tag{19}$$

In the first order of k, we have from equation (10)

$$(L\delta u_1)_\xi = -3\mathrm{i}\,\delta u_{0\xi\xi} - \mathrm{i}\tilde{w}_1\,\delta u_0 - 6\mathrm{i}\,\alpha B^2 \delta u_0 . \tag{20}$$

The constant K has to be equal to zero because of the equation obtained by integration of (20) over the period λ:

$$K(\tilde{w}_1 + \mathrm{const}\,1) = 0 .$$

We can also multiply (20) by B and integrate over the period to obtain

$$K(\tilde{w}_1 + \mathrm{const}\,2) = 0 ,$$

and hence

$$K = 0 . \tag{21}$$

The homogeneous equation corresponding to (20) is solved by

$$\delta u_{1,1} = B_\xi , \qquad \delta u_{1,2} = B_\xi \int \frac{\mathrm{d}\xi}{B_\xi^2} .$$

So, the general solution of equation (20) may be rewritten as

$$\delta u_1 = B_\xi(1 - \tfrac{1}{2}\mathrm{i}\,\xi) + (D - \tfrac{1}{2}\mathrm{i}\,l)\Psi + \mathrm{i}\,\hat{R}\Phi - \tfrac{1}{2}\mathrm{i}(\tilde{w}_1 + c)\phi , \tag{22}$$

where D is an arbitrary constant and

$$\phi \equiv B_\xi \int \frac{B^2\,\mathrm{d}\xi}{B_\xi^2} = \kappa\xi B_\xi + Q_2(\xi) , \qquad \hat{R} = -(a + \mathrm{i}\,R) . \tag{23a, b}$$

We remove secular expressions in (22) to determine D:

$$D = \frac{\mathrm{i}}{2\beta}\{\beta l + (\tilde{w}_1 + c)\kappa - 2\hat{R}\gamma + 2\}, \tag{24}$$

and

$$\delta u_1 = B_\xi + (D - \tfrac{1}{2}\mathrm{i}\, l)Q_0 + \mathrm{i}\,\hat{R}Q_1 - \tfrac{1}{2}\mathrm{i}(\tilde{w}_1 + c)Q_2 . \tag{25}$$

Equation (10) yields, in second order,

$$(L\delta u_2)_\xi = 3B_{\xi\xi} - 3\mathrm{i}\,\delta u_{1\xi\xi} - \mathrm{i}\tilde{w}_1\,\delta u_1 - \mathrm{i}\tilde{w}_2\, B_\xi - 6\mathrm{i}\,\alpha B^2\delta u_1 . \tag{26}$$

We then determine

$$\langle f\rangle = \frac{1}{\lambda}\int_0^\lambda f\, \mathrm{d}\xi . \tag{27}$$

Equation (26) integrates over the period to give

$$\tilde{w}_1\langle\delta u_1\rangle + 6\alpha\langle B^2\delta u_1\rangle = 0 . \tag{28}$$

If we multiply (26) by B on the left, integrate over the period and use the self-adjoint property of the operator L, we get

$$\langle B\,\mathrm{d}\xi\, L\delta u_2\rangle = -\langle B_\xi\, L\delta u_2\rangle = -\langle L\delta u_2\, B_\xi\rangle = -\langle\delta u_2\, LB_\xi\rangle = 0 . \tag{29}$$

Thus, the compatibility condition for $(L\delta u)_\xi = f$, and f periodic, is $\langle fB\rangle = 0$. This is the condition used after (20) and (26). Then we get

$$3\mathrm{i}\langle BB_{\xi\xi}\rangle + 3\langle B\,\delta u_{1\xi\xi}\rangle + \tilde{w}_1\langle B\,\delta u_1\rangle + 6\alpha\langle B^3\,\delta u_1\rangle = 0 . \tag{30}$$

From (24) and (28), we have

$$\hat{R} = \frac{\beta(\tilde{w}_1 + c)(6\alpha\langle B^2 Q_2\rangle + \tilde{w}_1\langle Q_2\rangle)}{2\beta(6\alpha\langle B^2 Q_1\rangle + \tilde{w}_1\langle Q_1\rangle) - 2\gamma(\langle Q_0\rangle\tilde{w}_1 + 6\alpha\langle B^2 Q_0\rangle)}$$
$$+ \frac{(\tilde{w}_1\langle Q_0\rangle + 6\alpha\langle B^2 Q_0\rangle)\{2 - \kappa(\tilde{w}_1 + c)\}}{2\beta(6\alpha\langle B^2 Q_1\rangle + \tilde{w}_1\langle Q_1\rangle) - 2\gamma(\langle Q_0\rangle\tilde{w}_1 + 6\alpha\langle B^2 Q_0\rangle)} . \tag{31}$$

Substituting (31) into (30), after some straightforward calculations, we obtain

$$a_2\,\tilde{w}_1^3 + b_2\,\tilde{w}_1^2 + c_2\,\tilde{w}_1 + d_2 = 0 . \tag{32}$$

3. Numerical Results

Equation (32) is the main result of this paper. Within the limitations of the mKdV model it is a very general test for stability of nonlinear waves. Three arbitrary parameters a, c and l are restricted by existing solution regimes. Let us consider the simpler case of $\alpha = -1$ and $a = 0$. Then $l = 0$ and $l = c^2/4$ correspond to linear and shock waves respectively (see Fig. 1). The parameter l changes between these two limitations. Fig. 4 presents numerically obtained plots of the real parts of $\tilde{w}_1$ for

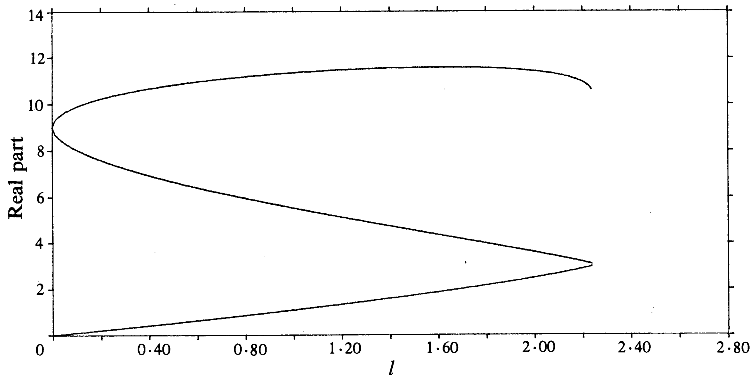

Fig. 4. Real part of the roots of equation (32) for $c = -3$, $a = -1$ and $a = 0$. The imaginary part is equal to zero.

