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A Formal Time-domain Approach to Cold Magnetised Plasmas

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Abstract

Representations of the electromagnetic and the average velocity field for a cold magnetised plasma are derived in terms of scalar potential functions. These Hertz potentials are solutions of a coupled system of integro-differential equations of second order. Different from other approaches, the analysis is carried out in the time domain and is therefore especially suited for the investigation of transient wave phenomena. Furthermore, the dielectric tensor operator of the plasma is derived. After solving the system of integro-differential equations for a special limiting case, the applicability of the method presented is demonstrated and generalisations are discussed.

1. Introduction

By looking at the literature on electromagnetic waves one finds two different viewpoints in treating problems arising. As many electromagnetic sources operate in the steady-state time-harmonic regime, wave phenomena are frequently investigated in a time-harmonic approach (in mathematical terms this means the basic equations in space and time are subjected to a Fourier transformation with respect to time). Within the last decade increasing interest has emerged in the study of transient wave phenomena. In this context one is therefore definitely interested in the time-dependent behaviour of the electromagnetic fields. [For an introduction to the techniques and applications of transient electromagnetic fields see Felsen (1976).]

The present paper presents a time-domain approach to the electrodynamics of a cold magnetised electron plasma. It is shown that the electromagnetic field equations and the equation of motion for the velocity field can be reduced to a system of integro-differential equations for two scalar potentials (and certain auxiliary functions). Secondly, the forthcoming mathematical procedure serves to present the successful application of the inverse operator technique (Felsen and Marcuvitz 1973) to mathematical problems in physics.

The basic partial differential equations are

$$\epsilon_0\, \partial \boldsymbol{E}(\boldsymbol{x}, t)/\partial t - \nabla \times \boldsymbol{H}(\boldsymbol{x}, t) - n_0\, q\boldsymbol{v}(\boldsymbol{x}, t) = -\boldsymbol{J}(\boldsymbol{x}, t), \tag{1}$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{x}, t) + \mu_0\, \partial \boldsymbol{H}(\boldsymbol{x}, t)/\partial t = -\boldsymbol{M}(\boldsymbol{x}, t), \tag{2}$$

$$n_0\, q\boldsymbol{E}(\boldsymbol{x}, t) + n_0\, m\{(\partial/\partial t)\mathbf{I} - \omega_{\mathrm{c}}(\boldsymbol{b} \times \mathbf{I})\} . \boldsymbol{v}(\boldsymbol{x}, t) = 0. \tag{3}$$

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Equations (1) and (2) are Maxwell's equations for the electric and magnetic field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ in the presence of applied electric and magnetic current density distributions $\boldsymbol{J}$ and $\boldsymbol{M}$. The vacuum dielectric constant is ϵ_0 and μ_0 is the vacuum permeability. The background electron charge and mass density are given by $n_0\, q$ and $n_0\, m$ (n_0 is the electron number density). The electromagnetic fields are coupled to the average velocity field $\boldsymbol{v}$ by the electron current $n_0\, q\boldsymbol{v}$ acting as a source term in Maxwell's equations. Finally (3) is the linearised equation of motion for the velocity field ($\mathbf{I}$ is the unit tensor). The plasma is rendered anisotropic by a static magnetic field given by $\boldsymbol{B}_0 = B_0\, \boldsymbol{b}$ ($\boldsymbol{b}$ is a unit vector). The electron gyrofrequency is $\omega_c = qB_0/m$. In this paper it is assumed that n_0 and B_0 are constant, but the method presented can be generalised to inhomogeneous media with n_0 and B_0 being functions of the coordinate in the direction of $\boldsymbol{b}$.

Equations (1)–(3) are valid in a cold magnetised electron plasma, i.e. the background pressure of the plasma (which is proportional to the plasma temperature) is assumed to be zero. Furthermore, only plasma electrons are treated as mobile and the equation of motion has been linearised by omitting the term $(\boldsymbol{v}\,.\nabla)\boldsymbol{v}$.

The most important application of this model is found in the Earth's ionosphere where the plasma temperature is (almost) zero and the static magnetic field is the Earth's magnetic field.

