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Fermi–Dirac Equations

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Abstract

Assuming that space–time is accompanied by hidden anticommuting coordinates, we have constructed 'fermionic' generalisations of the Dirac equation; these equations involve matrices (which can be construed as operating in an internal space) multiplying the Grassmann derivatives. We discuss several models, of varying degrees of complexity, with 'internal symmetries' including Sp(2) and SU(N). By appending the space–time Dirac operator, one is led to mass spectra with quantised values, suggesting that this approach may provide a model relating generations to internal symmetries.

1. Introduction

It is an attractive notion that any hidden additional space–time coordinates are fermionic in character rather than bosonic. An assumption of this type produces strongly constrained theories and leads to models with *finite* internal degrees of freedom; they are thus more amenable to experimental verification than models based on extra bosonic coordinates, where an infinite tower of states is usually entrained and one has to ensure that the higher excitations are sufficiently massive so as not to conflict with the known low-energy particle spectrum.

The idea is not new. It originated in Fermi–Bose supersymmetry and has been applied to superparticles; for a review with a phenomenological emphasis see Ross (1985), while the technical complexities are discussed by, for instance, Lindstrom *et al.* (1990 and references therein). The idea has also been applied to superstrings (see e.g. Green *et al.* 1987), as well as providing a framework for extended BRST symmetry and the ghost spectrum in gauge theory (Bonora and Tonin 1981; Delbourgo and Jarvis 1982; Twisk and Zhang 1988). The concept has been advocated as a nice way of handling spin and picturing internal symmetry, with the choice of coordinates and superwave functions reflected in the resultant gauge group and the associated particle representations (Delbourgo 1988). It is even possible to contemplate a Kaluza–Klein generalisation of general relativity which encompasses fermionic coordinate extensions (Delbourgo and Zhang 1988).

In two earlier papers (Delbourgo *et al.* 1989, 1990) we examined the consequences of a Grassmann scheme for quantum-mechanical models where the Hamiltonian H is a hermitian function of two (or more) fermionic coordinates. Generally H can be written as a harmonic, quadratic function of pairs of Grassmann variables

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and their conjugate momenta plus anharmonic terms which are *finite* in extent because of the terminating character of Taylor expansions of anticommuting quantities. As a result, problems of this type are always completely soluble in principle and often in practice.

In this paper, we would like to follow in Dirac's footsteps and look for a 'square root' of the harmonic Hamiltonian,

$$H = \sum_k (p_k^1 p_k^2 + x_k^1 x_k^2)\,,$$

which is the sum of k pairs of conjugate fermionic variables. Notice that we are not allowed to take the Hamiltonian as only the square of the Grassmann momenta p, because this is not strictly hermitian (the hermitian conjugate of the Grassmann variable x is the differential operator d/dx); the addition of the square of coordinates is essential for restoring hermiticity. In this respect we are departing from Dirac's brief. However, Moshinsky and Szczepaniak (1989) have demonstrated that this is not a very radical departure by square-rooting the bosonic harmonic Hamiltonian. Here we want to carry out the same thing but in a fermionic setting, which is why we have entitled our article a study of Fermi–Dirac equations.

When the square root of the Hamiltonian is obtained in the form $\mathcal{G}.D$ where D is some linear combination of x and p, and $\mathcal{G}$ are the associated 'internal' matrices, it is generally true that the square $(\mathcal{G}.D)^2$ equals a constant plus an operator whose eigenvalues sum to zero. We show this in the next section. There we also present the simplest model of this sort; it has an invariance which might be considered 'rotations around the z axis in symplectic space'; it does not have full Sp(2) invariance, but rather functions as a dynamical operator, very similarly to its role in the O(4,2) formalism for the hydrogen atom (Wybourne 1974; Barut and Bohm 1970).

Our preference is a different Hamiltonian which is an invariant under combined Sp(2) rotations of coordinates and 'internal spin', as that is in direct analogy to the Lorentz invariance of the ordinary Dirac equation. Therefore we construct in Section 3 an appropriate linear combination of coordinates and momenta, multiplied into related matrices, which possesses this Sp(2) symmetry. At first we do so for $k = 1$.

The generalisation to higher k values may be done in more than one way. In Section 3*b* we consider the most straightforward approach. This has a permutation symmetry among the Grassmannian coordinates of different index; it has the unusual feature that the component operators for the different coordinates anticommute rather than commute.

A more common internal symmetry group is SU(N). It turns out that there is more than one way to write the SU(N) generators within this framework. Two of these methods are demonstrated in Sections 4 and 5; both lead to the *same* invariant Hamiltonian. In the final section we adjoin these Grassmann coordinates to space–time and consider the full Fermi–Dirac equation to determine the repercussions for the mass matrix.

2. Grassmann Coordinates and Matrices

(a) Internal Space Operators

Let us begin by briefly summarising our notation. We are dealing with coordinate pairs of fermionic variables x_k^1 and x_k^2 and their conjugate momenta,

$$p_k^2 = i\partial/\partial x_k^1 = -p_{k1}, \quad p_k^1 = -i\partial/\partial x_k^2 = p_{k2}\,, \tag{1}$$

connected with the raising and lowering index rules,

$$p_{kr} = \eta_{rs} p_k^s; \quad \eta_{21} = \eta^{12} = 1\,, \tag{2}$$

and in agreement with the 'Heisenberg commutation relations',

$$\{x_k^r, p_l^s\} = i\eta^{rs}\delta_{kl}, \quad \{x_k^r, x_l^s\} = \{p_k^r, p_l^s\} = 0\,. \tag{3}$$

For this purpose, note that the index k is a spectator, simply serving to count the number of independent pairs.

