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Mott–Schwinger Effect in the Elastic Scattering of Neutrons from ^{209}Bi

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Abstract

The Mott–Schwinger potential arising from the interaction of the magnetic moment of a neutron incident upon an electric field (of ^{209}Bi) is found to have a profound effect upon the elastic scattering cross sections and polarisations at 0·5, 14·5 and 24 MeV. These effects are evaluated by specific solution of the relevant Schrödinger equations (for 100 partial waves) and with the Born approximation used to define the influence upon all higher ones. These 'exact' results agree with the estimations made under approximation in the past, but not with 'exact' results calculated by a different method.

1. Introduction

The influence of the Mott–Schwinger (MS) effect in neutron elastic scattering from nuclei is investigated, and its effect upon the differential cross sections and polarisations is considered in particular. The MS effect is the result of an (electromagnetic) interaction between the magnetic moment of a moving neutron and the electric field of the target nucleus, which gives rise to a spin–orbit potential having $1/r^3$ radial dependence. Previous studies (Mott 1929; Schwinger 1948; Sample 1956; Eriksson 1957; Monahan and Elwyn 1964; Hogan and Seyler 1969) have shown that this MS interaction can have a marked effect upon nucleon scattering observables at small angles and at all energies (to 130 MeV). Consequently it may also be an important effect in charge exchange (p, n) scattering at 0°, the cross sections and spin depolarisations from which have been used to define Gamow–Teller strengths (Goodman and Bloom 1982). These strengths are the weights given to the Gamow–Teller operator which mediates the beta decay in nuclei, and they are related to the value of the axial vector coupling constant g_A as well. To date, all analyses of the 0° (p, n) cross section have not included the Mott–Schwinger interaction and, therefore, estimates of the Gamow–Teller strengths found using them may be in serious error.

The influence of the electromagnetic interaction between the spin magnetic moment of a projectile and the electric field of the target nucleus was first recognised by Mott (1929) when he observed its effect upon the polarisation of electrons scattered from nuclei. Twenty years later, Schwinger (1948) noted the importance of the same interaction in his assessment of the scattering of neutrons

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from nuclei. He used the Born approximation to determine that the scattering amplitude due to that electromagnetic interaction alone is

$$f_{sc}(\theta) = -i\left(\frac{Ze^2\mu_n}{2mc^2}\right)\cot(\tfrac{1}{2}\theta)\,\sigma\cdot\mathbf{n}\,, \tag{1}$$

where the normal vector is defined by

$$\mathbf{n} = \frac{\mathbf{k_0}\times\mathbf{k}}{k^2\sin\theta}\,. \tag{2}$$

He also estimated the effect of the interaction upon the polarisation of fast neutrons and noted that there was a well-defined signature with a sharp negative peak at small scattering angles. The importance of those studies was to find a mechanism for the production of a strongly polarised beam of fast neutrons. In a later study, Sample (1956) treated the Mott–Schwinger (MS) potential, as it was called by then, as a first-order perturbation to the scattering. Using hard-sphere wavefunctions as a basis, he confirmed the effect of the MS interaction in producing a negative peak polarisation at small scattering angles. Likewise, Eriksson (1957) found that the MS effect has a noticeable effect upon the polarisation for $\theta < 10°$ from the elastic scattering of 130 MeV protons from nuclei. It is also of note that the MS interaction was included in the recent optical model analyses of low-energy cross sections and polarisations of neutrons scattered elastically from ^{208}Pb (Roberts *et al.* 1991), but no specific details of how that effect was handled or of the exact effects of the MS term in the data analyses were given.

Computer advances allowed studies subsequent to those of Schwinger (1948), Sample (1956) and Eriksson (1957) to be more sophisticated. Continued advances have led to our investigations. But two studies of the past, namely those of Monahan and Elwyn (1964) and of Hogan and Seyler (1969), are of significance. As we use aspects of the techniques developed in those papers, a brief review of those methods is given in the next section.

In Section 3, we present a brief discussion of the Schrödinger equations and their solutions for the scattering of spin $\frac{1}{2}$ particles from a central field that also includes a spin–orbit potential. The non-spin-flip and spin-flip scattering amplitudes and their specification in partial wave series are given preparatory to delineation of the effects of including the long-range MS (spin–orbit) interaction. The standard optical model potential that was used in the calculations is then given in Section 3*a* and the MS interaction is defined in Section 3*b*. The actual calculated phase shifts and how the scattering amplitudes can be formed are considered in Section 3*c*.

The results of our calculations and a discussion of them are presented in Section 4 and the conclusions we have drawn are given in Section 5.

2. Previous Analyses of the MS Effect on Scattering

Two previous studies on the effects of the MS interaction are reviewed in this section. In the first (Monahan and Elwyn 1964), a formalism was set up that circumvented numerical problems that otherwise made their calculations intractable. These difficulties were associated with the long-range nature of the potential, specifically the fact that a very large number of partial waves contribute. Further, to calculate phase shifts accurately by numerical solution of the Schrödinger equations, a very large matching radius is required. In fact matching radii in the region of the electron cloud are needed to ensure accurate numerical values. Also, the contributions from very large partial waves, small as any may be, are such that this spin–orbit MS potential alone requires one to sum to infinity the scattering partial wave amplitudes to define accurately the complete scattering amplitude at very small scattering angles. Monahan and Elwyn circumvented these difficulties by using a Born approximation to evaluate MS potential corrections to the nuclear potential scattering phase shifts. They considered the total scattering potential as

$$V(r) = V_1(r) + V_2(r)\,, \tag{3}$$

where

$$\begin{aligned} V_1(r) &= V_{nuc} + V_{ms};\ V_2(r) = 0 \qquad && r < r_c \\ V_1(r) &= 0;\ V_2(r) = V_{ms} && r > r_c\,, \end{aligned} \tag{4}$$

for a cutoff radius r_c. Then they used the integral form of the radial wavefunction for total angular momentum j, viz.