Section 2 presents a reduction of the system (1)–(3) to a system of integro-differential equations for the electromagnetic field components which are parallel to the direction of the static magnetic field $\boldsymbol{B}_0$. In Section 3 scalar Hertz potentials are introduced and it is shown how the electromagnetic and the dynamical velocity field can be represented in terms of these scalar potentials. The solution of the integro-differential equations for an isotropic plasma is presented in Section 4. Section 5 demonstrates the applicability and possible generalisations of the method presented and finally the dielectric and the conductivity tensor operator are derived in the Appendix.

2. Reduction of the Field Equations

The axis of the static magnetic field $\boldsymbol{B}_0$ is the only direction distinguished within the field equations. It is therefore natural to decompose all vector fields into transverse and longitudinal components with respect to this axis, namely

$$\boldsymbol{E} = \boldsymbol{E}^{\mathrm{t}} + E\boldsymbol{b}, \qquad \boldsymbol{H} = \boldsymbol{H}^{\mathrm{t}} + H\boldsymbol{b}, \qquad \boldsymbol{v} = \boldsymbol{v}^{\mathrm{t}} + v\boldsymbol{b},$$

$$\boldsymbol{J} = \boldsymbol{J}^{\mathrm{t}} + J\boldsymbol{b}, \qquad \boldsymbol{M} = \boldsymbol{M}^{\mathrm{t}} + M\boldsymbol{b}, \tag{4}$$

with

$$\nabla = \nabla^{\mathrm{t}} + (\partial/\partial x_b)\boldsymbol{b} \qquad (x_b = \boldsymbol{x}\,.\,\boldsymbol{b}) \tag{5}$$

for the derivative operator. In the same way the three vector equations (1)–(3) are decomposed into longitudinal and transverse parts. By calculating the scalar product of $\boldsymbol{b}$ with (1)–(3) one finds that ($\partial_t \equiv \partial/\partial t$)

$$\epsilon_0(\partial_t E) - \nabla^{\mathrm{t}}.(\boldsymbol{H}^{\mathrm{t}} \times \boldsymbol{b}) - n_0\, qv = -J, \tag{6}$$

$$\nabla^{\mathrm{t}}.(\boldsymbol{E}^{\mathrm{t}} \times \boldsymbol{b}) + \mu_0(\partial_t H) = -M, \tag{7}$$

$$n_0\, qE + n_0\, m(\partial_t v) = 0. \tag{8}$$

Applying $\boldsymbol{b} \times \mathbf{I}$ to (1)–(3) results in three two-component vector equations which are scalarised by calculating their transversal divergence ∇^{t}:

$$\epsilon_0 \, \partial_t \{\nabla^{\mathrm{t}} . (\boldsymbol{b} \times \boldsymbol{E}^{\mathrm{t}})\} - \nabla_{\mathrm{t}}^2 H + \partial \nabla^{\mathrm{t}} . \boldsymbol{H}^{\mathrm{t}} / \partial x_b - n_0 \, q \nabla^{\mathrm{t}} . (\boldsymbol{b} \times \boldsymbol{v}^{\mathrm{t}}) = -\nabla^{\mathrm{t}} . (\boldsymbol{b} \times \boldsymbol{J}^{\mathrm{t}}), \tag{9}$$

$$\nabla_{\mathrm{t}}^2 E - \partial \nabla^{\mathrm{t}} . \boldsymbol{E}^{\mathrm{t}} / \partial x_b + \mu_0 \, \partial_t \{\nabla^{\mathrm{t}} . (\boldsymbol{b} \times \boldsymbol{H}^{\mathrm{t}})\} = -\nabla^{\mathrm{t}} . (\boldsymbol{b} \times \boldsymbol{M}^{\mathrm{t}}), \tag{10}$$

$$n_0 \, q \nabla^{\mathrm{t}} . (\boldsymbol{b} \times \boldsymbol{E}^{\mathrm{t}}) + n_0 \, m \partial_t \{\nabla^{\mathrm{t}} . (\boldsymbol{b} \times \boldsymbol{v}^{\mathrm{t}})\} + n_0 \, m \omega_{\mathrm{c}} (\nabla^{\mathrm{t}} . \boldsymbol{v}^{\mathrm{t}}) = 0 \tag{11}$$

