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On Parametric and Dispersive Integrals

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Abstract

Parametric and dispersive representations of self-energy integrals for particles of arbitrary mass in any dimension look very different. We establish their equivalence explicitly and suggest ways in which the parametric form might prove suitable for tackling Schwinger–Dyson equations in gauge theories.

1. The Problem

We have known for a long time how to write down and evaluate self-energy integrals in relativistic field theories. The most popular method is due to Feynman (1949) and involves introducing 'Feynman parameters' in order to combine propagators and then integrating over (2ℓ-dimensional) momentum space ('tHooft and Veltman 1972), thereby leaving a one-dimensional scalar integral over the parameters; the final parametric representation neatly embodies the singularity properties of the self-energy. Another familiar approach exploits the well-understood singularity structure of the momentum-space integrals (Eden *et al.* 1966; Nakanishi 1970), by constructing a dispersion relation over the absorptive part of the self-energy, which is itself rather easily computed. Both methods are clearly correct, but having said that, an inspection of the dispersive and parametric integrals does not reveal many similarities. In fact they look very different, especially when applied to arbitrary-mass intermediate states and when written in any dimension 2ℓ. This short paper shows how to establish *directly* the relation between the two representations, and discusses the consequences for gauge models at the end.

In order to exhibit the problem, consider the basic self-energy function

$$\Sigma(p^2) \equiv \frac{-\mathrm{i}}{(2\pi)^{2\ell}} \int \frac{\mathrm{d}^{2\ell}k}{[k^2 - m_1^2][(p-k)^2 - m_2^2]}\,. \tag{1}$$

It is a one-loop integral, requiring but a single Feynman parameter α. Combining denominators and integrating over momentum, according to the standard dimensional continuation rules, one arrives at the usual parametric representation of Σ,

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$$\Sigma_{\mathrm{P}}(p^2) = \frac{\Gamma(2-\ell)}{(4\pi)^\ell}\int_0^1 \mathrm{d}\alpha\,[m_1^2\alpha + m_2^2(1-\alpha) - p^2\alpha(1-\alpha)]^{\ell-2}\,, \tag{2}$$

which can be re-expressed as

$$\Sigma_{\mathrm{P}}(p^2) = \frac{(p^2)^{\ell-2}\,\Gamma(2-\ell)}{(4\pi)^\ell}\int_0^1 \mathrm{d}\alpha\,[(\alpha-\alpha_+)(\alpha-\alpha_-)]^{\ell-2}\,, \tag{3}$$

where

$$2p^2\,\alpha_\pm(p^2) = p^2 + m_2^2 - m_1^2 \pm \Delta(p^2, m_1^2, m_2^2),$$

$$\Delta^2(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca\,. \tag{4}$$

On the other hand, the Landau (1959) and Cutkosky (1960) rules tell us that, regarded as a function of p^2, Σ has a branch cut extending from $(m_1+m_2)^2$ to ∞, with a discontinuity given by

$$\begin{aligned}\Im\Sigma(p^2) &= \pi(2\pi)^{-\ell}\int \mathrm{d}^{2\ell}k\,\delta_+[k^2 - m_1^2]\,\delta_+[(p-k)^2 - m_2^2]\\ &= \frac{\pi^{3/2-\ell}\,2^{3-4\ell}\,\Delta^{2\ell-3}}{\Gamma(\ell-\frac{1}{2})(p^2)^{\ell-1}}\,\theta[p^2 - (m_1+m_2)^2]\,,\end{aligned} \tag{5}$$

that can be determined fairly easily by explicit calculation.

This ties in perfectly with the discontinuity due to the integrand of the parametric integral (3), which is nonzero in the range $\alpha_- \le \alpha \le \alpha_+$:

$$\mathrm{Disc}[(\alpha-\alpha_+)(\alpha-\alpha_-)]^{\ell-2} = 2\mathrm{i}\sin[\pi(\ell-2)][(\alpha_+-\alpha)(\alpha-\alpha_-)]\,. \tag{6}$$

Changing variables from α to u, via $\alpha = \alpha_-(1-u) + \alpha_+ u$, one recovers the answer (5) precisely. Assuming that $\ell > 1$, say, one proceeds to write a dispersive representation of the self-energy by making $(\ell-1)$ subtractions* and obtains the dispersive integral for the self-energy,

$$\Sigma_{\mathrm{D}}(p^2) = \mathcal{P}_{[\ell-1]}(p^2) + \int_{(m_1+m_2)^2}^\infty \mathrm{d}s\,\frac{2^{3-4\ell}\,\pi^{1/2-\ell}}{(s-p^2-\mathrm{i}\epsilon)}\,\frac{(p^2/s^2)^{\ell-1}\,\Delta^{2\ell-3}(s)}{\Gamma(\ell-\frac{1}{2})}\,, \tag{7}$$

where $\mathcal{P}$ stands for a polynomial in p^2 of degree $\ell-1$, connected with possible renormalisation terms.