$$\begin{aligned} \psi_j(r) = j_l(kr)&\left[A_j(r_c) - k\int_{r_c}^{r} dx\; x^2 U_j(x) n_l(kx)\psi_j(x)\right] \\ &+ n_l(kr)\left[B_j(r_c) + k\int_{r_c}^{r} dx\; x^2 U_j(x) j_l(kx)\psi_j(x)\right], \end{aligned} \tag{5}$$

where

$$U_j(x) = \frac{2m}{\hbar^2} V(r)\langle \mathbf{l}\cdot\sigma\rangle\,. \tag{6}$$

At the nuclear cutoff, $\psi_j(r_c)$ satisfies

$$\psi_j(r_c) = A_j(r_c) j_l(kr_c) + B_j(r_c) n_l(kr_c)\,. \tag{7}$$

The Born approximation is used by replacing the wavefunction within the integrals with

$$\psi_j(r) = A_j(r) j_l(kr) + B_j(r) n_l(kr)\,. \tag{8}$$

Then, with ζ_j being the phase shifts found by matching the numerically obtained logarithmic derivatives at r_c to the free particle ones, the full phase shifts can be deduced from

$$\tan\delta_j = \frac{(1+b_j)\tan\zeta_j - a_j}{(1-b_j) + c_j\tan\zeta_j}\,, \tag{9}$$

where the parameters a_j, b_j and c_j are defined by

$$a_j = k \int_{r_c}^{\infty} U_j(x) j_l^2(kx) x^2 dx\,, \tag{10}$$

$$b_j = k \int_{r_c}^{\infty} U_j(x) j_l(kx) n_l(kx) x^2 dx\,, \tag{11}$$

$$c_j = k \int_{r_c}^{\infty} U_j(x) n_l^2(kx) x^2 dx\,. \tag{12}$$

These integral expressions for a_j, b_j and c_j are analytic for the MS potential.

Using this scheme with a phenomenological Woods–Saxon optical potential for V_{nuc}, Monahan and Elwyn calculated the polarisation at $\theta = 24°$, 86° and 118°, for 0·3 to 0·9 MeV neutrons elastically scattered from Zr, Nb, Mo and Cl. At 86° and 118° the polarisations calculated with and without the MS interaction were in reasonable agreement with each other and with the measured values. However at 24°, the measured polarisations were consistently more negative than either calculated values. But the MS effect was most important in this case, accounting for a substantial part of the polarisation measurements at 24° .

The second major study of the time was that made by Hogan and Seyler (1969) who used a method that was based on the work of Calogero (1963). They made calculations of the cross section and polarisations for the elastic scattering of 0·5, 1·0, 7·0, 14·5 and 24·0 MeV neutrons from Al, Mn, Nb and Bi. This method used the equivalent integral representation of the Schrödinger equation,

$$\psi_l(\rho) = j_l(\rho) - \int_0^{\rho} [j_l(\rho) n_l(\rho') - n_l(\rho) j_l(\rho')]\; U(\rho')\psi(\rho')\rho'^2 d\rho'\,, \tag{13}$$

and a solution for $\psi_l(\rho)$ of the form

$$\psi_l(\rho) = C_l(\rho) j_l(\rho) - S_l(\rho) n_l(\rho)\,. \tag{14}$$

Hogan and Seyler found two coupled differential equations for $C_l(\rho)$ and $S_l(\rho)$, namely

$$C_l'(\rho) = \; - U(\rho)\rho^2 n_l(\rho)\; [C_l(\rho) j_l(\rho) - S_l(\rho) n_l(\rho)]\,, \tag{15}$$

$$S_l'(\rho) = \; - U(\rho)\rho^2 j_l(\rho)\; [C_l(\rho) j_l(\rho) - S_l(\rho) n_l(\rho)]\,. \tag{16}$$

For the MS potential, specifically

$$C_l'(\rho) = \; - \frac{\lambda_{l,j}}{\rho} [C_l(\rho) n_l(\rho) j_l(\rho) - S_l(\rho) n_l^2(\rho)]\,, \tag{17}$$

$$S_l'(\rho) = \; - \frac{\lambda_{l,j}}{\rho} [C_l(\rho) j_l^2(\rho) - S_l(\rho) n_l(\rho) j_l(\rho)]\,, \tag{18}$$

where

$$\lambda_{l_j} = \frac{Ze^2\mu_n k}{mc^2} l \qquad \text{if } j = l + \tfrac{1}{2}$$

$$= -\frac{Ze^2\mu_n k}{mc^2}(l+1) \qquad \text{if } j = l - \tfrac{1}{2}\,. \tag{19}$$

From this, Hogan and Seyler were able to obtain an exact formula for the asymptotic phase shifts in terms of the phase shifts evaluated at the nuclear cutoff ρ_c, where only the MS potential is significant. Hogan and Seyler noted that the partial wave summation that gives the non-spin-flip scattering amplitude $g(\theta)$ converged quickly, but that for the spin-flip one $h(\theta)$ did not. To overcome the slow convergence, they employed a method that had been used by Sample (1956) to turn an infinite sum into an expression involving a $\cot\frac{1}{2}\theta$ term. We use the same closure summation.