$$(\nabla^2 = \nabla_{\mathrm{t}}^2 + \partial^2 / \partial x_b^2).$$

The remaining equations are derived by applying the divergence operator ∇ to (1)–(3). One arrives at

$$\epsilon_0 \, \partial_t (\nabla^{\mathrm{t}} . \boldsymbol{E}^{\mathrm{t}}) + \epsilon_0 \, \partial_t (\partial E / \partial x_b) - n_0 \, q (\nabla^{\mathrm{t}} . \boldsymbol{v}^{\mathrm{t}}) - n_0 \, q \partial v / \partial x_b = -\nabla^{\mathrm{t}} . \boldsymbol{J}^{\mathrm{t}} - \partial J / \partial x_b, \tag{12}$$

$$\mu_0 \, \partial_t (\nabla^{\mathrm{t}} . \boldsymbol{H}^{\mathrm{t}}) + \mu_0 \, \partial_t (\partial H / \partial x_b) = -\nabla^{\mathrm{t}} . \boldsymbol{M}^{\mathrm{t}} - \partial M / \partial x_b, \tag{13}$$

$$n_0 \, q (\nabla^{\mathrm{t}} . \boldsymbol{E}^{\mathrm{t}}) + n_0 \, q (\partial E / \partial x_b) + n_0 \, m \partial_t (\nabla^{\mathrm{t}} . \boldsymbol{v}^{\mathrm{t}}) + n_0 \, m \partial_t (\partial v / \partial x_b) - n_0 \, m \omega_{\mathrm{c}} \{\nabla^{\mathrm{t}} . (\boldsymbol{b} \times \boldsymbol{v}^{\mathrm{t}})\} = 0. \tag{14}$$

The nine scalar equations (6)–(14) replace the three vector equations (1)–(3) as the fundamental differential equations.

From (6)–(8) one derives

$$E = \{\partial_t / \epsilon_0 (\partial_t^2 + \omega_{\mathrm{p}}^2)\} \{\nabla^{\mathrm{t}} . (\boldsymbol{H}^{\mathrm{t}} \times \boldsymbol{b}) - J\}, \tag{15}$$

$$H = (1/\mu_0 \, \partial_t) \{\nabla^{\mathrm{t}} . (\boldsymbol{b} \times \boldsymbol{E}^{\mathrm{t}}) - M\}, \tag{16}$$

$$v = -(q / m \partial_t) E, \tag{17}$$

where the electron plasma frequency $\omega_{\mathrm{p}}^2 = n_0 \, q^2 / m \epsilon_0$ has been introduced. In these equations the time derivative operator $\partial / \partial t$ has been treated like an algebraic quantity. This is a purely formal way of introducing an integral operator because (Felsen and Marcuvitz 1973)

$$(1/\partial_t) f(t) \equiv \int_{\tau = -\infty}^{t} f(\tau) \, \mathrm{d}\tau. \tag{18}$$

Equations (15)–(17) show the interesting result that the longitudinal field components E, H and v can be derived from the transverse components $\boldsymbol{E}^{\mathrm{t}}$, $\boldsymbol{H}^{\mathrm{t}}$ and $\boldsymbol{v}^{\mathrm{t}}$. This is due to the special structure of Maxwell's equations and is valid for a very general class of electromagnetic media (Weiglhofer 1987*b*).

By eliminating all terms including the transverse components $\boldsymbol{E}^{\mathrm{t}}$, $\boldsymbol{H}^{\mathrm{t}}$ and $\boldsymbol{v}^{\mathrm{t}}$, a system of equations for E and H only can be derived from (6)–(14) (v is eliminated by virtue of 17):

$$\tilde{H}_{\mathrm{e}}\, E + \tilde{L}_{\mathrm{e}}\, H = \tilde{s}_{\mathrm{e}}(\boldsymbol{J}, \boldsymbol{M}), \qquad \tilde{L}_{\mathrm{m}}\, E + \tilde{H}_{\mathrm{m}}\, H = \tilde{s}_{\mathrm{m}}(\boldsymbol{J}, \boldsymbol{M}), \tag{19, 20}$$