All of this material is extremely well known to anyone who has carried out a one-loop computation, and we have not dwelt on the detailed derivations of (2) and (7) for that reason. Having also reassured ourselves that the absorptive parts of (2) and (7) coincide, we should need no convincing that the dispersive

* Strictly, the integer part of $\ell-1$. These subtractions are associated with a certain number of renormalisation constants.

and parametric representations *must* be strictly equivalent. But declaim as we might, a glance at (2) and (7) shows this statement to be far from obvious. This is the problem we wish to address: the aim of this note is to prove that one can proceed from Σ_{P} to Σ_{D} directly, without first calculating the absorptive part and relying on the analytic properties of Σ.

2. The Solution

Before tackling the general case, we first treat two elementary examples, as they bring in ideas that will be applied later. To begin with, suppose that one particle is massless so that *

$$\Sigma_{\mathrm{P}}(p^2) = \frac{\Gamma(2-\ell)}{(4\pi)^\ell} \int_0^1 \mathrm{d}\alpha \, [\alpha(m^2 - p^2(1-\alpha))]^{\ell-2} \,, \tag{8}$$

$$\Sigma_{\mathrm{D}}(p^2) = \mathcal{P}(p^2) + \int_{m^2}^\infty \mathrm{d}s \, \frac{(s-m^2)^{2\ell-3}}{(s-p^2)} \, 2^{3-4\ell} \, \pi^{1/2-\ell} \, (p^2/s^2)^{\ell-1} \,. \tag{9}$$

To prove that (8) equals (9), let $\xi = p^2/m^2$ and suppose $\ell \gg 2$ initially. (Later we shall continue to arbitrary ℓ.) Integrate (8) by parts n times. The boundary terms in α produce a (specific) polynomial in p^2 of degree n and thus the parametric integral reduces to

$$\Sigma_{\mathrm{P}}(p^2) = \frac{m^{2\ell-4}}{(4\pi)^\ell} \Bigg[\mathcal{P}_n + \frac{\Gamma(2-\ell-n)\,\Gamma(\ell-1)}{\xi^{-n}\,\Gamma(\ell-n-1)} \times \int_0^1 \mathrm{d}\alpha \, (1-\xi+\xi\alpha)^{\,\ell-2-n} \, \alpha^{\ell-2+n} \Bigg] \,. \tag{10}$$

Continue this answer to non-integral $n = \ell - 1$, so that

$$\Sigma_{\mathrm{P}}(p^2) = \frac{m^{2\ell-4}}{(4\pi)^\ell} \left[\mathcal{P}_{\ell-1}(\xi) + \frac{\Gamma(\ell-1)}{\Gamma(2\ell-2)} \int_0^1 \mathrm{d}\alpha \, \xi^{\ell-1} \, \alpha^{2\ell-3}/(1-\xi+\xi\alpha) \right] . \tag{11}$$

Finally, change variable to $s = m^2/(1-\alpha)$ and use the duplication formula for gamma functions to prove that the answer for the parametric integral *exactly* matches (9), up to subtractions terms—which is all we can prove anyway.

For the next exercise, take the two internal propagator masses to be equal, $m_1 = m_2 = m$, whereupon

$$\Sigma_{\mathrm{P}}(p^2) = \frac{\Gamma(2-\ell)}{(4\pi)^\ell} \int_0^1 \mathrm{d}\alpha \, [m^2 - p^2\alpha(1-\alpha)]^{\ell-2} \,, \tag{12}$$

* Hereafter we assume that p^2 contains a positive imaginary part in order to define which side of the cut we are on.

$$\Sigma_{\rm D}(p^2) = \mathcal{P}_{[\ell-1]} + \int_{4m^2}^{\infty} {\rm d}s\, \frac{2^{3-4\ell}\,(p^2/s^2)^{\ell-1}}{\pi^{\ell-\frac{1}{2}}\,\Gamma(\ell-1/2)}\, \frac{s(s-4m^2)^{\ell-3/2}}{(s-p^2)}\,. \tag{13}$$