The major findings of the Hogan–Seyler study were that: (1) the effect upon the differential cross section is a strong divergence at angles less than $\approx 2°$; (2) polarisations have a characteristic negative peak at small angles; (3) the polarisation is affected observably at much larger scattering angles than previous studies had shown, although primarily at angles where minima occur in the cross section (4) for the energies and nuclei considered, the MS influence tended to be larger with larger Z and smaller energies and (5) the use of the Born approximation to compute the part of the phase shift due to the MS potential was entirely adequate for all cases considered.

Our computation facilities to solve the radial Schrödinger equations are such that it is feasible to evaluate explicitly and accurately the logarithmic derivatives of the scattering wavefunctions for the standard nuclear-plus-MS potentials for very large partial wave numbers and extreme matching radii. Thus, while the infinite sum approach of Hogan and Seyler is still used in these calculations, the maximum l values and the matching radii are taken as extreme to check details of the analyses to an accuracy far better than any in the past, and with fewer approximations.

3. Potential Scattering with Spin $\frac{1}{2}$ Projectiles: The Mott–Schwinger Interaction in Particular

The Schrödinger equation for scattering by a central potential including a spin–orbit interaction is

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V_0(r) + V_{so}(r)\mathbf{l}\cdot\sigma - E\right]\Psi_{k\nu}(\mathbf{r}) = 0\,, \tag{20}$$

for which it is convenient to expand the solutions in partial wave form using the generalised spherical harmonics that are eigenfunctions of the spin–orbit operator. Specifically (with $E = \hbar^2k^2/2\mu$), that expansion is

$$\Psi_{k\nu}(\mathbf{r}) = \sum_{lj}\sqrt{4\pi(2l+1)}\; i^l \frac{1}{r} u_{lj}(kr)\mathcal{Y}^m_{l\frac{1}{2}j}\,, \tag{21}$$

but to be physical it must have an asymptotic form

$$\Psi_{k\nu}(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} e^{ikz}\chi_{\frac{1}{2}\nu} + \frac{e^{ikr}}{r}\sum_{\mu} f_{\mu\nu}(\theta_{sc})\chi_{\frac{1}{2}\mu} \tag{22}$$

by which spin-flip events are allowed. This partial wave expansion of the Schrödinger equation leads to the sets of radial equations

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} - U_0(r)\begin{pmatrix} +lU_{so}(r) \\ -(l+1)U_{so}(r)\end{pmatrix}\right] u_{lj}(kr) = 0 \tag{23}$$

for $j = l \pm \frac{1}{2}$ respectively.

The standard matching of the logarithmic derivatives of the actual solutions of the radial equations to those of the asymptotic forms specify the phase shifts in terms of which the non-spin-flip scattering amplitudes $g(\theta)$ and the spin-flip ones $h(\theta)$ are given by

$$g(\theta) = f_{\nu\nu}(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}[(l+1)e^{2i\delta_l^{(+)}} + le^{2i\delta_l^{(-)}} - (2l+1)]\ P_l(\theta)\,, \tag{24}$$

and with $\nu \neq \mu$,

$$h(\theta) = (-)^{\frac{1}{2}-\nu} if_{\nu\mu}(\theta) = \frac{1}{2k}\sum_{l=0}^{\infty}[e^{2i\delta_l^{(+)}} - e^{2i\delta_l^{(-)}}]\ P_l^1(\theta) \tag{25}$$

when, as is customary, the scattering plane is taken as the x–z plane with the z axis being the beam direction. The cross sections are then defined by

$$\frac{d\sigma}{d\Omega} = |g(\theta)|^2 + |h(\theta)|^2\,, \tag{26}$$

and the polarisations by

$$P(\theta) = \frac{2\Re(g^*(\theta)h(\theta))}{d\sigma/d\Omega}\,. \tag{27}$$

It is also known that other possible measurements can be made that will reflect other characteristic products (Wolfenstein 1956; Feshbach 1960). Specifically those functions are

$$X(\theta) = \frac{2\Im(h^*(\theta)g(\theta))}{d\sigma/d\Omega}\,, \tag{28}$$

$$Y(\theta) = -\frac{|h(\theta)|^2}{d\sigma/d\Omega}\,. \tag{29}$$

In fact the X variable is exactly the spin rotation function $Q(\theta)$, while a combination of X, Y, and P define the results one can obtain with triple scattering experiments such as $S(\theta)$; both measurables as specified by Glauber and Osland (1979).

(3a) Basic Optical Model Potential

Most nucleon–nucleus scattering has been analysed by means of optical model interactions—a central-field one-body scattering problem. While there have been attempts recently to define those potentials from 'first principles', i.e. by folding NN interactions with a density distribution, most analyses have been made using a totally phenomenological form of the interaction. Nevertheless, that phenomenological optical model potential reflects both the matter distribution in the nucleus and the character of the hadronic interactions between the constituent nucleons of the target and the projectile nucleon, and so will be a short-range quantity. The usual potentials are taken to be of the form

$$V_{om}(r) = V_c(r) - V_0 f(r, r_0, a_0) - iW_0 f(r, r_w, a_w) - i4W_d f'(r, r_d, a_d) + (V_{so} + iW_{so})\left(\frac{\hbar}{m_\pi c}\right)^2 \frac{1}{r} f'(r, r_{so}, a_{so}), \qquad (30)$$

where $f(r, r_x, a_x)$ are Woods–Saxon functions. In the studies to be made, W_0 and W_{so} will both be set to zero, and the Coulomb potential, $V_c(r)$ is, as usual, that due to a uniformly charged sphere of radius R_c ($\approx 1\cdot 3A^{\frac{1}{3}}$), viz.

$$\begin{aligned} V_c(r) &= \frac{Ze^2}{2R_c}\left(3 - \frac{r^2}{R_c^2}\right), && \text{if } r < R_c \\ &= \frac{Ze^2}{r}, && \text{if } r \geq R_c . \end{aligned} \qquad (31)$$

While the optical potential so specified has been used with considerable success in many analyses of elastic scattering data, special circumstances exist for which it is very limited. Very forward angle scattering is just such a circumstance. Correction terms must be taken.