where the integro-differential operators $\tilde{H}_{\mathrm{e}}$, $\tilde{H}_{\mathrm{m}}$, $\tilde{L}_{\mathrm{e}}$ and $\tilde{L}_{\mathrm{m}}$ are given by

$$\tilde{H}_{\mathrm{e}} = \nabla_{\mathrm{t}}^2 + (\tilde{\epsilon}/\tilde{\epsilon}_1)\partial^2/\partial x_b^2 - (\tilde{\epsilon}/c^2)\partial^2/\partial t^2, \tag{21}$$

$$\tilde{H}_{\mathrm{m}} = \nabla^2 - \{(\tilde{\epsilon}_1^2 - \tilde{\epsilon}_2^2)/\tilde{\epsilon}_1 c^2\}\partial^2/\partial t^2, \tag{22}$$

$$\tilde{L}_{\mathrm{e}} = -\mu_0(\tilde{\epsilon}_2/\tilde{\epsilon}_1)\partial^2/\partial t\, \partial x_b, \tag{23}$$

$$\tilde{L}_{\mathrm{m}} = \epsilon_0(\tilde{\epsilon}\tilde{\epsilon}_2/\tilde{\epsilon}_1)\partial^2/\partial t\, \partial x_b. \tag{24}$$

The *dielectric integral operators* $\tilde{\epsilon}$, $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ are defined by

$$\tilde{\epsilon} = 1 + \omega_{\mathrm{p}}^2/\partial_t^2, \tag{25}$$

$$\tilde{\epsilon}_1 = 1 + \omega_{\mathrm{p}}^2/(\partial_t^2 + \omega_{\mathrm{c}}^2), \tag{26}$$

$$\tilde{\epsilon}_2 = -\omega_{\mathrm{c}}\, \omega_{\mathrm{p}}^2/\{\partial_t(\partial_t^2 + \omega_{\mathrm{c}}^2)\}, \tag{27}$$

and $c^2 = 1/\epsilon_0\mu_0$ is the vacuum light velocity. The source terms $\tilde{s}_{\mathrm{e}}$ and $\tilde{s}_{\mathrm{m}}$ are comprised of lengthy combinations of the sources $\boldsymbol{J}$ and $\boldsymbol{M}$.

Unfortunately the transverse field components cannot be found from the longitudinal ones without solving a second set of integro-differential equations. Thus, in the next section scalar potentials are introduced to accomplish a full representation of the electromagnetic and velocity fields.

3. Scalar Hertz Potentials and Field Representations

By a special application of the Helmholtz theorem (see for example Plonsey and Collin 1961) the two-component transverse fields are decomposed into the transverse gradient of a scalar function and the transverse curl of a one-component longitudinal vector. The defining equations for the scalar potentials $\hat{u}(\boldsymbol{x}, t)$, $\hat{v}(\boldsymbol{x}, t)$, $w_1(\boldsymbol{x}, t)$ and $w_2(\boldsymbol{x}, t)$ are

$$\boldsymbol{E}^{\mathrm{t}} = -\mu_0\, \partial_t(\nabla^{\mathrm{t}} \times \hat{v}\boldsymbol{b} + \nabla^{\mathrm{t}} w_1), \tag{28}$$

$$\boldsymbol{H}^{\mathrm{t}} = \epsilon_0\, \tilde{\epsilon}_1\, \partial_t(\nabla^{\mathrm{t}} \times \hat{u}\boldsymbol{b} + \nabla^{\mathrm{t}} w_2). \tag{29}$$