All of this may look more familiar if one defines creation and annihilation operators,

$$\begin{aligned} A_k^\dagger &= (x_k^1 + ip_k^1)/\sqrt{2} = (\partial/\partial x_k^2 + x_k^1)/\sqrt{2}\,, \\ A_k &= (x_k^2 - ip_k^2)/\sqrt{2} = (\partial/\partial x_k^1 + x_k^2)/\sqrt{2}\,, \end{aligned} \tag{4}$$

$$\begin{aligned} B_k^\dagger &= i(x_k^2 + ip_k^2)/\sqrt{2} = i(-\partial/\partial x_k^1 + x_k^2)/\sqrt{2}\,, \\ B_k &= i(x_k^1 - ip_k^1)/\sqrt{2} = i(-\partial/\partial x_k^2 + x_k^1)/\sqrt{2}\,. \end{aligned} \tag{5}$$

The harmonic Hamiltonian can be re-expressed as $\sum_k (A_k^\dagger A_k + B_k^\dagger B_k)$ if one so wishes. However, for the most part we shall stick to the coordinate–momentum operators rather than Fock space combinations.

Acting on the Grassmann variables are the Sp(2) generators,

$$\begin{aligned} S_1 &= i(x^1p^1 - x^2p^2) = (x^1\partial/\partial x^2 + x^2\partial/\partial x^1)\,, \\ S_2 &= (x^1p^1 + x^2p^2) = -i(x^1\partial/\partial x^2 - x^2\partial/\partial x^1)\,, \\ S_3 &= -i(x^1p^2 + x^2p^1) = (x^1\partial/\partial x^1 - x^2\partial/\partial x^2)\,. \end{aligned} \tag{6}$$

These obey the standard spin algebra rules. For later use, we should point out the existence of 'quasispin' operators which are quadratic in momenta or coordinates that also obey the commutation rules of angular momentum, and which include the harmonic Hamiltonian:

$$\begin{aligned} L_1 &= x^1x^2 + p^1p^2 = x^1x^2 + \partial^2/\partial x^2\partial x^1\,, \\ L_2 &= i(-x^1x^2 + p^1p^2) = i(-x^1x^2 + \partial^2/\partial x^2\partial x^1)\,, \\ L_3 &= -i(x^1p^2 - x^2p^1) = x^1\partial/\partial x^1 + x^2\partial/\partial x^2 - 1\,. \end{aligned} \tag{7}$$

It should be noted that all the $\vec{L}$ operators are Sp(2) invariants (i.e. they are unaffected by the action of the $\vec{S}$ operators). In particular, the scale operator L_3 measures the degree of an x-monomial.

(b) Internal Space Matrices

The aim of this paper is to obtain a square root of the Grassmannian harmonic Hamiltonian in much the same way that Dirac tackled the relativistic energy equation. We are looking for an operator $\mathcal{G}.D$ whose square produces the quadratic H plus possibly other operators which commute with it. The D are linear combinations of Grassmann coordinates and/or momenta, while $\mathcal{G}$ are internal space matrices, direct analogues of the Dirac gamma matrices. Since

$$4(\mathcal{G}_i D_i)^2 = [\mathcal{G}_i, \mathcal{G}j][D_i, D_j] + \{\mathcal{G}_i, \mathcal{G}_j\}\{D_i, D_j\},$$

we can reduce the square to the product of two commutators by requiring either that $\{D_i, D_j\} = 0$ *or* that $\{\mathcal{G}_i, \mathcal{G}_j\} = 0$. Furthermore we would like

$$[D_i, D_j] \propto \eta_{ij}(H + O)$$

where O vanishes or at least commutes with H. One may even relax the conditions by ensuring that when $i = j$ the anticommutators of $\mathcal{G}_i$ and of D_i reduce to the identity, in which case the square is still given by the product of two commutators up to an additive constant. The various models that we shall study attempt to satisfy the above requirements or variants thereof. In any case, since by necessity $4(\mathcal{G}_i D_i)^2$ contains a commutator of $\mathcal{G}$, that part of the square has zero trace; so the sum of its eigenvalues is zero.

As a matter of fact, a set of natural internal matrices $\mathcal{G}$ does exist. Because all wavefunctions can be expressed as linear combinations of the basic states,

$$(1 + x^2 x^1)/\sqrt{2}, \quad x^1, \quad x^2, \quad (1 - x^2 x^1)/\sqrt{2},$$

we may determine the action of coordinates x and momenta p in this basis and extract a set of corresponding matrices,

$$\sqrt{2}\mathcal{X}^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \sqrt{2}\mathcal{X}^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\sqrt{2}\mathcal{P}^1 = \begin{pmatrix} 0 & 0 & -i & 0 \\ i & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad \sqrt{2}\mathcal{P}^2 = \begin{pmatrix} 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & -i \\ 0 & i & 0 & 0 \end{pmatrix}. \tag{9}$$

Obviously, the anticommutation relations between the matrices $\mathcal{X}$ and $\mathcal{P}$ are precisely the same as those for the original variables x and p. The same applies to the matrix representations of the Fock space operators, namely $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A} = (\mathcal{X}^2 - i\mathcal{P}^2)/\sqrt{2}$, and so on. For later use we should record that the 'ground state' spinor on which these matrices act is

$$\chi_0^{\mathrm{T}} = (1,0,0,1)/\sqrt{2}\,,$$

the first two excited spinors (obtained by applying $\mathcal{X}^r$ to χ_0) are

$$\chi_1^{\mathrm{T}} = (0,1,0,0), \qquad \chi_2^{\mathrm{T}} = (0,0,1,0)\,,$$

while the 'highest weight' spinor (annihilated by the $\mathcal{X}$) is

$$\chi_4^{\mathrm{T}} = (1,0,0,-1)/\sqrt{2}\,. \tag{10}$$

(c) A 'Dynamical' Hamiltonian Model

Our first model uses a couple of D and a corresponding pair of internal matrices $\mathcal{G}$. For simplicity we identify the two D with appropriate creation and annihilation combinations:

$$\begin{aligned} D_1 &= x^1 - ip^2 = \partial/\partial x^1 + x^1\,, \\ D_2 &= -ix^2 - p^1 = i(\partial/\partial x^2 - x^2)\,. \end{aligned} \tag{11}$$