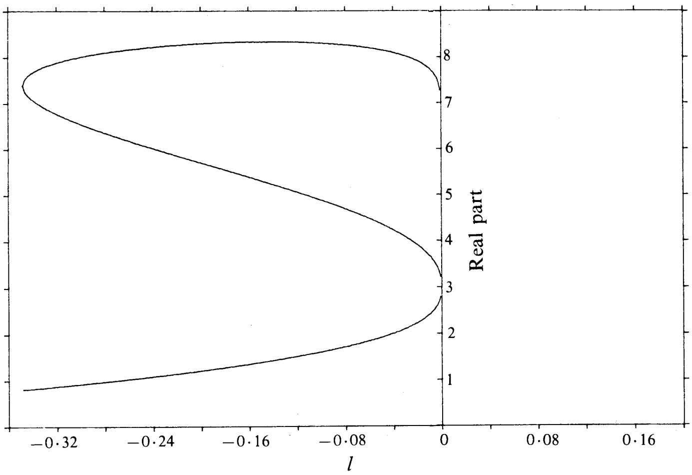

Fig. 5. As for Fig. 4, but $a = -1$.

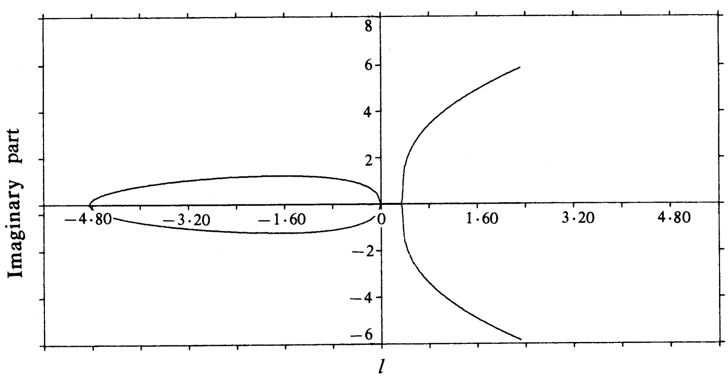

Fig. 6. Imaginary parts of equation (32) for $c = 3$, $a = 1$ and $a = -1$. Waves are unstable in the regions $l_{\min} < l < l_{\mathrm{av}}$ and $l > l_{\max}$.

$c = -3$. Imaginary roots of equation (32) do not appear at all. Thus, both periodic and shock waves are stable. Secondly, we also have the case of $\alpha = -1$, $c = -3$ and $a = -1$ (see Fig. 2). The double root of equation (4) appears at $l = l_{\min} < l_{\mathrm{av}} < l_{\max}$. The cases $l = l_{\min}$ and $l = l_{\mathrm{av}}$ correspond to a linear wave and soliton respectively. For $l_{\min} < l < l_{\mathrm{av}}$, there are periodic waves.

The real parts of $\tilde{w}_1$ are plotted against l in Fig. 5 for the case $a = -1$. In this region periodic waves and solitons are stable. The case of $\alpha = 1$ is more complicated. We choose $c = 3$ and $a = -1$ in further calculations and distinguish $l_{\min} < l_{\mathrm{av}} < l_{\max}$. In the regions $l_{\min} < l < l_{\mathrm{av}}$ and $l > l_{\max}$ all waves are unstable, whereas for $l_{\mathrm{av}} < l < l_{\max}$ periodic waves are stable. The imaginary parts of $\tilde{w}_1$ are shown in Fig. 6.

In conclusion we note that unstable mKdV waves exist for $\alpha = 1$ in the regions $l_{\min} < l < l_{\mathrm{av}}$ and $l > l_{\max}$.

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References

Bateman, G. (1980). 'MHD Instabilities' (MIT Press: London).
Fornberg, B., and Whitham, G. B. (1978). *Phil. Trans. R. Soc. London* A **289**, 373.
Infeld, E. (1981). *Acta Phys. Polon.* A **60**, 623.
Infeld, E. (1985). *J. Plasma Phys.* **33**, 171.
Infeld, E., Frycz, P., and Czerwińska-Lenkowska, T. (1985). *Lett. Nuovo Cimento* **44**, 537.
Infeld, E., and Rowlands, G. (1979*a*). *Acta Phys. Polon.* A **56**, 329.
Infeld, E., and Rowlands, G. (1979*b*). *Proc. R. Soc. London* A **366**, 537.
Infeld, E., and Rowlands, G. (1980). *Z. Phys.* B **37**, 277.
Infeld, E., Rowlands, G., and Hen, H. (1978). *Acta Phys. Polon.* A **54**, 131.
Lashmore-Davies, C. N., and Stenflo, L. (1979). *Plasma Phys.* **21**, 735.
Murawski, K. (1986). *Z. Naturforsch.* **41a**, 1186.
Shen, M. C., and Zhong, X.-C. (1981). *J. Mech.* **20**, 789.
Storer, R. G. (1983). *Plasma Phys.* **25**, 1279.

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