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Dirac Equation in the Field of a Charged Cosmic String and in the Field of a Point Charge with $Z > 137$

T. J. Allen and L. J. Tassie

Department of Theoretical Physics, Research School of Physical Sciences and Engineering, Australian National University, G.P.O. Box 4, Canberra, A.C.T. 2601, Australia.

Abstract

In both spherical and cylindrical coordinates, the radial Dirac equation can be written in the form of a Schrödinger equation with an effective potential. It is shown that the difficulties at $r \to 0$ for the Dirac equation in the field of a point charge for $Z > 137$ are the same as those for the Schrödinger equation with a $1/r^2$ potential. The effective potential is used to show that similar difficulties do not arise for the field of a line charge, so allowing the consideration of the motion of electrons in the field of a charged superconducting cosmic string without considering the internal structure of the string.

1. Introduction

Interesting effects occur for relativistic wave equations with strong external fields, and have been discussed extensively in Greiner *et al.* (1985), which contains references to the earlier literature. Most treatments are one-dimensional or with spherical symmetry. A possible source of strong fields is a charged superconducting cosmic string (Allen 1990), so that one needs to treat the case of cylindrical symmetry corresponding to a straight infinite piece of charged cosmic string. The Klein–Gordon equation for this case has already been treated (Allen and Tassie 1991) and it has been shown that the vacuum is unstable for a sufficiently large charge. In the present paper we look at the Dirac equation for an electron in the electric field of a straight line charge. We first revise the case of an electron in the field of a point charge and use the method of the effective potential to show that the difficulties at short distances arising in this case for the Dirac equation for $Z > 137$ are essentially the same as those for the Schrödinger equation with a $1/r^2$ potential. We then show that no difficulties arise at short distances for the field of a line charge.

2. Electron in the Field of a Point Charge

The eigenvalues are given by the well-known result (e.g. Dirac 1958)

$$W/mc^2 = (\gamma + n)\{\zeta^2 + (\gamma + n)^2\}^{-1/2}, \tag{1}$$

where

$$\gamma = \{(j + \tfrac{1}{2})^2 - \zeta^2\}^{1/2}, \tag{2}$$

$$\zeta = Ze^2/\hbar c, \tag{3}$$

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n is an integer and $j\hbar$ is the total angular momentum of the electron. At small r the radial wavefunctions are $\sim r^{\gamma-1}$.

For $j=\frac{1}{2}$, we have $\gamma=\{1-\zeta^2\}^{1/2}$. For $\zeta>1$, γ is imaginary giving unacceptable wavefunctions. These difficulties can be overcome by replacing the point charge by a finite charge distribution. However, further insight into the problem can be obtained by following the procedure (Mott and Massey 1965, p. 228) of bringing the radial wave equation into the Schrödinger form with an effective potential,

$$\frac{d^2G_\ell}{dr^2}+\left(\frac{1}{\hbar^2c^2}(W^2-m^2c^2)-\frac{\ell(\ell+1)}{r^2}-U_\ell(r)\right)G_\ell=0\,, \tag{4}$$

where $j=\ell+\frac{1}{2}$. For $j=\ell-\frac{1}{2}$, ℓ is replaced by $-\ell-1$. The effective potential $U_\ell(r)$ is given by

$$U_\ell(r)=\frac{2W}{\hbar^2c^2}V-\frac{V^2}{\hbar^2c^2}-\frac{\ell+1}{r}\frac{\alpha'}{\alpha}+\frac{3}{4}\left(\frac{\alpha'}{\alpha}\right)^2-\frac{1}{2}\frac{\alpha'}{\alpha}\,, \tag{5}$$

where

$$\alpha=\frac{1}{\hbar c}(W-V+mc^2)\,. \tag{6}$$

For $V=-Ze^2/r$, we have

$$U_\ell(r)=\frac{2W\zeta}{r}-\frac{\zeta^2}{r^2}-\frac{\ell\zeta}{\alpha r^3}+\frac{3}{4\alpha^2}\frac{\zeta^2}{r^4}\,. \tag{7}$$

Then, as $r\rightarrow 0$, we get

$$U_\ell(r)\rightarrow-\frac{\zeta^2}{r^2}+\frac{\ell}{r^2}+\frac{3}{4}\frac{1}{r^2}\,. \tag{8}$$

For $\ell=0$ and $\zeta^2>\frac{3}{4}$, the wave equation (4) is of the form of a Schrödinger equation with an attractive $1/r^2$ potential, a problem which has been discussed by Mott and Massey (1965, p. 41). Writing

$$U_\ell(r)=\beta/r^2 \tag{9}$$

gives

$$\beta=-\zeta^2+\ell+\tfrac{3}{4}\,. \tag{10}$$

Mott and Massey showed that there are wavefunctions well-behaved at the origin for

$$\beta>-\tfrac{1}{4}$$

but for $\ell=0$ and $\beta<-\frac{1}{4}$ the wavefunction near the origin is proportional to $\exp(\pm ia\log r)$, which is inadmissible because there is no criterion as to which solution to take. Case (1950) has shown how to overcome the mathematical difficulty for the Schrödinger case for a $1/r^2$ potential and the Dirac case for a $1/r$ potential. As we have seen, these two cases are the same. The physical difficulty must be overcome by modifying the potential at short distances.