Thus $D_1^2 = D_2^2 = 1$ and $\{D_1, D_2\} = 0$. Also both D's are hermitian. In order to ensure that $\{\mathcal{G}_1, \mathcal{G}_2\} = 0$, we make the simplest two-dimensional choice, namely $\mathcal{G}_1 = \sigma_2$, $\mathcal{G}_2 = \sigma_3$ signifying an 'internal spin space' with two degrees of freedom. This way one arrives at

$$\begin{aligned} (\mathcal{G}.D)^2 &= (\sigma_2 D_1 + \sigma_3 D_2)^2 = i\sigma_1[D_1, D_2] + D_1^2 + D_2^2 \\ &= 2\sigma_1(\partial^2/\partial x^2 \partial x^1 + x^1 x^2 - x^1\partial/\partial x^2 - x^2\partial/\partial x^1) + 2 \\ &= 2\sigma_1(H - S_1) + 2\,. \end{aligned} \tag{12}$$

In this model, the extra operator O equals the first spin component S_1, which indeed commutes with H.

Because the eigenvalues of H are ± 1 on the bosonic states and 0 on fermionic states in equation (8), while the eigenvalues of S_1 are the reverse (± 1 on fermionic combinations $x^1 \pm x^2$ and 0 on bosonic states), we conclude that the full eigenvalues of $(\mathcal{G}.D)^2$ are $2 \pm 2\sigma_1$ whether the states are Bose or Fermi and hence the range of eigenvalues is 0, 2 and 4. We notice in this model that the Hamiltonian is associated with the matrix σ_1 and is only invariant under rotations about the first axis. It is, in fact, invariant under the *full* Sp(2) rotation about this axis, $S_1 + \sigma_1$.

Since the Hamiltonian is not invariant under the full Sp(2) group, it is a 'dynamical' operator of this group including the 'spin'. This concept of a dynamical symmetry including the Hamiltonian is well known in a number of contexts. Our Fermi–Dirac equation is merely another example, albeit in an unusual setting.

One may develop this idea and double the internal space by extending the D with another hermitian pair,

$$D_3 = i(\partial/\partial x^1 - x^1), \quad D_4 = (\partial/\partial x^2 + x^2)$$

and finding another matrix pair $\mathcal{G}_3$ and $\mathcal{G}_4$ which anticommute with the previous $\mathcal{G}$. In the sections below we develop this idea, yielding models with full Sp(2) symmetry, and with SU(N) symmetry.

3. Sp(2) Symmetric Models

(a) Basic Case

Recall that a natural set of spinorial matrices exists in the form of $\mathcal{X}$ and $\mathcal{P}$ of equation (9). For them one can also construct a triple of Sp(2) spin matrices:

$$\begin{aligned} \mathcal{S}_1 &= i(\mathcal{X}^1\mathcal{P}^1 - \mathcal{X}^2\mathcal{P}^2)\,, \\ \mathcal{S}_2 &= (\mathcal{X}^1\mathcal{P}^1 + \mathcal{X}^2\mathcal{P}^2)\,, \\ \mathcal{S}_3 &= -i(\mathcal{X}^1\mathcal{P}^2 + \mathcal{X}^2\mathcal{P}^1)\,, \end{aligned} \tag{13}$$

and by the same token there arise the 4×4 quasi-spin matrix analogues,

$$\begin{aligned} \mathcal{L}_1 &= \mathcal{X}^1\mathcal{X}^2 + \mathcal{P}^1\mathcal{P}^2\,, \\ \mathcal{L}_2 &= i(-\mathcal{X}^1\mathcal{X}^2 + \mathcal{P}^1\mathcal{P}^2)\,, \\ \mathcal{L}_3 &= -i(\mathcal{X}^1\mathcal{P}^2 - \mathcal{X}^2\mathcal{P}^1)\,, \end{aligned} \tag{14}$$

which stay invariant under $\mathcal{S}$ rotations.

We are now guaranteed that the hermitian linear combination

$$\mathcal{G}.D = -i\eta_{rs}(\mathcal{X}^r p^s + x^r\mathcal{P}^s) \tag{15}$$

is Sp(2) invariant under combined coordinate–spin rotations generated by the full generators $S + \mathcal{S}$. *A fortiori* its square will also be Sp(2) symmetric; in fact the result can be manoeuvred into the pleasing form,

$$(\mathcal{G}.D)^2 = 1 - \vec{S}.\vec{\mathcal{S}} - \vec{L}.\vec{\mathcal{L}}, \tag{16}$$

where the last term on the right is also quasi-spin invariant.

It only remains to find the eigenspectrum. This is readily done by splitting the operator in question into the sum of two commuting parts, $\mathcal{G}.D = U + V$, where

$$U = -i(\mathcal{X}^1 p^2 + x^1\mathcal{P}^2), \quad V = i(\mathcal{X}^2 p^1 + x^2\mathcal{P}^1), \tag{17}$$

and determining the eigenfunction of each part, ψ_u and ψ_v, with eigenvalues λ_u and λ_v respectively. Nevertheless we should point out that the total eigenvalue of the product wavefunction $\psi_u\psi_v$ equals $\lambda = \lambda_u \pm \lambda_v$; the possible change in sign is due to the fact that the eigenstates ψ_u are sometimes fermionic; passing the operator V through the product can induce this curious sign reversal.

