Aust. J. Phys., 1992, **45**, 1–20

Particle Motion in Longitudinal Waves. I Subluminal Waves

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Sydney, N.S.W. 2006, Australia.

Abstract

The classical equations of motion for a particle moving in a parallel longitudinal wave of arbitrary phase speed are discussed and the case of subluminal waves is considered in detail. Motion of both trapped and untrapped particles is explored with particular reference to the ability of a wave to accelerate particles to relativistic energy. The particle orbit is found in both closed and expanded forms, taking the electric field into account exactly. Expressions are also found for the 'drift velocity' of a particle, which is an important quantity because it is a constant of the particle motion that describes the motion of the centre of oscillation.

1. Introduction

This is the first of two papers in which the motion of a charged particle in the field of an electrostatic wave of arbitrary phase speed and amplitude is explored in detail. Waves with phase speed less than the speed of light in vacuo are called subluminal, waves with phase speed greater than the speed of light are called superluminal and waves with phase speed equal to the speed of light are called luminal waves. In this paper we treat motion in subluminal waves. Motion in luminal and superluminal waves is treated in Part II (see the following paper).

There are two motivations for the calculations presented in this paper and in Part II. The first is to extend the understanding of particle motion in electromagnetic fields to the case of longitudinal waves. Gunn and Ostriker (1971) presented calculations for the transverse wave case and Melrose (1978) considered briefly the emission by particles in an arbitrary longitudinal electric field to first order in a perturbation expansion in the field. Krishan and Sivaram (1983) attempted to expand the work of Melrose to oblique wave–particle trajectories. Neither the work of Melrose nor Krishan and Sivaram can be used to consider strong electric fields since by assumption the field is small enough to be treated as a perturbation. A solution for the equation of motion for a relativistic particle moving in a longitudinal wave is also vital for solving the self-consistent field problem, that is finding a distribution of particles which can create the longitudinal wave in which they move. Finding such a solution to Vlasov's equation and simultaneously Maxwell's equations allows the consideration of the emission by particles in a strong coherent plasma wave.

The second motivation is possible application of a linear acceleration mechanism in astrophysics. One application is in radio emission from pulsars where the

0004-9506/92/010001$05.00

superstrong magnetic field makes gyromagnetic motions unimportant due to the short lifetime of the first excited state. The idea is that a strong coherent plasma wave causes particles to emit radiation due to linear acceleration in the wave. Having an exact solution to the particle motion allows very strong electric fields to be treated and thus stronger emission. In the literature a favoured emission mechanism for radio pulsars is curvature emission by particles moving along curved magnetic field lines. However, it is difficult to obtain the necessary coherence or brightness of the radiation since maser amplification is not possible (Blandford 1975). In the linear acceleration mechanism, maser action is possible (Melrose 1978). Another application is in the treatment of electrostatic double layers (e.g. Kuijpers 1990) which can be modelled as a few wavelengths of a coherent, self-consistent plasma wave. The theory developed here is valid for one dimensional particle motion with or without a magnetic field. These applications of the theory will be discussed elsewhere. Here we concentrate on giving solutions for the particle motion, treating the electric field exactly and including relativistic effects. We deal only with the case of particle motion along the direction of wave propagation; but the results can be generalised to allow a perpendicular motion.

The format of this paper is as follows. In Section 2 we write down the equation of motion for a particle in the field of an electrostatic wave moving parallel to the particle motion and give first integrals for the subluminal, superluminal and luminal wave cases. These first integrals are simple combinations of the total particle energy and the particle momentum. In Section 3 we solve for the particle velocity as a function of the phase of the wave in the subluminal case and determine the conditions for particle trapping. Section 4 deals with the exact solutions of the particle orbit. In the subluminal case there are two solutions, one for untrapped particles and the other for trapped particles. From these exact solutions one may identify the 'drift velocity' of an untrapped particle and what we call the 'bounce speed' of trapped particles. These quantities demonstrate the effect of the electric field on the constant part of the particle velocity (the velocity of the 'centre of oscillation'). In order to proceed further, in calculating the Fourier transform of the particle current for example, an expansion of the orbit is required. This is developed in Section 5. In the untrapped case we expand first in harmonics of the wave, obtaining integrals for the coefficients introduced, including an integral form for the drift velocity. The first of these coefficients is also given analytically. The asymptotic behaviour of these coefficients is determined in the trapping limit and an expansion which is useful in the weak field case is given. In the trapped particle case we expand in harmonics of the bounce motion of the particle and write down an integral form of the average 'bounce speed' of the particle. The behaviour of the coefficients introduced is determined for a particle which is almost untrapped and for a particle at rest in the trough of the wave. We use units with $\hbar = c = 1$ throughout this paper.

2. Equations of Motion

We start from the equation of motion for a particle under the influence of the Lorentz force in an arbitrary reference frame

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})\,, \tag{1}$$

where $\mathbf{v}$, $\mathbf{p}$ and q are the particle velocity, momentum and charge and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. In this paper we consider only the case of $\mathbf{E}$, $\mathbf{v}$ and $\mathbf{B}$ all parallel whence the equation becomes one dimensional

$$\frac{dv}{dt} = \frac{qE}{m\gamma^3}. \tag{2}$$

Although the magnetic field and gyromagnetic effects are now irrelevant (2) is a useful model for fermions in a strong magnetic field since their perpendicular energy is quickly radiated away and their ground state is independent of B [by contrast the ground state for bosons depends on B (see e.g. Witte *et al.* 1988; Rowe 1991) and so this model cannot be used]. The equation in this form has been discussed before (e.g. Melrose 1978); the appearance of the electric field linearly on the right hand side being used to recursively generate terms in an expansion for the particle orbit. This method fails for strong electric fields ($|q\phi| > m\gamma$, where ϕ is the electric potential). It is possible to generalise the above discussion to particle motion in a non-parallel electric field (excluding a magnetic field), and to motion *following* an arbitrarily curved field line with the electric field parallel to the magnetic field. We do not give the details here; suffice it to say that the *calculations* are similar to those presented below (this is not to say that the results contain no additional information).

In solving (2) we choose an electric field of the form

$$\mathbf{E} = E_0 \sin\{K(z - z_0) - \Omega(t - t_0)\}\hat{\mathbf{z}}, \tag{3}$$

which represents a longitudinal wave of amplitude E_0 and phase speed $v_\phi = \Omega/K$ moving in the positive z direction. On substituting (3) into (2) one may rewrite the resulting equation as a system of three equations which can then be solved, as follows. Defining

$$\psi = \Omega(t - t_0) - K(z - z_0), \tag{4}$$

one has

$$\frac{d\psi}{dt} = \Omega - Kv, \tag{5}$$

$$\frac{d\psi}{dz} = \frac{\Omega}{v} - K, \tag{6}$$

$$\frac{dv}{d\psi} = -\frac{q}{m\gamma^3}\frac{E_0 \sin\psi}{\Omega - Kv}. \tag{7}$$

Equation (5) is non-trivial for $K \neq 0$ as is (6) for $\Omega \neq 0$ and (7) corresponds to (2). There are only two independent equations and so there are two independent constants of the motion (one is a momentum or energy constant and the other is a phase like constant giving the initial particle position). One of these constants is obtained directly from (7)

$$\gamma(1 - v_\phi v) + \frac{qE_0}{mK}\cos\psi = C_1, \tag{8}$$

or

$$\gamma\left(v - \frac{1}{v_\phi}\right) - \frac{qE_0}{m\Omega}\cos\psi = C_2. \tag{9}$$

The form (8) is suitable for $K \neq 0$ and has the dimensions of energy, while the second form is suitable for $\Omega \neq 0$ and has dimensions of momentum. The significance of the constants C_1 and C_2 is determined as follows.