This time change variable from α to u, via $u = 1$–2α, and again use the abbreviation $\xi = p^2/m^2$. Here the parametric integral simplifies to

$$\Sigma_{\rm P}(p^2) = \frac{m^{2\ell-4}\,\Gamma(2-\ell)}{(4\pi)^\ell} \int {\rm d}u\,[1-\xi(1-u^2)/4]^{\ell-2}\,. \tag{14}$$

To make further progress, consider the generic integral

$$I = \int_0^1 {\rm d}u\,\Gamma(2-\ell)\,[A+Bu^2]^{\ell-2}\,.$$

By carrying out a series of integrations by parts with respect to u, as before, this can be reduced to a boundary polynomial in B plus a residual integral:

$$I = \mathcal{P}_n(B) + \frac{(2B)^n\,\Gamma(2+n-\ell)}{(2n-1)!!} \int_0^1 {\rm d}u\,u^{2n}\,[A+Bu^2]^{\ell-n-2}\,.$$

Next, continue to $n = \ell - 1$, with the interpretation

$$N!! = \Gamma(1+N/2)\,\pi^{-1/2}\,2^{(N+1)/2}\,,$$

in order to reduce the generic integral to

$$I = \mathcal{P}_{\ell-1} + \frac{(2B)^{\ell-1}\,\pi^{\frac{1}{2}}}{\Gamma(\ell-\frac{1}{2})\,2^{\ell-1}} \int_0^1 {\rm d}u\,u^{2\ell-2}/(A+Bu^2)\,.$$

As the last step, convert to the variable s via $u^2 = 1 - 4m^2/s$. In this manner we arrive at the dispersive integral (13) exactly.

We are now in a position to tackle the arbitrary-mass case, but to see what manoeuvres are needed let us initially specialise to $\ell = 1$, when the denominators in p^2 of $\Sigma_{\rm D}$ and $\Sigma_{\rm P}$ are obviously alike:

$$\Sigma_{\rm P}(p^2) = \int_0^1 {\rm d}\alpha/4\pi\,[m_1^2\alpha + m_2^2(1-\alpha) - p^2\alpha(1-\alpha)]\,, \tag{15}$$

$$\Sigma_{\rm D}(p^2) = \int_{(m_1+m_2)^2}^{\infty} {\rm d}s/2\pi\,\Delta(s)\,(s-p^2)\,. \tag{16}$$

By inspection, it is pretty clear that we must make the change of variable from α to s via

$$s = m_1^2/(1-\alpha) + m_2^2/\alpha\,.$$

As the parameter α ranges from 0 to 1, s starts off at ∞, reaches a minimum of $s_0 = (m_1+m_2)^2$ at $\alpha_0 = m_2/(m_1+m_2)$ and then climbs back up to ∞. In fact, solving for α in terms of s, one gets the two branches

$$\alpha_\pm(s) = [s + m_2^2 - m_1^2 \pm \Delta(s)]/2s\,,$$

$$\Delta^2(s) = [s-(m_1+m_2)^2][s-(m_1-m_2)^2]\,. \tag{17}$$

Also worth noting is the differential relation

$$\mathrm{d}s = [m_1^2 - m_2^2 - s + 2\alpha s]\,\mathrm{d}\alpha/\alpha(1-\alpha) = \mp\Delta(s)\,\mathrm{d}\alpha/\alpha(1-\alpha) \tag{18}$$

on the $\alpha_\mp$ branch. Because the limits get swapped, this variable change allows us to discard any terms which are odd in $\Delta(s)$, which will be important in what follows.

If we now increment to $\ell \simeq 2$ and carry out an integration by parts on α, we pick up the usual boundary terms plus a residual integral of the form

$$\int \mathrm{d}\alpha\,(\alpha-\tfrac{1}{2})[(\alpha-\tfrac{1}{2})+(m_1^2-m_2^2)/p^2][m_1^2\alpha+m_2^2(1-\alpha)-p^2\alpha(1-\alpha)]^{\ell-1}\,,$$

which can be simplified to

$$\int \mathrm{d}\alpha\,[\Delta(s)/s]^2/2[m_1^2\alpha+m_2^2(1-\alpha)-p^2\alpha(1-\alpha)]\,,$$

remembering that only even terms in Δ will contribute. For larger values of ℓ we have to make n successive integrations by parts, each time picking up a factor $\Delta(s)/s$ in the integrand. By this means the general parametric integral can be cast in the form

$$\Sigma_{\mathrm{P}}(p^2) = \mathcal{P}_n(p^2) + \frac{(p^2)^{\ell-1}\,\Gamma(2+n-\ell)\,\Gamma(\ell-1)}{(4\pi)^\ell\,\Gamma(\ell+n-1)}\int\frac{\mathrm{d}s}{\Delta(s)}\,\frac{[\Delta(s)/s]^{2n}}{(s/p^2-1)^{2-\ell+n}}\,. \tag{19}$$