3. Wave Equation for Spin-$\frac{1}{2}$ Particles in the Field of a Line Charge

We use the same representation of the Dirac matrices as Greiner *et al.* (1985). Because the system possesses cylindrical symmetry, the z-component of angular momentum, m_j, is a good quantum number. The eigenvalue equation for m_j thus holds:

$$\left(\frac{1}{i}\frac{\partial}{\partial\phi}+\tfrac{1}{2}\sigma_z\right)\Psi = m_j\Psi. \tag{11}$$

With

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{12}$$

equation (11) yields

$$\begin{aligned} \left(\frac{1}{i}\frac{\partial}{\partial\phi}+\tfrac{1}{2}\right)\phi_{1,3} &= m_j\psi_{1,3}, \\ \left(\frac{1}{i}\frac{\partial}{\partial\psi}-\tfrac{1}{2}\right)\psi_{2,4} &= m_j\psi_{2,4}, \end{aligned} \tag{13}$$

and therefore

$$\Psi = \begin{pmatrix} R_1(r)e^{i(m_j-\frac{1}{2})\phi} \\ R_2(r)e^{i(m_j+\frac{1}{2})\phi} \\ R_3(r)e^{i(m_j-\frac{1}{2})\phi} \\ R_4(r)e^{i(m_j+\frac{1}{2})\phi} \end{pmatrix} e^{ikz}. \tag{14}$$

Setting $\hbar = c = 1$, substitution of (14) into the Dirac equation yields

$$\begin{aligned} (W-V-m)R_1 &= kR_3 - i\left(\frac{d}{dr}+\frac{m_j+\frac{1}{2}}{r}\right)R_4, \\ (W-V-m)R_2 &= -i\left(\frac{d}{dr}-\frac{m_j-\frac{1}{2}}{r}\right)R_3 - kR_4, \\ (W-V+m)R_3 &= kR_1 - i\left(\frac{d}{dr}+\frac{m_j+\frac{1}{2}}{r}\right)R_2, \\ (W-V+m)R_4 &= -i\left(\frac{d}{dr}-\frac{m_j-\frac{1}{2}}{r}\right)R_1 - kR_2. \end{aligned} \tag{15}$$

If $k=0$, the four equations decouple into two sets of two equations,

$$\begin{aligned} (W-V-m)R_1 &= -i\left(\frac{d}{dr}+\frac{m_j+\frac{1}{2}}{r}\right)R_4, \\ (W-V+m)R_4 &= -i\left(\frac{d}{dr}-\frac{m_j-\frac{1}{2}}{r}\right)R_1, \end{aligned} \tag{16}$$

and

$$\begin{aligned}(W-V-m)R_2 &= -i\left(\frac{d}{dr}-\frac{m_j-\frac{1}{2}}{r}\right)R_3\,,\\ (W-V+m)R_3 &= -i\left(\frac{d}{dr}+\frac{m_j+\frac{1}{2}}{r}\right)R_2\,.\end{aligned} \tag{17}$$

Replacing m_j by $-m_j$ transforms the second pair of equations into the first. We note that the solution of the equations obtained from (16) by replacing m_j by $-m_j$ will in general not be degenerate with the original solution, but is degenerate with a solution of (17). So we need only consider (16).

Equations (15) have solutions with

$$\begin{aligned}R_3 &= k[(m^2+k^2)^{\frac{1}{2}}+m]R_1\,,\\ R_2 &= k[(m^2+k^2)^{\frac{1}{2}}+m]R_4\,,\end{aligned} \tag{18}$$

where R_1 and R_4 are solutions of

$$\begin{aligned}(W-V-(m^2+k^2)^{\frac{1}{2}})R_1 &= -i\left(\frac{d}{dr}+\frac{m_j+\frac{1}{2}}{r}\right)R_4\,,\\ (W-V+(m^2+k^2)^{\frac{1}{2}})R_4 &= -i\left(\frac{d}{dr}-\frac{m_j+\frac{1}{2}}{r}\right)R_1\,,\end{aligned} \tag{19}$$

which are of the form of equations (16) for $k=0$ but with m replaced by an effective mass $(m^2+k^2)^{1/2}$. So with no loss of generality we can confine the treatment to $k=0$.

