Aust. J. Phys., 1988, **41**, 629–31

On the Radiation from a Rotating Dipole

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Abstract

We reconsider a thought experiment recently described by Comay (1987) consisting of a rotating dipole emitting electromagnetic radiation. We show that there is no conflict with energy conservation, at least to tenth order in angular velocity.

Comay (1987) has considered the question of energy conservation of a rotating dipole (consisting of charges $\pm Q$ rigidly maintained a distance $d = 2$ apart and rotating with angular velocity ω), by expanding in powers of ω the tangential component of the three forces acting on one of the charges. He obtained the following results [equations (9), (17) and (18)]:

(1) The electromagnetic force due to the other charge:

$$f = -Q^2\{\tfrac{2}{3}\omega^3 - \tfrac{28}{15}\omega^5 + \mathrm{O}(\omega^7)\}. \qquad (1)$$

(2) The Lorentz–Dirac radiation reaction force:

$$F = -Q^2\{\tfrac{2}{3}\omega^3 + \tfrac{4}{3}\omega^5 + \mathrm{O}(\omega^7)\}. \qquad (2)$$

(3) The force corresponding to the loss of energy to radiation:

$$F_{\mathrm{r}} = I/\omega, \qquad (3)$$

where I is the total intensity of radiation from the rotating dipole.

For I, Comay used the Landau and Lifshitz (1983) result (p. 176, with $d_0 = 2$ and $c = 1$):

$$I = \tfrac{8}{3}Q^2\omega^4, \qquad (4)$$

and therefore claimed to have shown a violation of energy conservation, since the sum of the coefficients of the ω^5 terms in (1) and (2) above is $+\frac{8}{15}$ and not zero as expected from (4).

However, the discussion in Landau and Lifshitz (1983, Section 67) makes it clear that (4) is valid in the slow motion approximation only. We therefore wish to calculate

0004-9506/88/050629\$03.00

the next term in the expansion of I in powers of ω. For this we use a modification of the Landau–Lifshitz method (Section 74) used for synchrotron radiation by a single particle.

For generality let us consider N charges Q_j rotating in the same circle of radius r, with the same angular velocity ω and with initial angular positions ϕ_j. Then an obvious generalisation of the Landau–Lifshitz formulae (74.6) and (74.7) gives the following expressions for the nth Fourier coefficients of the x and y components of the vector potential of the radiation field:

$$A_{xn} = -\frac{\mathrm{i}\, v}{cR_0}\mathrm{e}^{\mathrm{i}kR_0}\sum_{j=1}^{N} Q_j \mathrm{e}^{\mathrm{i}n\phi_j} J'_n(n\beta\cos\theta), \tag{5}$$

$$A_{yn} = \frac{1}{\cos\theta\, R_0}\mathrm{e}^{\mathrm{i}kR_0}\sum_{j=1}^{N} Q_j \mathrm{e}^{\mathrm{i}n\phi_j} J_n(n\beta\cos\theta), \tag{6}$$

where R_0 is the distance to the origin, θ the latitude of the radiation vector $\boldsymbol{k}$, $v = \beta c$ the tangential velocity of the charges and J_n a Bessel function. The calculation of the angular distribution of radiation requires the quantity $|\boldsymbol{A}_n \times \boldsymbol{k}|^2$, so that defining

$$C(N, n) = \left|\sum_{j=1}^{N} Q_j \mathrm{e}^{\mathrm{i}n\phi_j}\right|^2, \tag{7}$$

we have

$$|\boldsymbol{A}_n \times \boldsymbol{k}|^2 = C(N, n)(A_{xn}^2 k^2 + A_{yn}^2 k^2 \sin^2\theta). \tag{8}$$

Therefore the intensity of radiation in the nth harmonic into the solid angle $\mathrm{d}\Omega$ is

$$\begin{aligned} \mathrm{d}I_n &= cR_0^2|\boldsymbol{A}_n \times \boldsymbol{k}|^2\, \mathrm{d}\Omega/2\pi \\ &= C(N, n)n^2\omega^2\{\tan^2\theta\, J_n^2(n\beta\cos\theta) + \beta^2 J'_n(n\beta\cos\theta)\}\, \mathrm{d}\Omega/2\pi c. \end{aligned} \tag{9}$$