By introducing (28), (29) into (15), (16) one immediately finds the longitudinal components

$$E = -(\tilde{\epsilon}_1/\tilde{\epsilon})\nabla_t^2\,\hat{u} - (1/\epsilon_0\,\tilde{\epsilon}\partial_t)J\,, \tag{30}$$

$$H = -\nabla_t^2\,\hat{v} - (1/\mu_0\,\partial_t)M\,. \tag{31}$$

The system of coupled integro-differential equations for the scalar Hertz potentials $\hat{u}$ and $\hat{v}$ is derived from (19), (20), (30) and (31) yielding

$$(\tilde{\epsilon}_1/\tilde{\epsilon})\tilde{H}_e\,\hat{u} + \tilde{L}_e\,\hat{v} = -(1/\epsilon_0\,\tilde{\epsilon}\partial_t)J + \tilde{q}_e(\boldsymbol{J}^t, \boldsymbol{M}^t)\,, \tag{32}$$

$$(\tilde{\epsilon}_1/\tilde{\epsilon})\tilde{L}_m\,\hat{u} + \tilde{H}_m\,\hat{v} = -(1/\mu_0\,\partial_t)M + \tilde{q}_m(\boldsymbol{J}^t, \boldsymbol{M}^t)\,. \tag{33}$$

The source terms $\tilde{q}_e$ and $\tilde{q}_m$ are given by

$$\tilde{q}_e(\boldsymbol{J}^t, \boldsymbol{M}^t) = -\partial\bar{u}/\partial x_b - \mu_0\,\partial_t\,\bar{\bar{v}}, \tag{34}$$

$$\tilde{q}_m(\boldsymbol{J}^t, \boldsymbol{M}^t) = \epsilon_0\,\partial_t(-\tilde{\epsilon}_2\,\bar{u} + \tilde{\epsilon}_1\,\bar{\bar{u}}) - \partial\bar{v}/\partial x_b\,. \tag{35}$$

The auxiliary functions $\bar{u}$, $\bar{\bar{u}}$, $\bar{v}$ and $\bar{\bar{v}}$ have been introduced to avoid terms of the form $(1/\nabla_t^2)\nabla^t\,.\,\boldsymbol{J}^t$ on the right-hand sides of (32) and (33), and to prevent the appearance of other than time-integrations in the coupled system of the scalar Hertz potentials. They are defined by

$$\boldsymbol{J}^t = -\epsilon_0\,\tilde{\epsilon}_1\,\partial_t\{\nabla^t\bar{u}(\boldsymbol{x}, t) + \nabla^t \times \bar{\bar{u}}(\boldsymbol{x}, t)\boldsymbol{b}\}\,, \tag{36}$$

$$\boldsymbol{M}^t = -\mu_0\,\partial_t\{\nabla^t\bar{v}(\boldsymbol{x}, t) + \nabla^t \times \bar{\bar{v}}(\boldsymbol{x}, t)\boldsymbol{b}\}\,. \tag{37}$$

With equations (9)–(14), (28)–(31) and (34)–(37) the functions w_1 and w_2 are found to be

$$w_1 = -(1/\mu_0\,\partial_t)\partial\hat{u}/\partial x_b + (\tilde{\epsilon}_2/\tilde{\epsilon}_1)\hat{v} - (1/\mu_0\,\partial_t)\bar{u}, \tag{38}$$

$$w_2 = (1/\epsilon_0\,\tilde{\epsilon}_1\,\partial_t)(\partial\hat{v}/\partial x_b + \bar{v})\,, \tag{39}$$

so that the transverse fields in terms of scalar potentials are

$$\boldsymbol{E}^t = \nabla^t(\partial\hat{u}/\partial x_b) - \mu_0\,\partial_t\{\nabla^t \times \hat{v}\boldsymbol{b} + (\tilde{\epsilon}_2/\tilde{\epsilon}_1)\nabla^t\,\hat{v}\} + \nabla^t\,\bar{u}, \tag{40}$$

$$\boldsymbol{H}^t = \epsilon_0\,\tilde{\epsilon}_1\,\partial_t(\nabla^t \times \hat{u}\boldsymbol{b}) + \nabla^t(\partial\hat{v}/\partial x_b) + \nabla^t\,\bar{v}. \tag{41}$$