By expanding the wavefunction ψ_u in the form

$$\psi_u = [\alpha + \beta x^1 + \gamma \mathcal{X}^1 + \delta x^1 \mathcal{X}^1]\chi_0$$

because it depends purely on the first Grassmann components, we may derive the four eigenvalues and wavefunctions

$$\begin{aligned}
\lambda_u &= 1: \qquad \psi_u = [x^1 + \mathcal{X}^1]\chi_0\,, \\
\lambda_u &= 0: \qquad \psi_u = \chi_0 \quad \text{or} \quad x^1\mathcal{X}^1\chi_0\,, \\
\lambda_u &= -1: \qquad \psi_u = [x^1 - \mathcal{X}^1]\chi_0\,.
\end{aligned} \tag{18}$$

Note that the wavefunctions are 4-component spinors in the fermionic variables because the bracketted quantities in (18) act on the ground state spinor χ_0. For instance,

$$\psi_u(1) = (x^1/\sqrt{2}, 1, 0, x^1/\sqrt{2})\,.$$

Similar sets can be found for V, with the second Grassmann component replacing the first. Paying proper attention to sign changes, the combined operator $U + V$ possesses the 5 eigenvalues and 16 eigenfunctions,

$$\begin{aligned}
\lambda &= 2: \qquad \psi = [x^1 + \mathcal{X}^1][x^2 - \mathcal{X}^2]\chi_0\,, \\
\lambda &= -2: \qquad \psi = [x^1 - \mathcal{X}^1][x^2 + \mathcal{X}^2]\chi_0\,, \\
\lambda &= 1: \qquad \psi = [x^1 + \mathcal{X}^1]\chi_0, \quad \psi = [x^2 + \mathcal{X}^2]\chi_0\,, \\
&\qquad \psi = [x^1 + \mathcal{X}^1](x^2\mathcal{X}^2)\chi_0, \quad (x^1\mathcal{X}^1)[x^2 + \mathcal{X}^2]\chi_0\,, \\
\lambda &= -1: \qquad \psi = [x^1 - \mathcal{X}^1]\chi_0, \quad \psi = [x^2 - \mathcal{X}^2]\chi_0\,, \\
&\qquad \psi = [x^1 - \mathcal{X}^1](x^2\mathcal{X}^2)\chi_0, \quad (x^1\mathcal{X}^1)[x^2 - \mathcal{X}^2]\chi_0\,, \\
\lambda &= 0: \qquad \psi = [1, x^1\mathcal{X}^1, x^2\mathcal{X}^2, x^1\mathcal{X}^1x^2\mathcal{X}^2]\chi_0\,, \\
&\qquad \psi = [x^1 + \mathcal{X}^1](x^2 + \mathcal{X}^2)\chi_0, \quad (x^1 - \mathcal{X}^1)[x^2 - \mathcal{X}^2]\chi_0\,.
\end{aligned} \tag{19}$$

(b) Extension to Higher k

We shall treat the case $k = 2$ in some detail and then sketch the results for larger k-values. The added 'normal' Grassmannian coordinates x_2^1, x_2^2 anticommute with each other and with the previous x_1^1, x_1^2 according to the relations in (3). When one considers the internal matrices $\mathcal{X}_2^1, \mathcal{X}_2^2$, however, one sees that a standard type construction will result in $\{\mathcal{X}_2^1, \mathcal{X}_2^2\} = 0$, but $[\mathcal{X}_2^i, \mathcal{X}_1^j] = 0$.

The reason for this is that the internal spaces 'attached' to x_1^i and x_2^j are similar to the spin spaces attached to two different particles in standard quantum mechanics. Just as the spin operators for different particles commute, the analogous symplectic matrices for different symplectic spaces will commute.

We will have a similar situation for spin-like matrices and wavefunctions for our internal coordinates $\mathcal{X}_1$ and $\mathcal{X}_2$; hence the matrices $\mathcal{G}_1$ and $\mathcal{G}_2$ commute even though the operators D_1 and D_2 made of normal Grassmannian coordinates anticommute. This has a number of very positive features, but it also introduces one or two complexities into the computation of the eigenfunctions of the extended Hamiltonian operator. Let us consider these in turn:

First, by using the $\mathcal{G}$ and D operators as defined in (15) for each coordinate k, our new operator (the square root of the new Hamiltonian) is $\Sigma_j \mathcal{G}_j.D_j$; this is the sum of operators for individual j which *anticommute* with each other. Hence the Hamiltonian becomes automatically a sum of Hamiltonians for the individual coordinates:

$$(\Sigma_j \mathcal{G}_j.D_j)^2 = \Sigma_k (\mathcal{G}_k.D_k)^2 . \tag{20}$$

The individual coördinate Hamiltonians *commute* with each other; hence if λ is an eigenvalue of the whole Hamiltonian, and λ_k is an eigenvalue of the kth Hamiltonian, the eigenvalues for larger numbers of dimensions can be computed from those of lower dimensions by

$$\lambda^2 = \Sigma_k \lambda_k^2 . \tag{21}$$

Hence mass contributions of these extra dimensions add in quadrature; that is a very nice feature of the system.

The eigenfunctions ψ of the whole operator $\Sigma_j \mathcal{G}_j.D_j$ are not, however, simple products of the eigenfunctions listed in (19); this is best illustrated by an example

$$\mathcal{G}_2.D_2(x_1^1 + \mathcal{X}_1^1) = (-x_1^1 + \mathcal{X}_1^1)\mathcal{G}_2.D_2 . \tag{22}$$

In other words, if we define $Y_2 = \mathcal{G}_2.D_2$, then for each ψ_1 such that $Y_1\psi_1 = \lambda_1\psi_1$, we have $Y_2\psi_1 = \psi_1' Y_2$, where ψ_1' may be different from ψ_1. Hence even though ψ_1 may be an eigenfunction of Y_1, and ψ_2 may be an eigenfunction of Y_2, the state $\psi_1\psi_2$ is not necessarily an eigenfunction of $Y_1 + Y_2$.