The Constants C_1 and C_2

In the case $v_\phi < 1$ one can consider the rest frame of the electrostatic wave ($v_\phi = 0$) in which the description of the physical system is simplest. From (8) one has

$$\gamma + \frac{qE_0}{mK_0} \cos \psi = h_0 \,, \tag{10}$$

where h_0 is a constant and the subscript 0 is used to stress that a quantity is as defined in the rest frame (with the exception of E_0 which is invariant in all parallel reference frames). The constant h_0 is then the total energy or Hamiltonian per unit mass of the particle in the rest frame ($h = \gamma - q\phi/m$). It must be a constant by virtue of the fact that there is no explicit time dependence in this frame.

Performing a Lorentz transform (see Appendix 1) on the quantities on the left hand side to the frame where the wave phase speed is v_ϕ, one regains (8) with

$$C_1 = \frac{h_0}{\gamma_\phi} \,, \tag{11}$$

where

$$\gamma_\phi = \frac{1}{(1 - v_\phi^2)^{\frac{1}{2}}} \,. \tag{12}$$

Thus the constant of the motion (8) is the total particle energy per unit mass in the wave rest frame written in terms of the new frame quantities.

In the case $v_\phi > 1$ one can consider the frame in which the electrostatic wave appears to be a uniform time varying field. In this frame v_ϕ is infinite and the physical system has its simplest description. From (9) one has

$$v\gamma - \frac{qE_0}{m\Omega_\infty} \cos \psi = b_\infty \,, \tag{13}$$

where the subscript ∞ is used to stress that a quantity is in the frame in which v_ϕ is infinite. The constant b_∞ represents the canonical momentum per unit mass of the particle ($p_c = v\gamma + qA/m$). This quantity is a constant in this frame by virtue of the fact that the system is independent of the space coordinate.

Performing a Lorentz transform (see Appendix 1 of Part II) to a frame where v_ϕ is finite gives

$$C_2 = \frac{b_\infty}{\gamma_\phi^*} \,, \tag{14}$$

where

$$\gamma_\phi^* = \frac{1}{(1 - 1/v_\phi^2)^{\frac{1}{2}}} \,. \tag{15}$$

The asterisk is used to distinguish between (12) and (15) which are in fact the γ-factors relating a wave rest frame to an observer's frame in the subluminal case (the relative velocity between frames is $-v_\phi$) and an infinite phase speed frame to an observer's frame in the superluminal case (the relative velocity between frames is $-1/v_\phi$). While γ_ϕ is related in a natural way to the wave phase speed, γ_ϕ^* is not. In this case the constant of the motion (9) is the canonical momentum per unit mass in the infinite phase speed frame written in terms of quantities in the arbitrary frame.

In the case $v_\phi = 1$ one has from (8) or (9)

$$\gamma(1-v) + \frac{qE_0}{m\Omega}\cos\psi = a\,, \tag{16}$$

where a is a constant and $\Omega = K$. There is no specific reference frame in this case. No further mention of luminal or superluminal waves is made until Part II.

3. Particle Motion

Solving (8) with (11) for the velocity of the particle, one has

$$v = \frac{v_\phi\gamma_\phi^2 \pm (h_0 - r_0\cos\psi)\{(h_0 - r_0\cos\psi)^2 - 1\}^{\frac{1}{2}}}{v_\phi^2\gamma_\phi^2 + (h_0 - r_0\cos\psi)^2}\,, \tag{17}$$

where

$$r_0 = \frac{qE_0}{mK_0}\,, \tag{18}$$

is a dimensionless constant. The solution with the upper sign represents a particle moving to the right in the wave rest frame and the solution with the lower sign a particle moving to the left. The phase ψ can only take values for which the quantity under the square root is non-negative and this is assured if

$$r_0\cos\psi \le h_0 - 1\,, \tag{19}$$

which is the condition that the particle momentum in the wave rest frame is always physical ($\gamma \ge 1$ in that frame). Suppose, without loss of generality, that $r_0 \ge 0$. Then we have

$$2N\pi + \cos^{-1}\left(\frac{h_0 - 1}{r_0}\right) \le \psi \le 2(N+1)\pi - \cos^{-1}\left(\frac{h_0 - 1}{r_0}\right), \tag{20}$$

which admits three possibilities. Untrapped particle motion occurs for $h_0 > 1+r_0$, trapped motion for $1-r_0 \le h_0 \le 1+r_0$ and there is no solution if $h_0 < 1-r_0$. In Fig. 1 plots of the particle velocity as a function of phase are given illustrating how particles can be trapped in the wave and how particles tend to be dragged along by a wave with nonzero phase speed.

The parameter r_0 determines the strength of the electric field; its value is half the change in the kinetic energy per unit mass during one period of the particle motion for untrapped particles in the wave rest frame. The corresponding change in kinetic energy for trapped particles is smaller depending on the degree of trapping. The range of values possible for r_0, in the case of pulsars, is considered in Section 6.

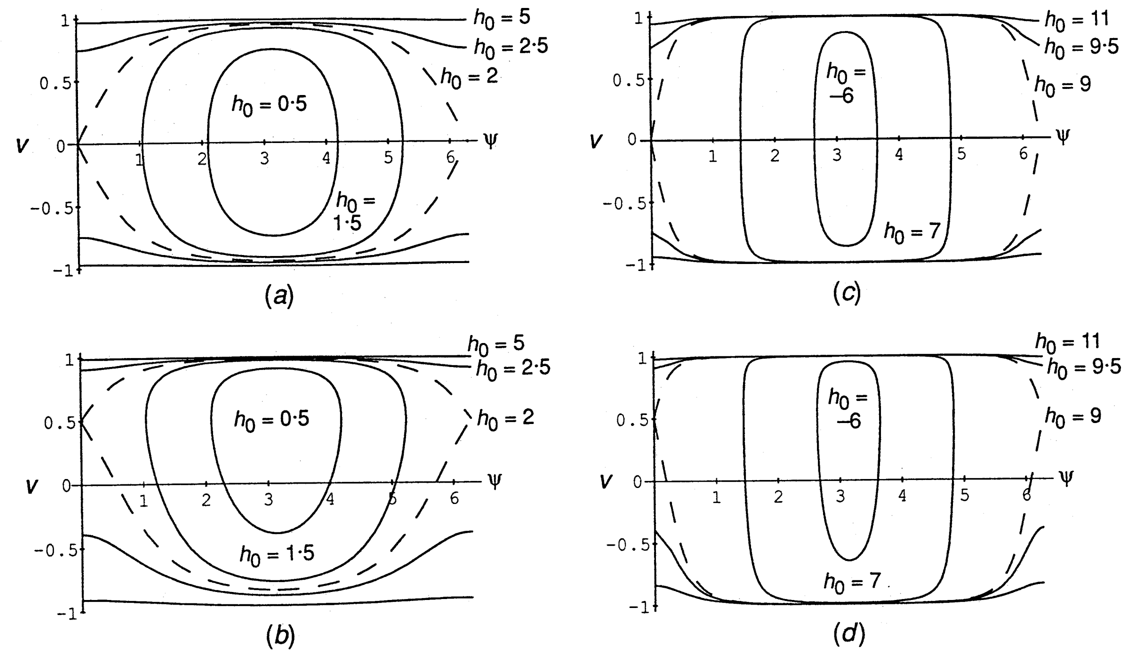

Fig. 1. Phase diagrams for particle motion in a subluminal longitudinal wave. In (*a*) and (*c*) $v_\phi = 0$ and in (*b*) and (*d*) $v_\phi = 0{\cdot}5$, while the parameter $r_0 = 1$ in (*a*) and (*b*) and $r_0 = 8$ in (*c*) and (*d*). Plotted is the particle velocity as a function of wave phase in radians for various values of the total particle energy per unit mass in the wave rest frame, h_0. Dashed lines represent the separatrix with trapped particle trajectories inside and untrapped particle trajectories outside.