The parameter values of the optical potential used in the present calculation are those used by Hogan and Seyler (1969), specifically

$$\begin{aligned} &V_0 = 47\cdot 20 - 0\cdot 27E \text{ MeV}, && W_d = 9\cdot 6 \text{ MeV}, && V_{so} = 7\cdot 2 \text{ MeV}, \\ &a_0 = a_d = 0\cdot 66 \text{ fm}, && a_s = 0\cdot 47 \text{ fm}, && r_0 = r_w = r_{so} = 1\cdot 27 \text{ fm}. \end{aligned} \qquad (32)$$

(3b) Mott–Schwinger Interaction

By taking the nonrelativistic limit of the Dirac equation for a neutron in an electromagnetic field, one finds a potential term of the form $\boldsymbol{\sigma}\cdot\mathbf{B}$ in the resulting Hamiltonian. Just as a point charge is subject to a force when it moves in a magnetic field, so a moving magnetic moment, such as the intrinsic neutron moment is envisaged to be, is also acted on by forces when that neutron moves in an electric field. If the electromagnetic interaction for a neutron incident upon a nucleus is identified with the (classical) potential energy of orientation of a magnetic dipole in a magnetic field, i.e.

$$V = \mathbf{m} \cdot \mathbf{B}\,, \tag{33}$$

one thus has

$$\mathbf{m} = -\frac{\mu_n e\hbar}{2mc}\boldsymbol{\sigma}\,, \tag{34}$$

$$\mathbf{B} = \frac{\mathbf{v}}{c} \times \mathbf{E}\,. \tag{35}$$

Then as

$$\mathbf{E} = \frac{Ze}{r^3}\mathbf{r} \longrightarrow \mathbf{B} = \frac{Ze}{c}\,\frac{\mathbf{v}\times\mathbf{r}}{r^3}\,, \tag{36}$$

by making the operator connection between quantum and classical angular momentum, i.e. $(\mathbf{r} \times m\mathbf{v} \rightarrow \hbar L)$, we get

$$\mathbf{B} = -\frac{Ze\hbar}{mcr^3}L\,.$$

Hence the MS correction to the Hamiltonian takes the form

$$V_{MS} = \frac{\mu_n Z e^2 \hbar^2}{2m^2c^2}\,\frac{\boldsymbol{\sigma}\cdot\mathbf{L}}{r^3} = \zeta\frac{\mathbf{L}\cdot\mathbf{s}}{r^3}\,, \tag{38}$$

where we define

$$\zeta = \frac{\mu_n Z e^2 \hbar^2}{m^2c^2}\,, \tag{39}$$

and so we have an element additional to the standard optical model potential and it varies the spin–orbit components. There is an inherent divergence with $1/r^3$ at the origin, but as the probe enters the nucleus the effective field is reduced by the decreasing effective charge, as is the case for the pure Coulomb interaction, so that the effective MS interaction will not diverge. Further, as the net strength of the MS potential (ζ) is quite small, it is not a problem to take the MS strength to be a constant within the Saxon radius, $\approx 1{\cdot}3A^{\frac{1}{3}}$ (Monahan and Elwyn 1964). This has been done in the present calculations.

(3c) Phase Shifts and Their Calculation

As demonstrated, the phase shifts are the essential quantities to specify observables. The phase shifts are obtained by matching the logarithmic derivatives of (internal) radial solutions of the appropriate Schrödinger equations to those of the known external (asymptotic) solutions. The radial equations resulting after the appropriate partial wave expansions are

$$\left[-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+\frac{l(l+1)}{r^2}+\frac{2m}{\hbar^2}V^{(\pm)}(r)-k^2\right]u_{l,k}(r)=0\,, \tag{40}$$

and which tend as

$$\begin{aligned} u_{l,k}^{(\pm)}(r) \underset{r\to\infty}{\to} & A_l^{(\pm)} j_l(kr) + B_l^{(\pm)} n_l(kr) \\ & = C_l \frac{1}{kr}\sin(kr - \tfrac{1}{2}l\pi + \delta_l^{(\pm)})\,, \end{aligned} \tag{41}$$

where $j_l(kr)$ and $n_l(kr)$ are the spherical Bessel and Neumann functions respectively. The $B_l^{(\pm)}$ are measures of the amount of scattering, and phase shifts are related to those numbers by

$$\tan\delta_l^{(\pm)} = -B_l^{(\pm)}/A_l^{(\pm)}\,. \tag{42}$$

Normally, the number of l values for which the phase shifts are significantly different from 0 and the radii at which internal to external (asymptotic) solution matching can be done are not excessive. But the MS interaction is long-range and so influences many partial waves. This can be understood also semiclassically as one can relate the impact parameter b to the angular momenta l and wave vector k by

$$b \sim l/k\,. \tag{43}$$

Hence for low energies and high l the impact parameter is very large. To evaluate asymptotic values one must then match wavefunctions at extreme radii. Despite the numerical difficulties of using 1000 partial waves and a matching radius of the order of 10^5 fm, computers make such calculations feasible. But, even so, the series for $h(\theta)$ does not converge rapidly enough. The saving grace is that for large partial waves, only the MS potential makes the phase shifts differ from zero and the Born approximation can be used for them. The Born approximation for the phase shifts is that, in the integral equation

$$\sin\delta_l^{(\pm)} = -k\int_0^\infty j_l(kr)\ \frac{2m}{\hbar^2}\,V_{ms}^{(\pm)}(r)u_{lk}^{(\pm)}(r)r\,dr\,, \tag{44}$$

the radial solutions are replaced by the free ones, viz.