Such an eigenfunction can, however, always be found from a linear combination of $\psi_1\psi_2$ and $\psi_1{}'\psi_2$. Notice that $(\psi_1{}')' = \psi_1$, and that since Y_1 and Y_2 anticommute, we can prove that $\psi_1{}'$ is an eigenstate of Y_1 with eigenvalue $-\lambda_1$ if ψ_1 is an eigenstate of Y_1 with eigenvalue $+\lambda_1$. Hence linear combinations of products of the states with eigenvalues $\pm\lambda_i$ can be used to form a basis in which to calculate the states with net eigenvalue λ such that $\lambda^2 = \Sigma_k \lambda_k^2$.

For example, the states

$$\{\alpha[x_1^1 + \mathcal{X}_1^1] + \beta[x_1^1 - \mathcal{X}_1^1]\}[x_2^1 + \mathcal{X}_2^1][x_2^1 - \mathcal{X}_2^2] \tag{23}$$

with $\beta = \alpha(1 \mp \sqrt{5})/2$ are eigenstates of the overall Hamiltonian with $\lambda = \pm\sqrt{5}$.

One might have some prejudice that a state with an even number of powers of x_i^k or $\mathcal{X}_i^k$ would be a 'boson' and one with an odd number of such powers would be a 'fermion'. Using this classification (which may or may not ultimately be useful), there are equal numbers of Fermi and Bose states.

For 2 pairs of Grassmannian coordinates, one therefore has 256 states, of which 128 are Fermi-type. There are 11 eigenvalues: $\pm\sqrt{8}, \pm\sqrt{5}, \pm 2, \pm\sqrt{2}, \pm 1$ and 0. There are 96 Fermi-type states with eigenvalues ± 1, and 32 'heavy' Fermi-type states with eigenvalues $\pm\sqrt{5}$.

Since there are 16 states for each coordinate dimension, the total number of states grows very rapidly like 16^n as more dimensions of symplectic space are added. Hence direct implementation of a higher symmetry by the addition of n such pairs of dimensions would require an additional selection rule to reduce the number of physical states. Alternatively, one may search for a more subtle representation of the symmetry in spaces of dimension lower than n. Work on this approach is currently under way.

4. SU(N) Symmetry: Approach I

(a) Introduction

As we discussed above, it is natural to generalise the result for one pair of Grassmannian coordinates x, y to the case of several symplectic dimensions by simply adding the symplectic Hamiltonians for the various pieces: $H = \Sigma H_i$ with (see equation 15)

$$H_i = x_i \frac{\partial}{\partial \mathcal{X}_i} + \mathcal{X}_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial \mathcal{Y}_i} + \mathcal{Y}_i \frac{\partial}{\partial y_i}\,. \tag{24}$$

(For the remainder of the paper we use x_k, y_k instead of x_k^1, x_k^2. This simplified notation stresses the index under discussion here—that of the different pairs.) Since the symplectic spinors for different dimensions commute, whereas the 'ordinary' symplectic coordinates anticommute, the Hamiltonians H_i anticommute. The eigenvalues then add in quadrature, $\lambda^2 = \Sigma_i \lambda_i^2$. This is perhaps unusual, but a complete theory can be composed in this way.

Although the Hamiltonian constructed in this way has permutation symmetry among the indices, it does not have SU(N) symmetry. In Subsection 4*b* we discuss and solve the problem of representing the SU(N) generators on the symplectic spinor coordinates.

In Subsection 4*c* we give a modification of the Hamiltonian which *does* have the desired SU(N) invariance. This Hamiltonian has the feature that the $\tilde{H}_i$ do commute with each other; the eigenvalues of the total H are then sums of the eigenvalues for the individual $\tilde{H}_i$.

(b) SU(N) Generators in Symplectic Spaces

'Ordinary' symplectic coordinates. When one deals with standard symplectic coordinates x_i, y_i such that $x_i y_j = -y_j x_i$, the basic SU(N) generators are well known (Delbourgo 1989). They are (for $i \neq j$)

$$S^G{}_i{}^j = x_i \frac{\partial}{\partial x_j} - y_j \frac{\partial}{\partial y_i} \tag{25}$$

and the commutators thereof. For example, the SU(2) generators are

$$\begin{aligned} S^G{}_1{}^2 &= x_1 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial y_1}; \qquad S^G{}_2{}^1 = x_2 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial y_2}\,; \\ S^G{}_3 &= \left[S^G{}_1{}^2, S^G{}_2{}^1\right] = x_1 \frac{\partial}{\partial x_1} x_2 \frac{\partial}{\partial x_2} - y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}\,. \end{aligned} \tag{26}$$

Internal space matrices. One might be tempted to simply make a copy of (25) using $\mathcal{X}_i$, $\mathcal{Y}_j$ instead of x_i, y_j, and add it to (25) in order to get the overall SU(N) generators. This procedure was successful in defining the symplectic group generators, equation (13). (For more than one coordinate x_i, one simply adds copies to equation 13.) This will not work, however, because $\mathcal{X}_1$ commutes rather than anticommutes with $\mathcal{X}_3$, so an imitation of (26) in the symplectic spinor space would need some additional way to specify that one should take the *anticommutator* of ${S_1}^2$ with ${S_2}^3$ but the *commutator* of ${S_1}^2$ with ${S_2}^1$.

The solution to this problem is to realise that, just as γ_5 anticommutes with all the ordinary gamma matrices, there is a 'γ_5-equivalent' matrix in the symplectic spin space. For a given set of symplectic coordinates x_i, y_i with associated spinors $\mathcal{X}_i, \mathcal{Y}_i$ we can form

$$\mathcal{Z}_i = \left(\frac{\partial}{\partial \mathcal{X}_i} + \mathcal{X}_i\right).\left(\frac{\partial}{\partial \mathcal{X}_i} - \mathcal{X}_i\right).\left(\frac{\partial}{\partial \mathcal{Y}_i} + \mathcal{Y}_i\right).\left(\frac{\partial}{\partial \mathcal{Y}_i} - \mathcal{Y}_i\right). \tag{27}$$

This has the feature that it anticommutes with all the four basic matrices $\mathcal{X}_i, \partial/\partial\mathcal{X}_i, \mathcal{Y}_i$ and $\partial/\partial\mathcal{Y}_i$. Hence insertion of the matrix can help change commutators into anticommutators as desired.