To obtain further insight, we transform equations (16) to the Schrödinger form with an effective potential, following the same procedure that Mott and Massey (1965) used for the spherically symmetric case. Defining

$$\alpha = W-V+m\,, \qquad \beta = W-V-m$$

and eliminating R_4 from equations (16) gives

$$\left[\frac{d^2}{dr^2}+\alpha\beta+\frac{1}{r}\frac{d}{dr}-\frac{\alpha'}{\alpha}\frac{d}{dr}-\frac{m_j^2-m_j+\frac{1}{4}}{r}+\frac{\alpha'}{\alpha}\frac{m_j-\frac{1}{2}}{r}\right]R_1=0\,. \tag{20}$$

The first derivative terms may be eliminated by the substitution

$$R_1=\alpha^{\frac{1}{2}}r^{-\frac{1}{2}}G(r) \tag{21}$$

giving an equation

$$\left[\frac{d^2}{dr^2}+\alpha\beta-\frac{m_j(m_j-1)}{r^2}+\frac{m_j}{r}\frac{\alpha'}{\alpha}-\frac{3}{4}\left(\frac{\alpha'}{\alpha}\right)^2+\frac{1}{2}\frac{\alpha''}{\alpha}\right]G=0\,, \tag{22}$$

which may be written as

$$\left[\frac{d^2}{dr^2}+W^2-m^2-\frac{m_j(m_j+1)}{r^2}-U_{m_j}(r)\right]G=0 \tag{23}$$

with an effective potential

$$U_{m_j}(r) = 2WV - V^2 - \frac{m_j}{r}\frac{\alpha'}{\alpha} + \tfrac{3}{4}\left(\frac{\alpha'}{\alpha}\right)^2 - \tfrac{1}{2}\frac{\alpha''}{\alpha}. \tag{24}$$

For a line charge the potential is

$$V = \chi \log r/r_0, \tag{25}$$

where

$$\chi = \frac{e\rho}{2\pi} \tag{26}$$

and where ρ is the charge per unit length. As in the Klein–Gordon case (Allen and Tassie 1991) it can be seen from (24) for the effective potential that a cut-off must be applied at large distances to enable bound states to form. The behaviour at large distances has been dealt with in more detail elsewhere (Allen 1991, 1992).

We examine here the short distance behaviour of the effective potential, as short-distance singularities can be troublesome for the Dirac equation:

$$\alpha = W + m - \chi \log(r/r_0),$$
$$\alpha' = -\chi/r, \qquad \alpha'' = \chi/r^2.$$

For small r, we get

$$\alpha \approx -\chi \log(r/r_0)$$

and

$$U_{m_j} \approx 2WV - V^2 - \frac{m_j - \frac{1}{2}}{r^2 \log r} + \tfrac{3}{4}\frac{1}{r^2(\log r)^2}.$$

The second term, $-V^2$, is attractive at small r but is less singular than the α' and α'' terms. It was already present in the Klein–Gordon case and presented no problems (Allen and Tassie 1991). The last term presents no problems because it is repulsive, despite being singular at $r = 0$. The third term is attractive for $m_j \geq \frac{3}{2}$ but its effect would be swamped by the repulsive centrifugal term, $m_j(m_j+1)/r^2$. Unlike the point charge, no difficulty occurs near $r = 0$ as the strength of the field increases. For $m_j = \frac{1}{2}$, the wavefunction at small r is given by

$$R_1 = r^{-3/2} \sum_{i,j=0}^{3} a_{ij} r^j \log^i r/r_0,$$

$$R_2 = r^{-3/2} \sum_{i,j=0}^{3} b_{ij} r^j \log^i r/r_0,$$

where a_{00} is arbitrary and the other nonzero a_{ij} and b_{ij} are

$$a_{02} = (4 - 4W^2 - 2\chi - 6W\chi - 3\chi^2)a_{00}/16,$$
$$a_{12} = \chi(4W + 3\chi)a_{00}/4, \qquad a_{22} = -\chi^2 a_{00}/4,$$
$$b_{02} = (2 - 2W - \chi)a_{00}/4, \qquad b_{12} = \chi a_{00}/2.$$

4. Discussion

To deal with the physics of an electron in the field of a nucleus with charge $z > 137$ it is not sufficient to treat the nucleus as a point charge, but it is necessary to consider the structure of the nucleus. Fortunately, the charged cosmic string can be considered as a line charge and it is not necessary to consider the very small thickness of the cosmic string. The results of numerical integration of the Dirac equation in the field of a charged string will be given elsewhere (Allen 1991, 1992).

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