$$u_{lk}^{(\pm)}(r) \approx r\ j_l(kr)\,. \tag{45}$$

Then, with the MS potential taken as

$$V_{ms}^{(\pm)}(r) = \zeta_l^{(\pm)}\frac{1}{r^3}\,, \tag{46}$$

where the scale includes the expectation of the associated $\mathbf{L}\cdot\mathbf{s}$ operator, i.e.

$$\begin{aligned} \zeta^{(\pm)} &= \tfrac{1}{2}\zeta\ l, && \text{if } \ j = l+\tfrac{1}{2} \\ &= -\tfrac{1}{2}\zeta\ (l+1), && \text{if } \ j = l-\tfrac{1}{2}, \end{aligned} \tag{47}$$

the Born approximation requires evaluation of

$$\sin\delta_l = -\frac{2mk}{\hbar^2}\zeta_l^{(\pm)} \int_0^\infty \frac{j_l^2(kr)}{r}dr\,.$$

The integral is analytic (Gradshteyn and Ryzhik 1980) as

$$\int_0^\infty \frac{J_\mu^2(kr)}{r^\lambda}\ dr = \frac{k^{\lambda-1}\,\Gamma(\lambda)\,\Gamma(\frac{1}{2}(2\mu-\lambda+1))}{2^\lambda\,\Gamma^2(\frac{1}{2}(\lambda+1))\,\Gamma(\frac{1}{2}(2\mu+\lambda+1))}, \tag{49}$$

with

$$j_l(kr) = \sqrt{\frac{\pi}{2kr}}J_{l+\frac{1}{2}}(kr)\,, \tag{50}$$

we then have

$$\int_0^\infty \frac{j^2_{\mu-\frac{1}{2}}(kr)}{r^{\lambda-1}}\ dr = \frac{\pi k^{\lambda-2}\,\Gamma(\lambda)\,\Gamma(\frac{1}{2}(2\mu-\lambda+1))}{2^{\lambda+1}\,\Gamma^2(\frac{1}{2}(\lambda+1))\,\Gamma(\frac{1}{2}(2\mu+\lambda+1))}\,. \tag{51}$$

For the particular case $\mu = l+\frac{1}{2}$ and $\lambda = 2$, the integral reduces to

$$\int_0^\infty \frac{j_l^2(kr)}{r}dr = \frac{1}{2l(l+1)}, \tag{52}$$

so that

$$\sin\delta_l^+ = -\frac{mk\zeta}{2\hbar^2(l+1)} \qquad \sin\delta_l^- = \frac{mk\zeta}{2\hbar^2 l}, \tag{53}$$

and for small values of the phase shifts, such as $l > l_{max}$, the scale factors in the partial wave summations vary as

$$\begin{aligned} g(\theta) &:\to 2i[(l+1)\delta_l^{(+)} + l\delta_l^{(-)}] = 0\,, \\ h(\theta) &:\to 2i[\delta_l^{(+)} - \delta_l^{(-)}] = -i\frac{mk\zeta}{\hbar^2}\,\frac{(2l+1)}{l(l+1)}\,. \end{aligned} \tag{54}$$

As an aside, it is clear from the Born approximation that, for small values of the phase shifts (large l values usually), as $\sin\delta_l \to \delta_l$, potentials varying as r^{-n}, when used with equation (51), give phase shifts that vary as $l^{(-n+1)}$.

Thus the summations over partial waves for $g(\theta)$ decrease rapidly with large l and inclusion of the MS interaction into the standard calculation of scattering phase shifts does not require excessive values of the maximum l. The non-spin-flip amplitudes with or without the MS interaction involved are then defined by the limited sum

$$g(\theta) = \frac{1}{2ik} \sum_{l=0}^{l_{max}} [(l+1)e^{2i\delta_l^{(+)}} + le^{2i\delta_l^{(-)}} - (2l+1)]\ P_l(\cos\theta)\,. \tag{55}$$

But this is not the case with the evaluation of the $h(\theta)$ amplitudes. In the partial wave summation for them, the weighting factors to the associated Legendre polynomials decrease only as l and so that sum is very slowly convergent. However, the Born series closes. Specifically, and as shown in the Appendix,

$$\cot \tfrac{1}{2}\theta = \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)}\ P_l^1(\theta)\,, \tag{56}$$

so that one can consider the expansion

$$h(\theta) = \frac{1}{2k} \sum_{l=1}^{l_{max}} \left[e^{2i\delta_l^{(+)}} - e^{2i\delta_l^{(-)}} + i\frac{mk\zeta}{\hbar^2}\ \frac{(2l+1)}{l(l+1)} \right] P_l^1(\theta) - i\frac{m\zeta}{2\hbar^2} \cot \tfrac{1}{2}\theta\,. \tag{57}$$

Although the maximum l value required for these summations may not be excessive (this is considered in the later discussion), it is to be stressed that the phase shifts involved are those in which the MS interaction has had effect. Thus the logarithmic matching to give those phase shifts by numerical solution of the radial Schrödinger equations must be taken at extreme distance to ensure that accurate values are used. Indeed, as the determining condition is that $1/r^3$ is small in comparison with $l(l+1)/r^2$, one need solve for matching radii of 5000 or more fermi.