We are then led to define the 'spinorial' contribution to the SU(N) generators as

$$\begin{aligned} {S^S}_i^{\ j} &= \left(\mathcal{X}_i\frac{\partial}{\partial \mathcal{X}_j} + \mathcal{Y}_j\frac{\partial}{\partial \mathcal{Y}_i}\right)\mathcal{Z}_i...\mathcal{Z}_{j-1} \quad \text{if} \quad i<j\,, \\ {S^S}_j^{\ i} &= -\left(\mathcal{X}_j\frac{\partial}{\partial \mathcal{X}_i} + \mathcal{Y}_i\frac{\partial}{\partial \mathcal{Y}_j}\right)\mathcal{Z}_i...\mathcal{Z}_{j-1} \quad \text{if} \quad i<j\,. \end{aligned} \tag{28}$$

By forming the sum ${S^G}_i^{\ j} + {S^S}_i^{\ j}$, and making all commutators of these with each other, we generate the entire algebra of SU(N).

It can easily be seen that these do not commute with the sum ΣH_i of the operators H_i in (24). If we take, for instance, just $H = H_1 + H_2$, the commutator of ${S^S}_1^{\ 2}$ with H_2 will lead to a messy expression which cannot be cancelled by the other terms. The 'natural' thing here would be the anticommutator.

(c) SU(N) Invariant Hamiltonian

Again, the thing to do is to convert some commutators into anticommutators. This can be guaranteed by a slight modification of H. We now choose

$$H = \Sigma_i H_i \mathcal{Z}_1\mathcal{Z}_2....\mathcal{Z}_{i-1}\,. \tag{29}$$

This has the feature that it commutes with all ${S_i}^j$ constructed from the sum of (26) and (28). Hence it commutes with all their commutators, and is an SU(N) invariant.

Furthermore, the 'sub-Hamiltonians' $H_i\mathcal{Z}_1\mathcal{Z}_2...\mathcal{Z}_{i-1}$ *commute* with each other. Hence eigenstates of the entire Hamiltonian may be formed from eigenstates of the individual coordinate Hamiltonian H_i. These were derived in Section 3, where we show they have eigenvalues ± 2, ± 1, and 0.

For two coordinates, consider a product eigenfunction of the form $\psi_T = \psi_1\psi_2$. Action on this of our Hamiltonian $H_1 + H_2\mathcal{Z}_1$ will yield $\lambda_1\psi_T + H_2\mathcal{Z}_1\psi_1\psi_2$. To use the fact that $H_2\psi_2 = \lambda_2\psi_2$, we must 'push' $H_2\mathcal{Z}_1$ through ψ_1. Fortunately, all the eigenstates of H_1 have a definite 'parity' under this operation. For instance, $H_2\mathcal{Z}_1(x_1 + \mathcal{X}_1) = -(x_1 + \mathcal{X}_1)H_2$.

Define P_i to be the 'parity' of ψ_i under commutation with $H_j\mathcal{Z}_i, i \neq j$. Then H as defined in (29) has eigenfunctions $\psi_1\psi_2...$ with eigenvalues

$$\lambda = \lambda_1 + (-1)^{P_1}\lambda_2 + (-1)^{P_1P_2}\lambda_3 + ... \tag{30}$$

We see, therefore, that although the Hamiltonian of (29) may appear rather ugly, its eigenfunctions and eigenvalues are simple to construct.

5. SU(N) Symmetry: Approach II

In the previous section, the SU(N) generators were constructed by first taking generators composed entirely of Grassmannian coordinates and then adding to them ones composed entirely of Grassmannian 'spin'. (The obvious analogy is orbital angular momentum plus spin angular momentum.) This is, however, not the only way to achieve operators which have the commutation relations of SU(N). In this section we display another approach, which combines the Grassmannians and their spins in a different way.

The Hamiltonian operator is the same as in the previous section. Our construction here demonstrates that in fact it has not only SU(N)×Sp(2) invariance, but also SO($4N$) invariance.

Let us begin with the H of (29), for two sets of Grassmannians:

$$\begin{aligned}\mathcal{H} &= H_1 + \mathcal{Z}_1H_2 \\ &= x_1\frac{\partial}{\partial\mathcal{X}_1} + \mathcal{X}_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial\mathcal{Y}_1} + \mathcal{Y}_1\frac{\partial}{\partial y_1} \\ &\quad + \mathcal{Z}_1\left(x_2\frac{\partial}{\partial\mathcal{X}_2} + \mathcal{X}_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial\mathcal{Y}_2} + \mathcal{Y}_2\frac{\partial}{\partial y_2}\right).\end{aligned} \tag{31}$$

The eigenstates are products $\psi_1^i\psi_2^j$ with eigenvalues $\lambda^i + (-1)^{P_1^i}\lambda^j$ where P_1^i is the parity for pushing $H_2\mathcal{Z}_1$ through ψ_1^i.