Equations (30), (31), (40) and (41) can be recast into the final form

$$\boldsymbol{E} = \tilde{\boldsymbol{\epsilon}}^{-1}.(\nabla \times \boldsymbol{\epsilon}).(\nabla \times \hat{u}\boldsymbol{b}) - (\mu_0\,\partial_t/\tilde{\epsilon}_1)\tilde{\boldsymbol{\epsilon}}^{T}.(\nabla \times \hat{v}\boldsymbol{b})$$
$$-(1/\epsilon_0\,\tilde{\epsilon}\partial_t)J\boldsymbol{b} + \nabla^t\,\bar{u}, \tag{42}$$

$$\boldsymbol{H} = \nabla \times (\nabla \times \hat{v}\boldsymbol{b}) + \epsilon_0\,\tilde{\epsilon}_1\,\partial_t(\nabla \times \hat{u}\boldsymbol{b})$$
$$-(1/\mu_0\,\partial_t)M\boldsymbol{b} + \nabla^t\,\bar{v}, \tag{43}$$

where $\tilde{\boldsymbol{\epsilon}}^{-1}$ and $\tilde{\boldsymbol{\epsilon}}^{T}$ are the inverse and transpose of the *dielectric tensor operator* $\tilde{\boldsymbol{\epsilon}}$.

In the Appendix (equation A7) it is shown that in the cold magnetised plasma under investigation $\tilde{\epsilon}$ is

$$\tilde{\epsilon} = \epsilon_0\{\tilde{\epsilon}_1 \mathbf{I} - \tilde{\epsilon}_2(\boldsymbol{b}\times\mathbf{I}) + (\tilde{\epsilon}-\tilde{\epsilon}_1)\boldsymbol{b}\boldsymbol{b}\}. \tag{44}$$

Finally the velocity field $\boldsymbol{v}$ is calculated from (3). Equation (A3) then leads to

$$\boldsymbol{v} = -\{q/m(\partial_t^2+\omega_c^2)\}\{\partial_t \mathbf{I} + \omega_c(\boldsymbol{b}\times\mathbf{I}) + (\omega_c^2/\partial_t)\boldsymbol{b}\boldsymbol{b}\}.\boldsymbol{E}. \tag{45}$$

Equations (42), (43) and (45) are the final representations of the fields $\boldsymbol{E}$, $\boldsymbol{H}$ and $\boldsymbol{v}$ in terms of scalar Hertz potentials $\hat{u}$ and $\hat{v}$ which are solutions of the set of coupled integro-differential equations (32) and (33). The appearance of the auxiliary functions $\bar{u}$ and $\bar{v}$ in the field representation is due to the existence of source terms transverse to the axis distinguished in the magnetised plasma. Their presence is a general feature of the chosen field representation in terms of the generalised Hertz vector components $\hat{u}$ and $\hat{v}$. A detailed discussion of the properties of these auxiliary functions may be found in Weiglhofer and Papousek (1987), Weiglhofer (1987*a*) and Felsen and Marcuvitz (1973).

4. Isotropic Plasma

As a test for the applicability of these results to definite calculations a longitudinal electric current distribution in an isotropic plasma is considered. (Note that despite the isotropy of the medium one still has to distinguish an axis for the field representation.) When $\omega_c = 0$ it follows that

$$\tilde{\epsilon} = \tilde{\epsilon}_1 = 1 + \omega_p^2/\partial_t^2, \qquad \tilde{\epsilon}_2 = 0, \tag{46}$$

and therefore the system (32) and (33) decouples into

$$\{\nabla^2 - (\partial_t^2+\omega_p^2)/c^2\}\,\hat{u}(x,t) = -(1/\epsilon_0\tilde{\epsilon}\partial_t)J(x,t), \tag{47}$$

$$\{\nabla^2 - (\partial_t^2+\omega_p^2)/c^2\}\,\hat{v}(x,t) = 0. \tag{48}$$

For a free-space radiation problem $\hat{v} \equiv 0$ and only (47) is of interest. Its solution in terms of the scalar Green function $g(x, x'; t, t')$ is

$$\hat{u}(x,t) = \int_{x',t'} g(x,x';t,t')\{(1/\epsilon_0\tilde{\epsilon}'\partial_{t'})J(x',t')\}\,\mathrm{d}^3x'\,\mathrm{d}t', \tag{49}$$

with g from

$$\{\nabla^2 - (\partial_t^2+\omega_p^2)/c^2\}\,g(x,x';t,t') = -\delta(x-x')\,\delta(t-t') \tag{50}$$

(δ is the Dirac delta function). The solution to (50) already found by Felsen and Marcuvitz (1973) is

$$g(x,x';t,t') = \delta(\tau - r/c)/4\pi r$$
$$-(\omega_p/4\pi c)J_1\{\omega_p(\tau^2 - r^2/c^2)^{\frac{1}{2}}\}H(\tau - r/c)/(\tau^2 - r^2/c^2)^{\frac{1}{2}}, \tag{51}$$

with the Heaviside step function $H(x)$, the Bessel function J_1, $\tau = t-t'$, and $r = |x-x'|$.