Calculations were performed using matching radii of differing (large) sizes and for the three different energies at which scattering of neutrons from ^{209}Bi was considered. The result was that the matching radii used in the calculations were the ones required not only to ensure that each calculation of the scattering gave convergent results, but also to ensure that the matching to the Born approximation estimates revealed just what maximum value of l is needed in each case in the limited sums that specify the scattering amplitudes in our method. It was found that by $l = 100$, all cases match to the Born estimate to five significant figures.

Diverse additional calculations were made to ensure that the choice of l_{max} in the summations did not affect the end results. For the energies considered, these summations were stable to one part in 10^6. The integration of the radial Schrödinger equations were repeated for various step sizes, as well as for various (extreme) matching radii to ensure an accuracy of one part in 10^4 for all partial waves to $l = 100$. In fact halving the step radius varied our results only in the seventh decimal.

The matching radii we have used extend to the region of the electron cloud of the atom and so one might contemplate that the screening effect of the electrons should be included in the calculations. But we seek an accurate solution of the nuclear problem with the low-value partial waves ($l \leq l_{max}$) only. The very large matching radii are required to give accurate values of those partial wave phase shifts, while electron screening is of importance at very large partial wave numbers. The actual electron screening, leading to the problem of neutron scattering from the atom, is an interesting topic in its own right, but the effects are small and influence essentially the way the phase shift values vary for the very large partial waves. That makes the infinite sum needed in the specification of $h(\theta)$ converge slightly better, with the result that very forward scattering does not diverge as $\cot\frac{1}{2}\theta$. But the influence is only at extremely small scattering angles (Monahan and Elwyn 1964) and will not influence in any significant way the results that are of specific interest to us.

4. Small-angle Expansions

Qualitatively, in the differential cross section calculations the prominent effect of the MS interaction occurs at small scattering angles, arising from the very high (∞) number of partial waves that are influenced by that potential. Essentially, the infinite sum in the Born approximation for $h(\theta)$ gives a $\cot\frac{1}{2}\theta$ dependence. But as the $g(\theta)$ summations generally give much larger values than those defining the $h(\theta)$ terms, the differential cross section is not significantly altered by the MS potential (except at very forward scattering angles). On the other hand, as the scattering angle tends to zero, $h(\theta)$ varies as $-\cot\frac{1}{2}\theta$ and the cross section varies as $\cot^2\frac{1}{2}\theta$, and thus the polarisation rapidly vanishes with small decreasing angle. But the balance of the MS and nuclear (spin–orbit) interaction effects give rise to a characteristic (negative) peak in the polarisations at small angles.

This characteristic peak can be understood from the MS effect in the scattering amplitudes by using small-angle expansions for

$$P_l(\theta) \underset{\theta\to 0}{\longrightarrow} 1 - \tfrac{1}{4}l(l+1)\theta^2\,,$$

$$P_l^1(\theta) \underset{\theta\to 0}{\longrightarrow} \tfrac{1}{2}l(l+1)\theta\,,$$

$$\cot(\tfrac{1}{2}\theta) \underset{\theta\to 0}{\longrightarrow} 2/\theta\,. \tag{58}$$

Then, with an appropriate limit l_{max} to every summation, and using

$$S_l^{(\pm)} = e^{2i\delta^{(\pm)}} = \Re_l^{(\pm)} + i\Im^{(\pm)}\,, \tag{59}$$

one finds that

$$g(\theta) \underset{\theta\to 0}{\longrightarrow} (a_1 + a_2\theta^2) + i(b_1 + b_2\theta^2)\,, \tag{60}$$

$$h(\theta) \underset{\theta\to 0}{\longrightarrow} c_1\theta + i(c_2\theta - c_3/\theta)\,, \tag{61}$$

where

$$a_1 = \frac{1}{2k} \sum_{l=0}^{l_{max}} [(l+1)\Im^{(+)} + l\Im^{(-)}] \,,$$

$$a_2 = -\frac{1}{8k} \sum_{l=0}^{l_{max}} l(l+1)[(l+1)\Im^{(+)} + l\Im^{(-)}] \,,$$

$$b_1 = \frac{1}{2k} \sum_{l=0}^{l_{max}} [(2l+1) - (l+1)\Re^{(+)} - l\Re^{(-)}] \,,$$

$$b_2 = -\frac{1}{8k} \sum_{l=0}^{l_{max}} l(l+1)[(2l+1) - (l+1)\Re^{(+)} - l\Re^{(-)}] \,,$$

$$c_1 = \frac{1}{4k} \sum_{l=1}^{l_{max}} l(l+1)[\Re^{(+)} - \Re^{(-)}] \,,$$

$$c_2 = \frac{1}{4k} \sum_{l=1}^{l_{max}} \left[\frac{(2l+1)mk\zeta}{\hbar^2} + l(l+1)[\Im^{(+)} - \Im^{(-)}] \right] ,$$

$$c_3 = \frac{m\zeta}{\hbar^2} \,. \tag{62}$$

Thus, in the small-angle limit, the cross sections and polarisations have the form

$$\frac{d\sigma}{d\Omega} = |g(\theta)|^2 + |h(\theta)|^2$$

$$\xrightarrow[\theta \to 0]{} \frac{1}{\theta^2}[c_3^2 + (a_1^2 + b_1^2 - 2c_2c_3)\theta^2] \,, \tag{63}$$

$$P(\theta) = \frac{2\Re(g^*(\theta)h(\theta))}{d\sigma/d\Omega}$$

$$\xrightarrow[\theta \to 0]{} -\frac{2b_1c_3\theta}{c_3^2 + (a_1^2 + b_1^2 - 2c_2c_3)\theta^2} \,. \tag{64}$$