Continuing this result to $n=\ell-1$ we end up with

$$\Sigma_{\mathrm{P}}(p^2) = \mathcal{P}_{\ell-1} + \frac{(p^2)^{\ell-2}\,\Gamma(\ell-1)}{(4\pi)^\ell\,\Gamma(2\ell-2)}\int_{(m_1+m_2)^2}^{\infty}\frac{\mathrm{d}s}{s}\,\frac{[\Delta(s)/s]^{2\ell-3}}{(s-p^2)}\,, \tag{20}$$

and upon applying the duplication formula for gamma functions, we can verify that this answer coincides with the dispersive integral (7).

3. Uses

The gauge technique (Salam 1963; Delbourgo 1979)—one of several methods of solving truncations of the Schwinger–Dyson equations in gauge models—employs the Lehmann (1954) dispersive representation of the full source propagator, such as the scalar

$$\Delta(p^2) = \int \mathrm{d}s\, \frac{\rho(s)}{s - p^2}\,. \tag{21}$$

By 'solving' the gauge identities one can obtain longitudinal approximations to various Green functions in terms of the spectral function ρ, which can then be determined self-consistently by substituting them in the field equations. But there are other ways to make the truncations, and there exist other solutions to the identities which better respect multiplicative renormalisability (Curtis and Pennington 1990, 1991). Whatever the advantages of one method over another may be, we can always improve the approximations/truncations by including transverse parts of Green functions (King 1983; Delbourgo and Zhang 1984); however the problem, as ever, is to find simple solutions (Delbourgo 1978) of the higher-point identities without the benefit of the dispersive representation. This is where the present work comes in handy.

We are all aware that, in perturbation theory at least, a parametric representation always exists. For example, in spinor electrodynamics the fermion self-energy to order e^2 is given by

$$\Sigma(p) = \frac{2\Gamma(2-\ell)}{(4\pi)^\ell} \int_0^1 \mathrm{d}\alpha\, [\gamma.p(1-\alpha)(\ell-1) - \ell m][m^2\alpha - p^2\alpha(1-\alpha)]^{\ell-2}\,. \tag{22}$$

This suggests that a parametric form of the propagator

$$S(p) = \int_0^1 \mathrm{d}\alpha\, \rho(\alpha)/[\gamma.p - m/\alpha]^{2-\ell} \tag{23}$$

may provide a better route for solving the Green-function gauge identities. This becomes pertinent when one tries to obtain a solution for the four-point function $G_{\mu\nu}$ in terms of the three-point function G_μ,

$$k'^{\mu}\, G_{\nu\mu}(p', p; k', k) = G_\mu(p+k, p) - G_\mu(p', p'-k)\,, \tag{24}$$

since the latter function contains a vertex which admits the parametric form

$$\Lambda_\mu(p', p) = \int \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma\, \delta(\alpha+\beta+\gamma-1)\, \chi_\mu(\alpha, \beta, \gamma, p, p')$$
$$\times\, [p'^2\alpha\beta + p^2\alpha\gamma + (p-p')^2\beta\gamma - m^2(1-\alpha)]^{\ell-3}\,,$$

where the polynomial χ can certainly be evaluated in perturbation theory. The crucial point is that it might be possible to solve the gauge identity for the longitudinal four-point function $G_{\mu\nu}$ self-consistently for the three-point spectral function χ, via the equation set

$$G_\mu(p', p)\,(\gamma.p - m_0) = S(p')\,\gamma_\mu - \frac{\mathrm{i}e^2}{(2\pi)^\ell} \int \mathrm{d}^{2\ell}k\, G_{\mu\nu}(p', p-k; p'-p, k)\,\gamma_\rho\, D^{\nu\rho}(k)\,,$$

$$(\gamma.p' - m_0)\, G_\mu(p',p) = \gamma_\mu\, S(p) - \frac{\mathrm{i}e^2}{(2\pi)^\ell} \int \mathrm{d}^{2\ell}k\, \gamma_\rho\, D^{\nu\rho}(k)\, G_{\mu\nu}(p'+k,p';p'-p,k)\,,$$

in other words for the *complete* vertex. This is likely to be a fruitful avenue for future research.

It is even conceivable that if we had a better idea about the singularity structure of confined particle propagators, like quarks and gluons, we might be able to give those sources a parametric representation too, and extend the technique to cover that case. For now this is just a gleam in the eye.

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