Now consider the SU(2) generators R_i and the Sp(2) generators S_j defined as follows:

$$\begin{aligned}R_+ &= \frac{1}{\sqrt{2}}\left[\mathcal{X}_1\frac{\partial}{\partial x_2} + \mathcal{Y}_1\frac{\partial}{\partial y_2} + \mathcal{Z}_1\left(x_1\frac{\partial}{\partial\mathcal{X}_2} + y_1\frac{\partial}{\partial\mathcal{Y}_2}\right)\right], \\ R_- &= \frac{1}{\sqrt{2}}\left[x_2\frac{\partial}{\partial\mathcal{X}_1} + y_2\frac{\partial}{\partial\mathcal{Y}_1} + \mathcal{Z}_1\left(\mathcal{X}_2\frac{\partial}{\partial x_1} + \mathcal{Y}_2\frac{\partial}{\partial y_1}\right)\right], \\ R_3 &= \tfrac{1}{2}\left[\mathcal{X}_1\frac{\partial}{\partial\mathcal{X}_1} + \mathcal{Y}_1\frac{\partial}{\partial\mathcal{Y}_1} + x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1}\right. \\ &\quad \left. - \left(\mathcal{X}_2\frac{\partial}{\partial\mathcal{X}_2} + \mathcal{Y}_2\frac{\partial}{\partial\mathcal{Y}_2} + x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2}\right)\right]\end{aligned} \tag{32}$$

and

$$
\begin{aligned}
S_{+} &= \frac{1}{\sqrt{2}}\left[x_1\frac{\partial}{\partial y_1} + x_2\frac{\partial}{\partial y_2} + \mathcal{X}_1\frac{\partial}{\partial \mathcal{Y}_1} + \mathcal{X}_2\frac{\partial}{\partial \mathcal{Y}_2}\right], \\
S_{-} &= \frac{1}{\sqrt{2}}\left[y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} + \mathcal{Y}_1\frac{\partial}{\partial \mathcal{X}_1} + \mathcal{Y}_2\frac{\partial}{\partial \mathcal{X}_2}\right], \\
S_3 &= \tfrac{1}{2}\left[x_1\frac{\partial}{\partial x_1} + \mathcal{X}_1\frac{\partial}{\partial \mathcal{X}_1} + x_2\frac{\partial}{\partial x_2} + \mathcal{X}_2\frac{\partial}{\partial \mathcal{X}_2}\right. \\
&\qquad \left. - \left(y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} + \mathcal{Y}_1\frac{\partial}{\partial \mathcal{Y}_1} + \mathcal{Y}_2\frac{\partial}{\partial \mathcal{Y}_2}\right)\right]. \qquad (33)
\end{aligned}
$$

Not only do we have $[\mathcal{H}, R_i] = 0 = [\mathcal{H}, S_j]$, but also $[R_i, S_j] = 0$. We therefore see that this Hamiltonian has SU(2)×Sp(2) invariance.

Actually, however, $\mathcal{H}$ has a larger invariance group than this. It has SO(8) invariance. This can be seen by explicitly constructing an SO(8) invariant using a trick discussed in Georgi (1984). In this method, sigma matrices for 4 commuting coordinates are used to construct an 8-dimensional vector representation of SO(8):

$$
\begin{aligned}
\Gamma_1 &= \sigma_2^1\sigma_3^2\sigma_3^3\sigma_3^4, \qquad & \Gamma_2 &= -\sigma_1^1\sigma_3^2\sigma_3^3\sigma_3^4, \\
\Gamma_3 &= \sigma_2^2\sigma_3^3\sigma_3^4, & \Gamma_4 &= -\sigma_1^2\sigma_3^3\sigma_3^4, \\
\Gamma_5 &= \sigma_2^3\sigma_3^4, & \Gamma_6 &= -\sigma_1^3\sigma_3^4, \\
\Gamma_7 &= \sigma_2^4, & \Gamma_8 &= -\sigma_1^4. \qquad (34)
\end{aligned}
$$

The matrices $M_{jk} = (1/4i)[\Gamma_j, \Gamma_k]$ have SO(8) commutation relations, and $[M_{jk}, \Gamma_l] = i(\delta_{jl}\Gamma_k - \delta_{kl}\Gamma_j)$ as required for the vector representation.

We now construct mutually commuting sigmas from our Grassmannian coordinates and spin matrices. We use separate 'gamma-5 equivalents' for the y and x coordinates; i.e. $z^y = -1 + 2y\partial/\partial y$ anticommutes with y and $\partial/\partial y$, whereas z^x plays the same role for x. Of course $z = z^y z^x$ anticommutes with all of these.

For the Grassmann spin matrices, therefore, the sigmas are

$$
\begin{aligned}
\Sigma_1^1 &= \left(\frac{\partial}{\partial \mathcal{X}_2} + \mathcal{X}_2\right)\mathcal{Z}_2^{\mathcal{Y}}, \quad & \Sigma_2^1 &= i\left(\frac{\partial}{\partial \mathcal{X}_2} - \mathcal{X}_2\right)\mathcal{Z}_2^{\mathcal{Y}}, \quad & \Sigma_3^1 &= -1 + 2\mathcal{X}_2\frac{\partial}{\partial \mathcal{X}_2}, \\
\Sigma_1^2 &= \left(\frac{\partial}{\partial \mathcal{Y}_2} + \mathcal{Y}_2\right), & \Sigma_2^2 &= i\left(\frac{\partial}{\partial \mathcal{Y}_2} - \mathcal{Y}_2\right), & \Sigma_3^2 &= -1 + 2\mathcal{Y}_2\frac{\partial}{\partial \mathcal{Y}_2}, \\
\Sigma_1^3 &= \left(\frac{\partial}{\partial \mathcal{X}_1} + \mathcal{X}_1\right)\mathcal{Z}_1^{\mathcal{Y}}, & \Sigma_2^3 &= i\left(\frac{\partial}{\partial \mathcal{X}_1} - \mathcal{X}_1\right)\mathcal{Z}_1^{\mathcal{Y}}, & \Sigma_3^3 &= -1 + 2\mathcal{X}_1\frac{\partial}{\partial \mathcal{X}_1}, \\
\Sigma_1^4 &= \left(\frac{\partial}{\partial \mathcal{Y}_1} + \mathcal{Y}_1\right), & \Sigma_2^4 &= i\left(\frac{\partial}{\partial \mathcal{Y}_1} - \mathcal{Y}_1\right), & \Sigma_3^4 &= -1 + 2\mathcal{Y}_1\frac{\partial}{\partial \mathcal{Y}_1}. \qquad (35)
\end{aligned}
$$