Clearly then, the polarisation is negative at small scattering angles due to the MS effect as the coefficients b_1 and c_3 are inherently positive. Furthermore, there is a maximum (negative) value to this polarisation when, at θ_m,

$$\left[\frac{dP(\theta)}{d\theta}\right]_{\theta_m} = 0 \longrightarrow \theta_m = \frac{c_3}{\sqrt{a_1^2 + b_1^2 - 2c_2c_3}} \,. \tag{65}$$

At that scattering angle then

$$d\sigma/d\Omega \rightarrow \; 2(a_1^2 + b_1^2 - 2c_2c_3)\,, \tag{66}$$

$$P(\theta_m) = \frac{-b_1}{\sqrt{a_1^2 + b_1^2 - 2c_2c_3}}\,. \tag{67}$$

The other functions also have low-angle limit forms, specifically

$$\begin{aligned} X(\theta) &= \frac{2\Im(h^*(\theta)g(\theta))}{d\sigma/d\Omega} \\ &\xrightarrow[\theta\to 0]{} \frac{a_1c_3\theta}{c_3^2 + (a_1^2 + b_1^2 - 2c_2c_3)\theta^2}\,, \end{aligned} \tag{68}$$

$$\begin{aligned} Y(\theta) &= \frac{-|h(\theta)|^2}{d\sigma/d\Omega} \\ &\xrightarrow[\theta\to 0]{} -\frac{c_3^2 + 2c_2c_3\theta^2}{c_3^2 + (a_1^2 + b_1^2 - 2c_2c_3)\theta^2} \\ &\xrightarrow[\theta\to 0]{} -1 + \theta^2\left(\frac{a_1^2 + b_1^2 - 4c_2c_3}{c_3^2}\right). \end{aligned} \tag{69}$$

5. Cross Sections and Polarisations: Effects of the MS Interaction

The prescription given before for the evaluation of measurables of nucleon–nucleus scattering when the Mott–Schwinger effect is included in the interaction Hamiltonian has been used to evaluate the differential cross sections and polarisations for scattering of neutrons from ^{209}Bi. The results for 0·5, 14·5 and 24·0 MeV scattering are considered specifically. In all results reported (other than when stated so specifically), the calculations were made using the limit value, $l_{max} = 100$.

The effect of MS in the differential cross sections $d\sigma/d\Omega$ is very prominent at small angles $< 2°$, and so the results are shown first for a large range of scattering angles and then for the very forward scattering ones alone. The cross section and polarisation results are displayed in all the figures in the top two sections respectively, while the variation of the X and Y functions are given in the bottom two segments. In the first three diagrams, the dashed and solid curves portray the results obtained with and without the MS interaction respectively.

The 0·5 MeV results are shown in Fig. 1. It is evident that the 'with' and 'without' MS cross sections, polarisations and X and Y variables are indistinguishable, except at the most forward of scattering angles. Various calculations of the 0·5 MeV scattering functions, all of which include fully the MS interaction, were made. They were obtained by using different values of l_{max} with the summations that define the scattering amplitudes. At this low energy the differences were negligible in all results for any maximum value between 40 and 100. Recall that the contributions from partial waves in excess of the cutoff are accounted for by the Born approximation corrections.

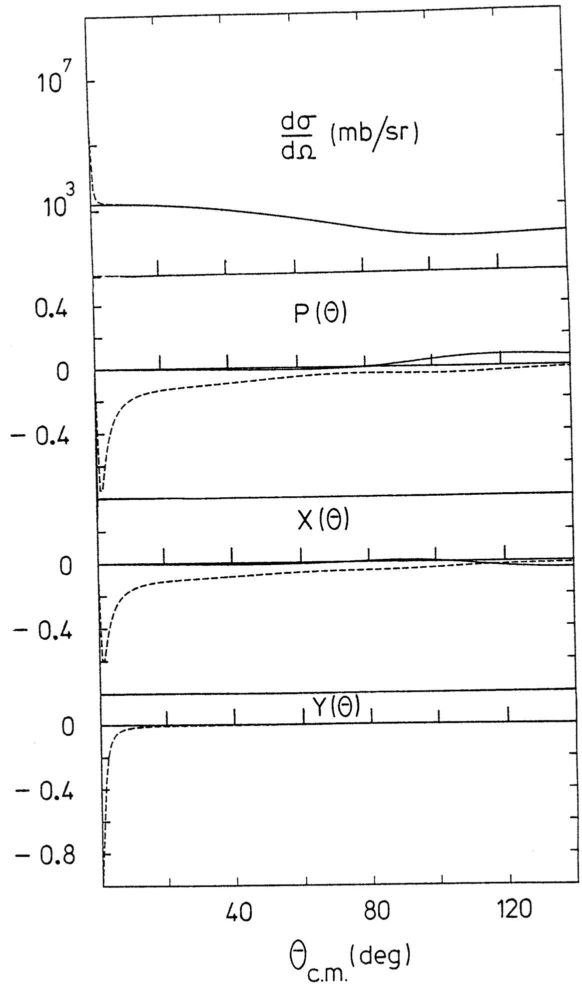

Fig. 1. Cross section (top) and polarisation (bottom) for 0·5 MeV neutrons elastically scattered from ^{209}Bi. The dashed and solid curves are the results of calculations made with and without the MS interaction.