A useful quantity to consider is the acceleration potential of the wave. One way to characterise this is by calculating the ratio of the maximum particle kinetic energy attained during the motion to the minimum particle kinetic energy, $R = \gamma_{\max}/\gamma_{\min}$ (this quantity contains more information about the particle acceleration than r_0 which is only the difference between two energies). We call R the energy gain of the particle. The particle undergoes this acceleration at least once each wave period. In the wave rest frame the values of the kinetic energy per unit mass at the ends of the orbit are $\gamma_a = h_0 + r_0$ and $\gamma_b = h_0 - r_0$ respectively. Note that γ_b may not be the minimum kinetic energy as the particle may change direction during the motion. After Lorentz transforming to a general frame one has the corresponding quantities

$$g_{a\pm} = \gamma_\phi\{\gamma_a \pm v_\phi(\gamma_a^2 - 1)^{\frac{1}{2}}\}\,, \tag{21}$$

$$g_{b\pm} = \gamma_\phi\{\gamma_b \pm v_\phi(\gamma_b^2 - 1)^{\frac{1}{2}}\}\,, \tag{22}$$

with the upper sign for particles moving positively in the rest frame and the lower sign for particles moving negatively. The energy gain in a general frame is given in Table 1 for untrapped particles and in Table 2 for trapped particles. It involves $g_{a\pm}$ and $g_{b\pm}$ in various combinations depending upon the value of the Lorentz boost—as this determines precisely how the orbit appears in the general frame.

The energy gain is shown in Fig. 2 for three values of r_0 in the wave rest frame and for $v_\phi^2 = \frac{3}{4}$. The qualitative features are as follows. The three main regions of interest in order of the magnitude of the acceleration possible in the

Table 1. Energy gain for untrapped particle motion in a subluminal longitudinal wave
The first and second columns give the values of r_0 and h_0 for which the particular result is valid. The third column describes the type of particle motion involved

Range of r_0	Range of h_0	Comments	$\gamma_{\max}/\gamma_{\min}$
All values	$h_0 > 1{+}r_0$	$v > v_\phi$	g_{a+}/g_{b+}
	$h_0 \geq \gamma_\phi{+}r_0$	$v < 0$	g_{a-}/g_{b-}
$r_0 \leq \dfrac{\gamma_\phi - 1}{2}$	$1{+}r_0 \leq h_0 \leq \gamma_\phi{-}r_0$	$0 < v < v_\phi$	g_{b-}/g_{a-}
	$\gamma_\phi{-}r_0 \leq h_0 \leq (\gamma_\phi^2{+}r_0^2/v_\phi^2)^{\frac{1}{2}}$	$v > 0$ mostly	g_{b-}
	$(\gamma_\phi^2{+}r_0^2/v_\phi^2)^{\frac{1}{2}} \leq h_0 \leq \gamma_\phi{+}r_0$	$v < 0$ mostly	g_{a-}
$\dfrac{\gamma_\phi - 1}{2} \leq r_0 \leq v_\phi^2\gamma_\phi^2$	$1{+}r_0 \leq h_0 \leq (\gamma^2{}_\phi{+}r_0^2/v_\phi^2)^{\frac{1}{2}}$	$v > 0$ mostly	g_{b-}
	$(\gamma_\phi^2{+}r_0^2/v_\phi^2)^{\frac{1}{2}} \leq h_0 \leq \gamma_\phi{+}r_0$	$v < 0$ mostly	g_{a-}
$r_0 \geq v_\phi^2\gamma_\phi^2$	$1{+}r_0 \leq h_0 \leq \gamma_\phi{+}r_0$	$v < 0$ mostly	g_{a-}

Table 2. Energy gain for trapped particle motion in a subluminal longitudinal wave

Range of r_0	Range of h_0	Comments	$\gamma_{\max}/\gamma_{\min}$
$r_0 \leq \dfrac{\gamma_\phi - 1}{2}$	$1{-}r_0 < h_0 < 1{+}r_0$	$v > 0$	g_{a+}/g_{a-}
$r_0 > \dfrac{\gamma_\phi - 1}{2}$	$1{-}r_0 < h_0 \leq \gamma_\phi{-}r_0$	$v > 0$	g_{a+}/g_{a-}
	$\gamma_\phi{-}r_0 \leq h_0 < 1{+}r_0$	$v > 0$ mostly	g_{a+}

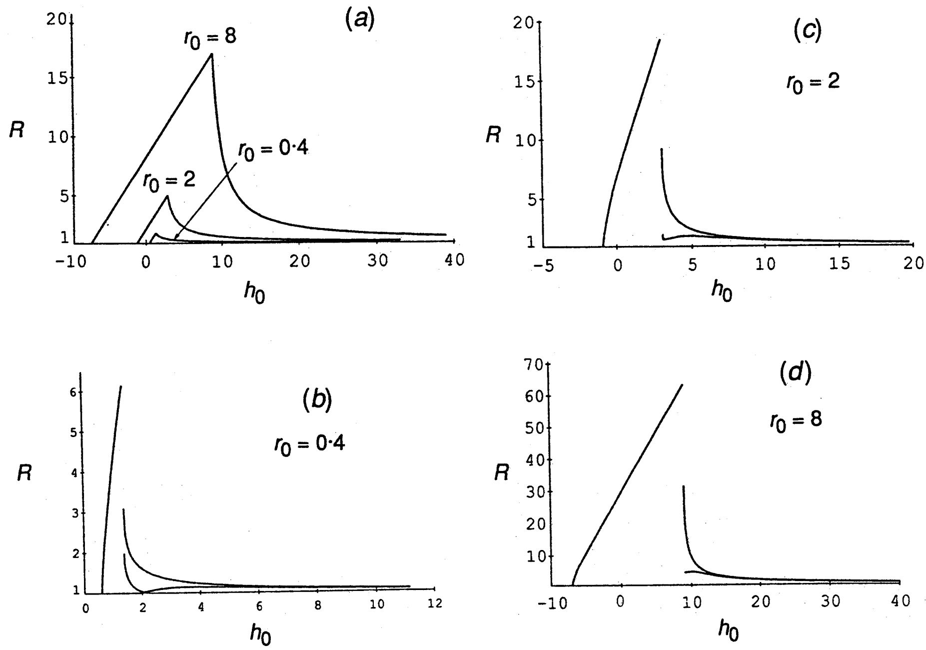

Fig. 2. Energy gain R for particles in a subluminal longitudinal wave as a function of h_0. In (a) $v_\phi = 0$ and curves are shown for three values of r_0. On the left of the peaks particles are trapped and on the right they are untrapped. In (b), (c) and (d) $v_\phi^2 = \frac{3}{4}$ and there are three curves for each value of r_0. In order of magnitude these curves are for trapped particles, untrapped particles with $v > v_\phi$ and untrapped particles with $v < v_\phi$. In all cases the h_0 axis is at $R = 1$.