Integrating over the sphere we obtain the total intensity

$$I_n = \frac{2C(N, n)\omega^2}{\beta c}\left(n\beta^2 J'_{2n}(2n\beta) - n^2(1-\beta^2)\int_0^\beta J_{2n}(2nx)\,\mathrm{d}x\right). \tag{10}$$

Since the integral in (10) cannot be evaluated in finite terms, we make use of the power series expansion of the Bessel function to find the dependence of I_n on β for small β. We have

$$J'_{2n}(2n\beta) = \frac{1}{4^n(2n-1)!}(2n\beta)^{2n-1} - \frac{2n+2}{4^{n+1}(2n+1)!}(2n\beta)^{2n+1} + \mathrm{O}(\beta^{2n+3}), \tag{11}$$

$$\int_0^\beta J_{2n}(2nx)\,\mathrm{d}x = \frac{n^{2n}}{(2n+1)(2n)!}\beta^{2n+1} - \frac{n^{2n+2}}{(2n+3)(2n+1)!}\beta^{2n+3} + \mathrm{O}(\beta^{2n+5}). \tag{12}$$

We now specialise to the case of the dipole, so that $\phi_1 = 0$, $\phi_2 = \pi$, $Q_1 = +Q$ and $Q_2 = -Q$; then

$$\begin{aligned} C(2, n) &= 4Q^2 \qquad (n \text{ odd}) \\ &= 0 \qquad (n \text{ even}). \end{aligned} \tag{13}$$

Inserting these expressions into (10) and choosing $r = c = 1$, so that $\beta = \omega$, we obtain

$$I_1 = Q^2\left(\frac{8}{3}\omega^4 - \frac{16}{15}\omega^6 + \frac{22}{105}\omega^8 - \frac{11}{567}\omega^{10} + O(\omega^{12})\right), \tag{14}$$

$$I_{2n} = 0,$$

$$I_3 = Q^2\left(\frac{486}{35}\omega^8 - \frac{729}{35}\omega^{10} + O(\omega^{12})\right), \tag{15}$$

$$I_{2n+1} = O(\omega^{4n+4}).$$

We see that the radiation is all at the fundamental up to order 8 in ω. To the lowest order it agrees with the result (4), while the total intensity to order 10 is

$$I_1 + I_3 = Q^2\left(\frac{8}{3}\omega^4 - \frac{16}{15}\omega^6 + \frac{296}{21}\omega^8 - \frac{59104}{2835}\omega^{10} + O(\omega^{12})\right). \tag{16}$$

A routine calculation of the next few terms in the expansions (1) and (2) gives

$$f = -Q^2\left(\frac{2}{3}\omega^3 - \frac{28}{15}\omega^5 + \frac{106}{21}\omega^7 - \frac{37112}{2835}\omega^9 + O(\omega^{11})\right), \tag{17}$$

$$F = -Q^2\left(\frac{2}{3}\omega^3 + \frac{4}{3}\omega^5 + 2\omega^7 + \frac{8}{3}\omega^9 + O(\omega^{11})\right). \tag{18}$$

We thus have

$$2(f+F)\omega = -Q^2\left(\frac{8}{3}\omega^4 - \frac{16}{15}\omega^6 + \frac{296}{21}\omega^8 - \frac{59104}{2835}\omega^{10} + O(\omega^{12})\right), \tag{19}$$

which agrees with (16). Therefore to order ω^{10} in intensity there is no disagreement with energy conservation.

We see that Comay's (1987) result is due to the use of (4) beyond its domain of applicability. Since the Lorentz–Dirac equation is derived by requiring the conservation of energy in the particle–field system, there is every reason to expect agreement to all powers of ω if the above expansions are continued.

Note added in proof:

It has come to my attention that Comay has retracted his original paper, pointing out the same error as the present paper [*Phys. Lett.* A **129**, 424 (1988)]. In addition, V. Hnizdo has calculated the sixth order term in the expansion of the total power radiated [*Phys. Lett.* A **129**, 426 (1988)]. The present paper uses essentially the same method as Hnizdo, but calculates a few more terms.

References

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Manuscript received 26 April, accepted 14 June 1988