And the corresponding sigmas for the Grassmann coordinates are

$$\begin{aligned}
\sigma_1^1 &= \left(\frac{\partial}{\partial x_2} + x_2\right) z_2^y z_1, \qquad \sigma_2^1 = i\left(\frac{\partial}{\partial x_2} - x_2\right) z_2^y z_1, \qquad \sigma_3^1 = -1 + 2x_2\frac{\partial}{\partial x_2},\\
\sigma_1^2 &= \left(\frac{\partial}{\partial y_2} + y_2\right) z_1, \qquad \sigma_2^2 = i\left(\frac{\partial}{\partial y_2} - y_2\right) z_1, \qquad \sigma_3^2 = -1 + 2y_2\frac{\partial}{\partial y_2},\\
\sigma_1^3 &= \left(\frac{\partial}{\partial x_1} + x_1\right) z_1^y, \qquad \sigma_2^3 = i\left(\frac{\partial}{\partial x_1} - x_1\right) z_1^y, \qquad \sigma_3^3 = -1 + 2x_1\frac{\partial}{\partial x_1},\\
\sigma_1^4 &= \left(\frac{\partial}{\partial y_1} + y_1\right), \qquad \sigma_2^4 = i\left(\frac{\partial}{\partial y_1} - y_1\right), \qquad \sigma_3^4 = -1 + 2y_1\frac{\partial}{\partial y_1}.
\end{aligned} \tag{36}$$

Starting with these, the equivalents to the Georgi Γ are

$$\begin{aligned}
G_1 &= \Sigma_2^1\Sigma_3^2\Sigma_3^3\Sigma_3^4, & g_1 &= \sigma_2^1\sigma_3^2\sigma_3^3\sigma_3^4,\\
G_2 &= -\Sigma_1^1\Sigma_3^2\Sigma_3^3\Sigma_3^4, & g_2 &= -\sigma_1^1\sigma_3^2\sigma_3^3\sigma_3^4,\\
G_3 &= \Sigma_2^2\Sigma_3^3\Sigma_3^4, & g_3 &= \sigma_2^2\sigma_3^3\sigma_3^4,\\
G_4 &= -\Sigma_1^2\Sigma_3^3\Sigma_3^4, & g_4 &= -\sigma_1^2\sigma_3^3\sigma_3^4,\\
G_5 &= \Sigma_2^3\Sigma_3^4, & g_5 &= \sigma_2^3\sigma_3^4,\\
G_6 &= -\Sigma_1^3\Sigma_3^4, & g_6 &= -\sigma_1^3\sigma_3^4,\\
G_7 &= \Sigma_2^4, & g_7 &= \sigma_2^4,\\
G_8 &= -\Sigma_1^4, & g_8 &= -\sigma_1^4.
\end{aligned} \tag{37}$$

The quantity $\Sigma_a G_a g_a$ is clearly invariant under commutation with the SO(8) generators $\mathcal{M}_{ij} = M_{ij} + m_{ij}$ where $M_{ij} = (1/4i)[G_i, G_j]$ and $m_{ij} = (1/4i)[g_i, g_j]$. The remarkable result for our purposes is that

$$\Sigma_a G_a g_a = 2\mathcal{H}. \tag{38}$$

The inclusion of further Grassmann variables is obvious. For three different Grassmannian sets, the construction produces a Hamiltonian invariant under SO(12), which contains SU(3)×Sp(2).

6. Grafting on Space–Time

The eventual purpose of this exercise is to tie in the internal degrees of freedom, namely the Grassmann x and p, with the space–time degrees of freedom through an extended Fermi–Dirac equation. Thus the total Dirac operator is to be regarded as some linear combination of $\gamma.P$ and $\mathcal{G}.D$. Because the space–time and Grassmann spin operators commute, it becomes obligatory to include a factor of γ_5 $(=\gamma_0\gamma_1\gamma_2\gamma_3)$ with the fermionic derivatives, in order that the space–time and Grassmannian terms add in quadrature when the overall Hamiltonian (square of our wavefunction operator) is calculated. This leads us to the full Fermi–Dirac equation,

$$(\gamma.P + \mu\gamma_5\mathcal{G}.D - m)\psi = 0, \tag{39}$$

where μ is an arbitrary mass scale factor.

Squaring the complete derivative operator ($\gamma_5^2 = -1$) then produces the mass spectrum,

$$M^2 = m^2 + (\mu\mathcal{G}.D)^2 = m^2 + (n\mu)^2; \quad n = 0, 1, 2\,, \tag{40}$$

with various degeneracies of eigenstates implied. For k additional pairs of Grassmannian variables, with a permutation symmetry invariant Hamiltonian, the mass spectrum will be

$$M^2 = m^2 + (\mu\mathcal{G}.D)^2 = m^2 + \Sigma_{j=1}^{j=k}(n_j\mu)^2; \quad n = 0, 1, 2\,. \tag{41}$$

For k additional pairs of Grassmannian variables with an SU(k) invariant Hamiltonian, the mass spectrum will be

$$M^2 = m^2 + \mu^2(\Sigma_{j=1}^{j=k} n_j)^2; \qquad n_j = \pm 2, \pm 1, 0\,. \tag{42}$$

We see, therefore, that the 'hidden degrees of freedom' in the symplectic spaces have immediate consequences for the mass spectrum. This suggests a tantalising possibility that 'families' of quarks and leptons might be 'explained' in this way. Study of this possibility is currently under way.

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