wave are: trapped particles, untrapped particles with velocities greater than v_ϕ and untrapped particles with velocities less than v_ϕ. The acceleration of trapped particles increases steadily as total particle energy per unit mass increases from the minimum value allowed ($h_0 = 1 - r_0$) to the trapping limit ($h_0 = 1 + r_0$). The maximum energy gain attained at $h_0 = 1 + r_0$ is

$$R_1 = \begin{cases} \dfrac{1 + 2r_0 + 2v_\phi\{r_0(1+r_0)\}^{\frac{1}{2}}}{1 + 2r_0 - 2v_\phi\{r_0(1+r_0)\}^{\frac{1}{2}}} & r_0 \leq \frac{1}{2}(\gamma_\phi - 1) \\ \gamma_\phi[1 + 2r_0 + 2v_\phi\{r_0(1+r_0)\}^{\frac{1}{2}}] & r_0 > \frac{1}{2}(\gamma_\phi - 1)\,, \end{cases} \tag{23}$$

which increases with both r_0 and v_ϕ. For untrapped particles with velocities exceeding v_ϕ, the maximum energy gain occurs at $h_0 \approx 1 + r_0$,

$$R_2 = 1 + 2r_0 + 2v_\phi\{r_0(1+r_0)\}^{\frac{1}{2}}\,, \tag{24}$$

which also increases with r_0 and v_ϕ. If $r_0 > (\gamma_\phi - 1)/2$ the ratio of the maximum trapped energy gain to the maximum untrapped energy gain is γ_ϕ which is dependent only on the wave phase speed (in the wave rest frame these maximum energy gains are equal). An important point is that the slope of R for particles with velocity greater than v_ϕ is negative, becoming arbitrarily steep at $h_0 = 1 + r_0$, so that R drops rapidly away as h_0 increases from its minimum value and thus to obtain an energy gain which is close to the maximum allowed for such particles, they must be very close to being trapped.

The situation is similar for particles with velocities less than v_ϕ except that the maximum value of R depends more critically on r_0 and it is generally much smaller than R_1 or R_2. The details are thus unimportant; qualitatively R drops away as h_0 increases (as in the case of particle velocities exceeding v_ϕ) and then slightly increases as particle velocities become negative only to decrease again towards a value of unity.

4. Exact Solutions For Particle Orbits

(a) Untrapped Particles

We start by calculating the orbit in the wave rest frame. It is then straightforward to perform a Lorentz transformation to an arbitrary frame. In the wave rest frame the equation for the orbit (5) can be written

$$\frac{dt}{dY} = \mp \pm \frac{1}{K_0} \frac{Y}{(Y^2-1)^{\frac{1}{2}}} \frac{1}{\{r_0 - (h_0 - Y)^2\}^{\frac{1}{2}}}, \tag{25}$$

with $Y = h_0 - r_0 \cos\psi$. The first choice of sign in (25) comes from the particle velocity (17) with the upper sign for positive velocities and the lower sign for negative velocities, and the second from the jacobian between Y and ψ with the upper sign for $\sin\psi$ positive and the lower sign for $\sin\psi$ negative. The choice of sign introduced by the jacobian implies that the exact solution for a complete period of the wave must be written in a piecewise manner.

In integrating we define the following parameters:

$$\xi_\pm = b_0^2 \pm r_0^2, \tag{26}$$

$$\Delta_0 = (\xi_-^2 - 4r_0^2)^{\frac{1}{2}}, \tag{27}$$

$$\alpha_0 = \left(\frac{\xi_+ - \Delta_0}{\xi_+ + \Delta_0}\right)^{\frac{1}{2}}, \tag{28}$$

$$k_0 = \left(\frac{\xi_- - \Delta_0}{\xi_- + \Delta_0}\right)^{\frac{1}{2}}, \tag{29}$$

$$\eta_0 = \frac{1}{\alpha_0} \frac{\xi_+ - \Delta_0 - 2h_0 r_0 \cos\psi}{\xi_+ + \Delta_0 - 2h_0 r_0 \cos\psi}, \tag{30}$$

$$p_0 = \frac{2r_0^2 \cos^2\psi - 2h_0 r_0 \cos\psi + \xi_-}{\xi_+ - 2h_0 r_0 \cos\psi}, \tag{31}$$

with $b_0 = (h_0^2 - 1)^{\frac{1}{2}}$ and the functions

$$F_0(\eta_0) = K(k_0) \pm F(\sin^{-1}\eta_0, k_0), \tag{32}$$

$$\Pi_0(\eta_0) = \Pi(\alpha_0^2, k_0) \pm \Pi(\sin^{-1}\eta_0, \alpha_0^2, k_0), \tag{33}$$

$$S_0(p_0) = \pm\left(\sin^{-1} p_0 - \frac{\pi}{2}\right), \tag{34}$$

with the upper sign for $\psi \in [2N\pi, (2N+1)\pi]$ and the lower sign for $\psi \in [(2N+1)\pi, 2(N+1)\pi]$ where F, K and Π are elliptic integrals [Byrd and Friedman (1954, pp. 8–10); the definiton of Π differs from that used by Gradshteyn and Ryzhik (1980, p. 905) in which α_0^2 is essentially replaced by $-\alpha_0^2$] and N is an integer. Again we use the subscript 0 to indicate the wave rest frame case. The parameter Δ_0 can also be written as

$$\Delta_0 = [\{(h_0-1)^2 - r_0^2\}\{(h_0+1)^2 - r_0^2\}]^{\frac{1}{2}}\,, \tag{35}$$

and it is only real if the particle is untrapped or if it is trapped with $h_0 < r_0 - 1$. It is also an important quantity in that all the other parameters which involve it tend to unity in the limit as Δ_0 tends to zero and it can be used to quantify the particle trapping limit. The solution to (25) is then

$$\begin{aligned} t_0 = \frac{N\pi}{\beta_{D0}K_0} \pm \frac{1}{K_0}\Bigg[&\frac{2+\xi_- - \Delta_0}{h_0\{2(\xi_- + \Delta_0)\}^{\frac{1}{2}}}F_0(\eta_0) \\ &+ \frac{\Delta_0}{h_0}\left\{\frac{2}{\xi_- + \Delta_0}\right\}^{\frac{1}{2}}\Pi_0(\eta_0) + \tfrac{1}{2}S_0(p_0)\Bigg] - \bar{t_0}\,, \end{aligned} \tag{36}$$

where the upper sign is for particles moving in the positive z direction and the lower sign is for particles moving in the negative z direction. The integer N is the number of completed *half* periods, $\bar{t_0}$ is an arbitrary constant and β_{D0} has the form of a drift velocity. In a general frame we have

$$t = \gamma_\phi\left(t_0 - \frac{\gamma_\phi v_\phi}{K}\psi\right). \tag{37}$$

The drift velocity β_{D0} describes the average motion of the particle over one wave period (it is defined as $\Delta z/\Delta t$ calculated for an integer multiple of half periods). In Section 5a, in which an expansion of the orbit is made, it is shown that this drift velocity contributes the linear part of the motion and is analogous to the motion of the guiding centre of a particle gyrating in a magnetic field. The analytic result for the drift velocity is obtained directly from (36) and (4). One has

$$\frac{1}{\beta_{D0}} = \pm\frac{2}{\pi}\left[\frac{2+\xi_- - \Delta_0}{h_0\{2(\xi_- + \Delta_0)\}^{\frac{1}{2}}}K(k_0) + \frac{\Delta_0}{h_0}\left\{\frac{2}{\xi_- + \Delta_0}\right\}^{\frac{1}{2}}\Pi(\alpha_0^2, k_0)\right]. \tag{38}$$