5. Applications

The scalar Hertz potential method can be applied to the analysis of transient electromagnetic wave phenomena in layered media where the boundaries between the plane layers are normal to the axis $\boldsymbol{b}$. In its simplest form a three layer problem is treated with the top and bottom layer being a vacuum and air respectively. The fields are excited by a steady-state source or an electromagnetic pulse in one of these two layers. The distortion and modification of the electromagnetic waves and pulses are studied after propagation through the middle layer, which is the anisotropic cold electron plasma simulating the Earth's ionosphere. (Detailed results will be presented in a forthcoming paper.) One of the difficulties of this approach is the fact that the number of matching conditions for the fields at the layer boundaries increases rapidly with the number of layers, thus increasing the algebraic complexity considerably. As mentioned above, the method can be generalised to inhomogeneous media with the basic physical parameters, the electron density n_0 and the static magnetic field B_0, being functions of the coordinate x_b. Therefore, the layered structure can be incorporated into the dielectric integral operators $\tilde{\epsilon}$, $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ in an analytic form. Then the radiation conditions at $x_b = \pm\infty$ are the only 'boundary' conditions in the model, thus making it an interesting alternative to the layered medium approach discussed above.

6. Conclusions and Outlook

A system of integro-differential equations for two scalar Hertz potentials has been derived for a cold magnetised electron plasma. It has been shown that the electromagnetic field and the dynamical velocity field can be represented in terms of these two potentials (plus two auxiliary functions due to current distributions transverse to the distinguished axis of the medium).

Thereby one is provided with a definite time-domain approach to the electrodynamics of a cold magneto-plasma. Future work will be devoted to an extension of the method outlined to the analytical treatment of warm magneto-plasmas where spatial-dispersion effects play a crucial role and complicate the analysis.

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Appendix: Dielectric and Conductivity Tensor Operators

The conductivity tensor operator $\tilde{\sigma}$ is defined by

$$j = n_0 \, q v = \tilde{\sigma} . E . \tag{A1}$$

With the help of (3) we get

$$\tilde{\sigma} = -\epsilon_0 \, \omega_p^2 \{ \partial_t \mathbf{I} - \omega_c (b \times \mathbf{I}) \}^{-1} . \tag{A2}$$

The inverse operator in (A2) can easily be calculated (see Chen 1983, example 1.7) giving

$$\tilde{\sigma} = -\epsilon_0 \, \omega_p^2 \{ \partial_t \mathbf{I} + \omega_c (b \times \mathbf{I}) + (\omega_c^2 / \partial_t) b b \} / (\partial_t^2 + \omega_c^2) , \tag{A3}$$

which is the desired result for equation (45).

Finally, the dielectric tensor operator $\tilde{\epsilon}$ establishes a relation between the electric field $E(x, t)$ and the electric flux density $D(x, t)$ via the definition

$$D(x, t) = \tilde{\epsilon} . E(x, t) . \tag{A4}$$

From (1) we derive

$$\partial D / \partial t = \epsilon_0 \, \partial E / \partial t - n_0 \, q v . \tag{A5}$$

With (A4) and (A1) this can be transformed into

$$\tilde{\epsilon} = \epsilon_0 \mathbf{I} - \tilde{\sigma} / \partial_t . \tag{A6}$$

With (A3) and the scalar dielectric operators (25)–(27) we finally find

$$\tilde{\epsilon} = \epsilon_0 \{ \tilde{\epsilon}_1 \mathbf{I} - \tilde{\epsilon}_2 (b \times \mathbf{I}) + (\tilde{\epsilon} - \tilde{\epsilon}_1) b b \} . \tag{A7}$$

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