Fig. 3 shows how the drift velocity varies as a function of the zero order velocity u_0 (corresponding to the total rest frame particle energy per unit rest mass or the particle velocity where the electric potential is zero), for four values of r_0. The most notable features are that the curves tend to straight lines with slope unity as u_0 tends to the speed of light and that, as u_0 approaches a value corresponding to $h_0 = 1 + r_0$, the drift velocity falls very rapidly to zero. The first feature illustrates that for high enough particle energy the effect of the wave becomes small and that then $\beta_{D0} \simeq u_0$, and the second feature illustrates the trapping of particles with energy $h_0 \leq 1 + r_0$. There is a relatively small range of zero order velocities for which the wave is neither strong enough to trap the particles nor weak enough to be treated as a small effect.

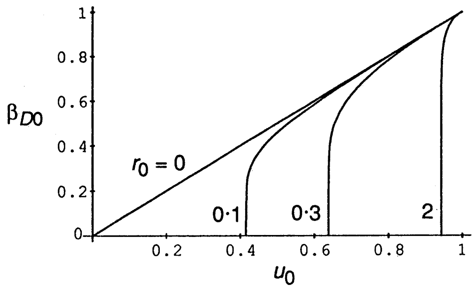

Fig. 3. Drift velocity of an untrapped particle in a subluminal wave as a function of the zero order velocity u_0, for four electric field strengths.

(b) Trapped Particles

In the trapped particle case equation (25) still applies, however many of the quantities defined in the previous solution now become complex. An alternative solution for this case with only real parameters involves replacing some of the previous definitions with

$$\Delta_T = \{16r_0(1+r_0-h_0)(1+r_0+h_0)\}^{\frac{1}{2}}\,, \tag{39}$$

$$k_T = \left[\frac{\{2(r_0+1)^2-\Delta_T\}^2-4h_0^2(1-r_0)^2}{\{2(r_0+1)^2+\Delta_T\}^2-4h_0^2(1-r_0)^2}\right]^{\frac{1}{2}}, \tag{40}$$

$$\alpha_T = \left\{\frac{2(r_0+1)^2-2h_0(1-r_0)-\Delta_T}{2(r_0+1)^2-2h_0(1-r_0)+\Delta_T}\right\}^{\frac{1}{2}}, \tag{41}$$

$$\eta_T = \frac{1}{\alpha_T}\frac{4h_0r_0-\Delta_T-4r_0(r_0+1)\cos\psi}{4h_0r_0+\Delta_T-4r_0(r_0+1)\cos\psi}, \tag{42}$$

$$p_T = \frac{r_0\cos^2\psi-h_0\cos\psi+1}{(r_0+1)\cos\psi-h_0}, \tag{42a}$$

$$\psi_T = \cos^{-1}\left(\frac{h_0-1}{r_0}\right), \tag{43}$$

and

$$F_T(\eta_T) = K(k_T)\pm F(\sin^{-1}\eta_T, k_T)\,, \tag{44}$$

$$\Pi_T(\eta_T) = \Pi(\alpha_T^2, k_T)\pm\Pi(\sin^{-1}\eta_T, \alpha_T^2, k_T)\,, \tag{45}$$

$$S_T(p_T) = \pm\left(\sin^{-1}p_T+\frac{\pi}{2}\right), \tag{46}$$

with the upper sign for $\psi\in[\psi_T,\pi]$ and the lower sign for $\psi\in[\pi, 2\pi-\psi_T]$ and where we use the subscript T to denote quantities in the trapped particle case. Note also the difference between (34) and (46).

The appropriate solution is now

$$t=\frac{N\pi}{2\Omega_B}\pm\frac{1}{K_0}\left[\frac{4h_0-\Delta_T}{2(r_0+1)\{(r_0+1)^2+4r_0-h_0^2+\Delta_T\}^{\frac{1}{2}}}F_T(\eta_T)\right. \tag{47}$$
$$\left.+\frac{\Delta_T}{(r_0+1)\{(r_0+1)^2+4r_0-h_0^2+\Delta_T\}^{\frac{1}{2}}}\Pi_T(\eta_T)-\tfrac{1}{2}S_T(p_T)\right],$$

with the upper sign for particles moving in the positive z direction (i.e. ψ from ψ_T to $2\pi-\psi_T$) and the lower sign for particles moving in the negative z direction (i.e. ψ from $2\pi - \psi_T$ to ψ_T) and where the integer N represents the number of quarter periods completed. In analogy with the determination of β_{D0} in the untrapped case one finds the bounce frequency Ω_B:

$$\frac{1}{\Omega_B} = \frac{4}{\pi K_0}\left[\frac{4h_0 - \Delta_T}{2(r_0+1)\{(r_0+1)^2 + 4r_0 - h_0^2 + \Delta_T\}^{\frac{1}{2}}} K(k_T) + \frac{\Delta_T}{(r_0+1)\{(r_0+1)^2 + 4r_0 - h_0^2 + \Delta_T\}^{\frac{1}{2}}} \Pi(\alpha_T^2, k_T)\right]. \tag{48}$$

The bounce frequency is defined as 2π times the reciprocal of the period of the particle motion and is shown in Fig. 4. Close to the trapping energy the bounce frequency rises abruptly from zero as the particle energy decreases. It then increases almost linearly as the particle energy decreases before increasing rapidly again close to the minimum allowable particle energy. The maximum bounce frequency is $\Omega_B = K_0 r_0^{\frac{1}{2}}$ which increases as the parameter r_0 increases. An interesting feature of Fig. 4 is that for a particle with zero energy ($h_0 = 0$), the bounce frequency is $\Omega_B = K_0$, independent of the electric field strength. The intersection of the Ω_B/K_0 curves implies that positive energy particles have bounce frequency less than K_0 and increasing with r_0, whereas negative energy particles have bounce frequency greater than K_0 and decreasing with r_0. Although we do not treat emission here we note that the emitted frequencies in the wave rest frame are the harmonics of this bounce frequency.

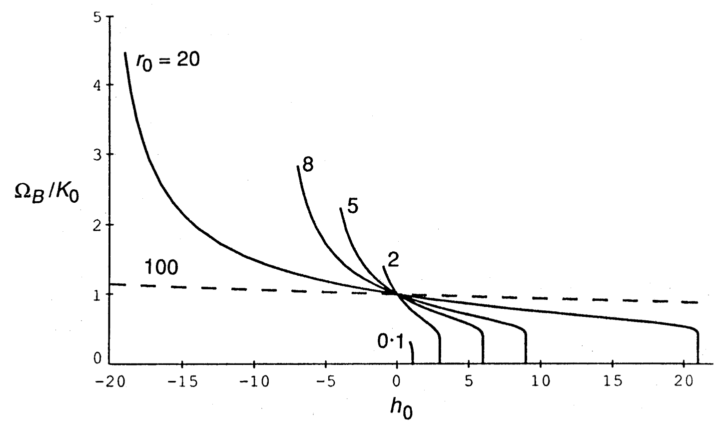

Fig. 4. Bounce frequency of trapped particles in a stationary longitudinal wave as a function of h_0 and for six values of the electric field strength parameter r_0. Part of the $r_0 = 100$ curve is shown (dashed).

5. Orbit Expansions

For the purposes of approximation, and of evaluating the Fourier transform of the single particle current, some form of expansion of the particle orbits is required. We employ two levels of expansion; the first is an expansion in harmonics of the electrostatic wave and the second is an expansion in the weak field limit.

(a) Untrapped Particles in Subluminal Waves

We start by expanding the particle orbit in the wave rest frame and then perform a Lorentz transformation to obtain the more general frame. We know from the exact solution that t is a function of ψ and so expanding in harmonics of ψ one must have

$$t_0 = -\sum_{p=1}^{\infty} C_p \sin(p\psi) - \frac{\psi}{K_0 \beta_{D0}} - \bar{t}_0 \,, \tag{49}$$

where the C_p are constants to be determined or alternatively

$$\frac{dt_0}{d\psi} = -\sum_{p=1}^{\infty} pC_p \cos(p\psi) - \frac{1}{K_0 \beta_{D0}} \,, \tag{50}$$

with the left hand side given by (5) with (17). Note that equation (50) includes only cosine terms since there can be no nonzero sine contributions from (17) and also that the expansion converges for untrapped particles with the C_p strictly decreasing in size with p. This reflects the fact that the particle motion is dominated by low harmonics (obviously the largest component in the particle motion must be the one with the same period as the wave). Orthogonality of the $\cos(p\psi)$ yields

$$C_p = \pm \frac{2}{p\pi K_0} \int_0^{\pi} \frac{h_0 - r_0 \cos\psi}{\{(h_0 - r_0 \cos\psi)^2 - 1\}^{\frac{1}{2}}} \cos(p\psi) d\psi \,, \tag{51}$$

$$\frac{1}{\beta_{D0}} = \pm \frac{1}{\pi} \int_0^{\pi} \frac{h_0 - r_0 \cos\psi}{\{(h_0 - r_0 \cos\psi)^2 - 1\}^{\frac{1}{2}}} d\psi \,. \tag{52}$$

In principle all of the constants C_p can be calculated analytically and the first of these is

$$C_1 = \pm \frac{2}{\pi K_0 r_0 h_0} \{(2 + \xi_- + \Delta_0)(2r_0^2 + \xi_- + \Delta_0)\}^{\frac{1}{2}} \{K(k_0) - E(k_0)\} \,, \tag{53}$$

where the complete elliptic integral of the second kind, $E(k)$, is introduced. The integral form (52) of the drift velocity, which is given analytically in Section 4*a*, shows that $1/\beta_{D0}$ is $1/v$ averaged over wave phase. The coefficients C_p decrease monotonically from infinity at $h_0 = 1 + r_0$, to zero as h_0 approaches becomes arbitrarily large. A Lorentz transform (see Appendix 1) applied to (49) yields the result in an arbitrary frame

$$t = -\sum_{p=1}^{\infty} \gamma_\phi C_p \sin(p\psi) - \frac{\psi}{K(\beta_D - v_\phi)} - \gamma_\phi^2 (\bar{t} - v_\phi \bar{z}) \,, \tag{54}$$

where β_D is the 'drift velocity' in the new frame and $\bar{t}$ and $\bar{z}$ are new arbitrary constants. The expansion outlined above goes beyond any given previously (Melrose 1978; Krishan and Sivaram 1983) and in particular it includes the electric field to all orders at each harmonic and as such it is indispensable in calculating the single particle current.

As discussed in Section 4a two opposite limits to consider are the particle trapping limit ($h_0 = 1 + r_0$), and the weak field limit ($h_0 \gg 1 + r_0$). In the first limit (51) and (52) gain non-integrable singularities at $\cos\psi = 1$ and thus all of the constants $1/\beta_{D0}$ and C_p tend to infinity. While it is not possible to obtain an expansion for these constants in the trapping limit, we can determine the nature of the singularity at $h_0 = 1 + r_0$. The singularity in the integrand occurs at $\psi = 0$ (i.e. where the particle velocity is zero) if $h_0 = 1 + r_0$, so we may write the integral in (51) and (52) as a sum of a divergent integral and a finite integral so that

$$C_p = \pm\frac{2}{p\pi K_0}\left\{\int_0^\delta F(\psi)\cos(p\psi)d\psi + \int_\delta^\pi F(\psi)\cos(p\psi)d\psi\right\}, \tag{55}$$

where $F(\psi)$ is the integrand in (52). As we approach trapping, the second integral approaches a finite limit and can be neglected. If δ is chosen such that $(p\delta)^2 \ll 2$ then $\cos(p\psi) \approx 1$ and it is obvious then that the behaviour of C_p in this limit is determined solely by the limiting behaviour of the integral in (52). Thus we have $\beta_{D0}C_p \approx 2/pK_0$ (these quantities are of most interest in evaluating the particle current, though we do not do this here). Defining $x = 1 - (1 + r_0)/h_0$, we can determine the behaviour of C_p and $1/\beta_{D0}$ for small x from (53). In the limit $x \to 0$, $\Delta_0 \to 2(1 + r_0)(2r_0x)^{\frac{1}{2}}$, $k_0 \to 1 - (1 + r_0)(2x/r_0)^{\frac{1}{2}}$, $K(k_0) \to \infty$ and $E(k_0) \to 1$. Using the result (Byrd and Friedman, p. 11)

$$\lim_{k_0\to 1}\left[K(k_0) - \ln\left\{\frac{4}{(1-k_0^2)^{\frac{1}{2}}}\right\}\right] = 0\,, \tag{56}$$

one obtains

$$C_1 \approx \pm\frac{1}{\pi K_0 r_0^{\frac{1}{2}}}\left[\ln\left\{\frac{32r_0}{(1+r_0)^2}\right\} - 4 - \ln x\right], \tag{57}$$

$$C_p \approx \mp\frac{\ln x}{p\pi K_0 r_0^{\frac{1}{2}}}\,, \tag{58}$$

$$\frac{1}{\beta_{D0}} \approx \mp\frac{\ln x}{2\pi r_0^{\frac{1}{2}}}\,, \tag{59}$$

where only the lowest order term in C_p and $1/\beta_{D0}$ and the two lowest order terms in C_1 are retained.

In the second limit, the integrand of (52) approaches the constant $1/u_0$ so that $\beta_{D0} \to u_0$ as expected, and $C_p \to 0$ for all values of p. In this weak field limit we expand (54) in powers of r_0/h_0 (Rowe 1990). The result converges for $|r_0/h_0| < 1$ which is the condition that in the rest frame the wave energy is less than the particle energy (this holds for all untrapped particles). To third order in r_0/h_0, one has in the general frame

$$t = -\frac{\gamma_\phi}{K_0 h_0^2 u_0^3}\left(\frac{r_0}{h_0}\right)\left[\left\{1+\frac{3}{8}\frac{5-u_0^2}{u_0^4}\left(\frac{r_0}{h_0}\right)^2\right\}\sin\psi + \frac{3}{8u_0^2}\left(\frac{r_0}{h_0}\right)\sin 2\psi \right.$$
$$\left. + \frac{5-u_0^2}{24u_0^4}\left(\frac{r_0}{h_0}\right)^2 \sin 3\psi\right] - \frac{\psi}{K(\beta_D - v_\phi)} + \text{const}\,, \qquad (60)$$

$$\frac{1}{\beta_{D0}} = \frac{1}{u_0}\left\{1+\frac{3}{4h_0^2 u_0^4}\left(\frac{r_0}{h_0}\right)^2\right\}, \qquad (61)$$

with $u_0 = \pm(1-1/h_0^2)^{\frac{1}{2}}$. The nth harmonic in the orbit expansion is of order $(r_0/h_0)^n$ with corrections of order $(r_0/h_0)^{n+2m}$ for all natural numbers m. The first order contribution to the orbit is the electric field multiplied by a factor which varies as $(h_0 u_0)^{-3}$, where $h_0 u_0$ is the zero order particle momentum. The modification of the drift velocity by the electric field is a second order effect. It is not possible to find directly an expansion of C_p for large p due to the way in which p appears in the integrand in (51). The dependence of the integrand on p can, however, be neglected in the trapping limit in which case $C_p \approx p^{-1}$. In the weak field limit the dependence of the integrand on p becomes important and C_p decreases more rapidly with p. It is shown in Appendix 2, where a full weak field expansion is given, that in this limit $(C_p \approx \text{const})^{-p}$.

(b) Trapped Particles in Subluminal Waves

We give only the harmonic expansion for the case of particles trapped in the wave and use the wave rest frame. The distance travelled by a particle is a more suitable variable than ψ to use in an expansion of the trapped orbit since it increases in time whereas ψ oscillates.

Let K' be the wave number for a half bounce (the part of the motion between successive values of zero velocity) and let Ω_B denote the bounce frequency, that is $2\pi/T_B$ where T_B is the full period of the motion. The definition of K' implies that $K'\Delta z = 2\pi$ where Δz is the distance travelled by a particle between points of zero velocity. In the rest frame $\psi = -K_0 z$ and from (20) we have then

$$\Delta z = \frac{2}{K_0}(\pi - \psi_T)\,, \qquad (62)$$

where ψ_T is defined by (43), and hence

$$K' = \frac{\pi K_0}{\pi - \psi_T}\,. \qquad (63)$$

The orbit expansion can now be written in the form

$$t(\zeta) = \sum_{p=1}^{\infty} T_p \sin(pK'\zeta) + \frac{K'\zeta}{2\Omega_B} + \bar{t}\,, \qquad (64)$$

where ζ represents the distance travelled by the particle. The expansion can be understood as follows: a particle starts with zero velocity at $\zeta = z_1$ say, then as it is moving positively, ζ is increasing monotonically with the displacement of the particle. At the point where $\zeta = 2\pi/K' + z_1$, the particle has zero velocity and has undergone half a period. For the second half of the period ζ increases in the same fashion and hence $t(\zeta)$ continues to increase as for the first half bounce. For some calculations, such as the particle current, it is necessary to have $\psi(\zeta)$ explicitly. One has

$$\psi = -K_0 \begin{cases} \zeta + z_1 - \dfrac{4N\pi}{K'} & \zeta \in \left(\dfrac{4N\pi}{K'}, \dfrac{4N\pi}{K'} + \dfrac{2\pi}{K'}\right) \\ -\zeta + z_2 + \dfrac{4N\pi}{K'} + \dfrac{2\pi}{K'} & \zeta \in \left(\dfrac{4N\pi}{K'} + \dfrac{2\pi}{K'}, \dfrac{4(N+1)\pi}{K'}\right), \end{cases} \tag{65}$$

for integer values of N with $z_1 = \psi_T/K_0$ and $z_2 = (2\pi - \psi_T)/K_0$.

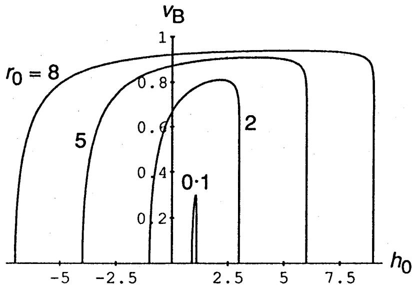

Fig. 5. Bounce speed of a trapped particle in a subluminal wave as a function of the total particle energy per unit mass h_0, for four electric field strengths.

It is now straightforward to derive the coefficients T_p and Ω_B, corresponding to C_p and β_{D0} in the untrapped case. One has

$$T_p = \frac{2}{p\pi K'}\int_0^\pi \frac{h_0 - r_0\cos\{(1-\psi_T/\pi)\phi + \psi_T\}}{([h_0 - r_0\cos\{(1-\psi_T/\pi)\phi + \psi_T\}]^2 - 1)^{\frac{1}{2}}}\cos(p\phi)d\phi\,, \tag{66}$$

$$\frac{K'}{2\Omega_B} = \frac{1}{\pi}\int_0^\pi \frac{h_0 - r_0\cos\{(1-\psi_T/\pi)\phi + \psi_T\}}{([h_0 - r_0\cos\{(1-\psi_T/\pi)\phi + \psi_T\}]^2 - 1)^{\frac{1}{2}}}d\phi\,, \tag{67}$$

where ψ_T is the wave phase at which the particle 'bounces'. It is not possible in general to obtain T_p analytically in terms of known functions due to the appearance of two cosines with different periods, however we have Ω_B from Section 4*b*. The construction of the orbit expansion implies that T_1 is the largest of the coefficients (as in the untrapped case the dominant term must be the one with the same period as the particle motion). It is clear that (66) and (67) reduce to (51) and (52) in the case $K' = K$. The quantity $2\Omega_B/K'$ which we call the bounce speed v_B appears naturally as the counterpart to β_{D0} in the untrapped case and is shown in Fig. 5. The figure illustrates how the energy range and the maximum bounce velocity of the trapped particles increases with r_0, and that the bounce velocity is relatively constant for much of the allowed energy range for high values of r_0.

In the trapped case there are also two limits, the first of which represents a particle coming to rest in the trough of the wave ($h_0 \to 1 - r_0$) and the second of which corresponds to a particle on the verge of becoming untrapped ($h_0 \to 1 + r_0$). The behaviour of T_p can be determined in both of these limits, in the same way as for the limiting behaviour of C_p. In the first limit $\psi_T \to \pi$ and the integrand in (67) becomes infinitely large over the entire range of integration. The singularity is not isolated in this case but it is obvious that the integral in (66) must be less than that in (67) due to the presence of the $\cos(p\phi)$ factor and the fact that the remaining factor is always positive. It is also apparent that $T_p > 0$ since the first factor is monotonically decreasing with ϕ. One has $\Omega_B \to K_0 r_0^{\frac{1}{2}}$ and hence $T_p \lesssim 1/pK_0 r_0^{\frac{1}{2}}$. In the second limit the integrands in (66) and (67) are singular only at $\phi = 0$ (where the particle velocity is zero) and this singularity is non-integrable if $h_0 = 1 + r_0$. This is obvious from the plots of the bounce speed in Fig. 5. The singular contribution to the integrals is independent of p and hence T_p and Ω_B have similar limiting behaviour which we deduce from (49):

$$\frac{1}{\Omega_B} \approx -\frac{\ln y}{\pi K_0 r_0^{\frac{1}{2}}}, \tag{68}$$

$$T_p \approx -\frac{\ln y}{p\pi K_0 r_0^{\frac{1}{2}}}, \tag{69}$$

where $y = 1 - h_0/(1 + r_0)$. This is the same limiting behaviour as for $2/K_0\beta_{D0}$ and C_p. In the trapped particle case, $T_p \lesssim p^{-1}$ in both limits.

6. Discussion

In this paper the motion of charged particles in the field of a subluminal longitudinal plasma wave is explored. We treat both trapped and untrapped particles and give exact orbits in both closed and expanded forms. In particular we treat the electric field exactly and include relativistic effects. These results are important in the theory of emission by particles accelerated by a strong plasma wave, which is to be developed in a later paper.

The wave strength parameter r_0 was introduced in Section 3. In SI units $r_0 = e\phi_0/mc^2$ for an electron or positron where ϕ_0 is the electric potential in the wave rest frame. Thus one has $r_0 = 1{\cdot}95 \times 10^{-6}\phi_0/(1\text{V})$. An electric potential of $\phi_0 = 10^6$ V in the wave rest frame corresponds to $r_0 \approx 2$, however in the case of pulsars potentials across the polar caps may be of the order of 10^{12} V (Ruderman and Sutherland 1975) and thus r_0 could be of order 10^6. Melrose (1978, 1986) invoked equipartition of energy between particles and the electric field and suggested that the potential across the polar caps would be unlikely to be neutralised by a plasma discharge to a potential less than $\phi_0 \approx \gamma mc^2/e$, where γ is the Lorentz factor of the particle distribution producing the wave so that $r_0 \approx \gamma$. The first estimate is an upper bound for r_0 and the second can be taken as a lower bound. The actual value for r_0 depends on the model for wave production in the pulsar magnetosphere. It is, however, not our aim to develop such a model here. In the case of pulsars then, a longitudinal wave is capable of accelerating electrons and positrons from rest to relativistic energies in less than half a wave period. Particle trapping is also more likely to be important for strong electric fields.

It was shown in Section 3 that a subluminal longitudinal wave can accelerate trapped particles more effectively than untrapped particles (in the sense that the energy gain is larger). Trapped particles have lower kinetic energy than the electric energy of the wave ($h_0 - 1 < r_0$) and so are strongly accelerated, whereas untrapped particles have greater kinetic energy than the electric energy of the wave ($h_0 - 1 > r_0$) and are less readily perturbed. The greater the wave phase speed, the greater the advantage of trapped particles over untrapped particles. The results also show that a larger range of total particle energy h_0 leads to appreciable acceleration in the trapped case compared with the untrapped case. In the untrapped case, particles must have energies very close to the trapping energy ($h_0 = 1 + r_0$) in order to be accelerated as much as possible by the wave. The results summarised above suggest that if r_0 is large (≈ 10 say) then emission from trapped particles dominates emission from untrapped particles. This needs to be clarified in the treatment of the emission.

Acknowledgments

I thank D.B. Melrose for inspiring the work presented here and for helpful comments on the manuscript, and I acknowledge useful suggestions from an anonymous referee.

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Appendix 1: Lorentz Transformations

In this paper, Lorentz transforms are used to transform results from a reference frame in which an electrostatic wave is at rest to another reference frame. The relations between quantities in the rest frame and those in a general frame are summarised below.

Quantities defined in the rest frame have subscript 0. The main quantities are E_0, K_0, v_0, ψ_0, γ_0, r_0 and β_{D0} where E_0 is an electric field, K_0 is a wave vector, v_0 and β_{D0} are velocities, ψ_0 is a phase, γ_0 is an energy and r_0 is a dimensionless parameter defined in Section 3. The wavevector and frequency of the wave in an arbitrary frame are

$$K = \gamma_R(K_0 - v_R\Omega_0)\,, \qquad \Omega = \gamma_R(\Omega_0 - v_R K_0)\,,$$

where v_R and γ_R are the relative velocity between the general frame and the wave rest frame and the corresponding γ-factor. The Lorentz transform is taken to be in the direction of the particle and wave motion. The frequency Ω_0 is zero by definition so that $v_R = -v_\phi$, where $v_\phi = \Omega/K$ is the wave phase speed in the new frame. We adopt the notation γ_ϕ for $\gamma_R = 1/(1-v_\phi^2)^{\frac{1}{2}}$.

Using the inverse transform one finds the wave frame quantities in terms of those in the observer's frame

$$E_0 = E\,, \tag{A1}$$

$$\psi_0 = \psi\,, \tag{A2}$$

$$K_0 = \gamma_\phi(K - v_\phi\Omega) = \frac{K}{\gamma_\phi}\,, \tag{A3}$$

$$v_0 = \frac{v - v_\phi}{1 - v_\phi v}\,, \tag{A4}$$

$$\beta_{D0} = \frac{\beta_D - v_\phi}{1 - v_\phi\beta_D}\,, \tag{A5}$$

$$\gamma_0 = \gamma_\phi\gamma(1 - v_\phi v)\,, \tag{A6}$$

$$r_0 = \gamma_\phi r\,. \tag{A7}$$

Appendix 2: Complete Weak Field Expansion

The complete weak field expansion for C_p is obtained as follows. Defining

$$F(y) = \frac{(1-y)h_0}{\{(1-y)^2h_0^2 - 1\}^{\frac{1}{2}}}\,, \tag{A8}$$

one has on expanding $F(y)$ for small y,

$$C_p = \pm\frac{1}{2^{p-1}pK_0}\left(\frac{r_0}{h_0}\right)^p \sum_{n=0}^{\infty} \frac{{}^{p+2n}C_n}{2^{2n}} \frac{F^{(p+2n)}(0)}{(p+2n)!}\left(\frac{r_0}{h_0}\right)^{2n}\,, \tag{A9}$$

where $F^{(n)}(0)$ denotes the nth derivative of $F(y)$ evaluated at $y = 0$ and ${}^{p+2n}C_n$ is a binomial expansion coefficient. The derivatives of $F(y)$ are given by

$$F^{(l)}(0) = \frac{(-)^l(l+1)!}{2^l u_0^{2l+1}} \sum_{n=[l/2]}^{l} G_{l,n} u_0^{2(l-n)}\,, \tag{A10}$$

where $[l/2]$ denotes the integer part of $l/2$ and

$$G_{l,n} = \frac{(-)^n(2n-1)!}{(n-1)!(l-n)!(2n-l+1)!} \times \begin{cases} 1 & n = 0 \\ 2 & n > 0\,. \end{cases} \tag{A11}$$

The summation in (A10) is a polynomial in u_0^2 with a factor $(1-u_0^2)$ for $l > 0$. The complete weak field expansion for $1/\beta_{D0}$ is

$$\frac{1}{\beta_{D0}} = \sum_{l=0}^{\infty} \frac{{}^{2l}C_l}{2^{2l}} \frac{F^{(2l)}(0)}{(2l)!}\left(\frac{r_0}{h_0}\right)^{2l}\,. \tag{A12}$$

In the weak field limit, the p dependence of the C_p is obtained by retaining only the highest order term in the expansion

$$\begin{aligned} C_p &\approx \frac{(-)^p(p+1)}{2^{2p-1}K_0 p u_0^{2p+1}}\left(\frac{r_0}{h_0}\right)^p \sum_{n=[l/2]}^{l} G_{l,n} u_0^{2(l-n)} \\ &\approx \mathrm{const}^{-p} \end{aligned} \tag{A13}$$

where const > 1. This dependence is stronger than the p^{-1} dependence in the strong field limit.

Manuscript received 25 June, accepted 8 October 1991