The 14·5 MeV results are shown in Fig. 2. In this case the MS effect is more evident at larger angles than at the lower energy discussed above. The cross sections do not seem to vary as dramatically as the polarisations, but it is really only a matter of scale. Clearly the MS effect alters the prediction of the polarisation noticeably and at most scattering angles to 140°. But it remains the forward scattering angle region that is the most affected and in this case the range of marked increase in the predicted cross section lies below 2° (as can be seen more clearly in Fig. 4). The actual l_{max} cutoff in the summations again is not serious (provided at least 40 partial waves are used). The MS effect upon the X and Y variables is also pronounced. The inclusion of MS can be seen across all angles; larger differences between 'with' and 'without' MS results occur frequently at angles where the polarisation results also differ the most.

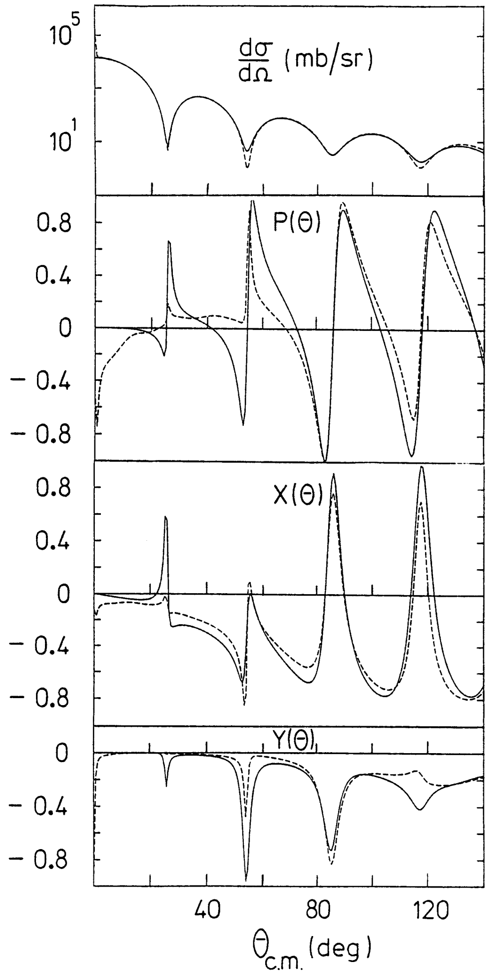

Fig. 2. As for Fig. 1, but for 14·5 MeV neutrons.

The 24·0 MeV results, shown in Fig. 3, are enhanced versions of the 14·5 MeV set, although there are now noticeable, but probably not experimentally measurable, differences at most scattering angles. The polarisation is very different when MS is included, as are the X and Y variables. The effects are essentially those observed at 14·5 MeV. Again 40 partial waves were adequate summation limits in the calculations of $g(\theta)$ and $h(\theta)$.

Of these results, those for polarisations differ from the Hogan and Seyler (1969) results in the 10–50° range for the 14·5 and 24·0 MeV sets. But the present results agree with the previous estimates. In view of the diverse calculations made to ensure stability and accuracy of predictions shown, this suggests that the Hogan and Seyler results are incorrect in this angle range for these energies.

With the exception of the very forward scattering angle region, the major effect of the MS interaction in the differential cross sections is to alter the minimum

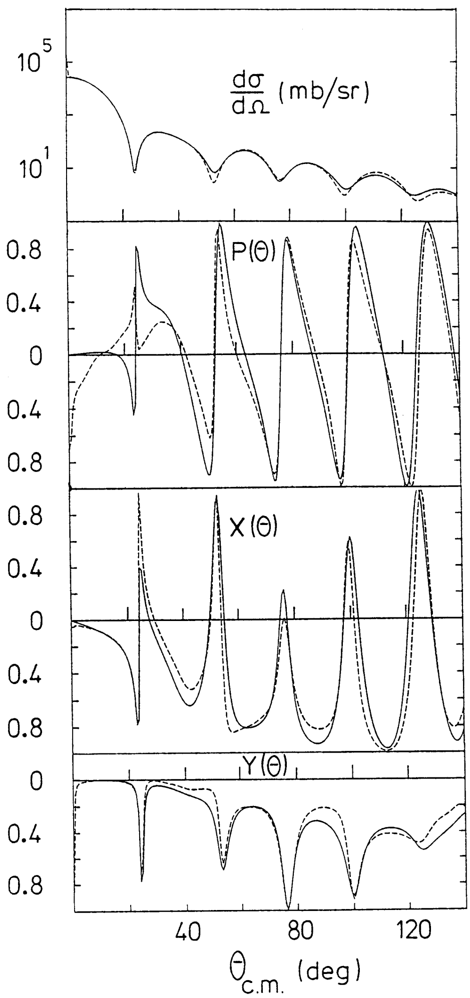

Fig. 3. As for Fig. 1, but for 24·0 MeV neutrons.

values. But for the other quantities there are more noticeable effects. In the polarisation, there are marked (observable) changes in the region of 20° that are due to the MS interaction. Observables related to the X and Y variables would also show similar differences but they are not as drastic. It is also the case that the marked MS effects in the spin-dependent measurables, at scattering angles greater than 4°, occur where the cross sections have minima.

Nevertheless, all results are severely affected by the inclusion of the MS interaction at very forward scattering angles. There the small-angle approximations are very useful. In Fig. 4, the full (MS included) results for all energies are shown with the 0·5, 14·5 and 24·0 MeV cases displayed by the solid, short-dashed and long-dashed curves respectively. The small-angle expansion formulae give results that are essentially the same to 4°, with only the polarisations at 14·5 and 24·0 MeV being noticeably different in magnitude (smaller) from 2 to 4°. The

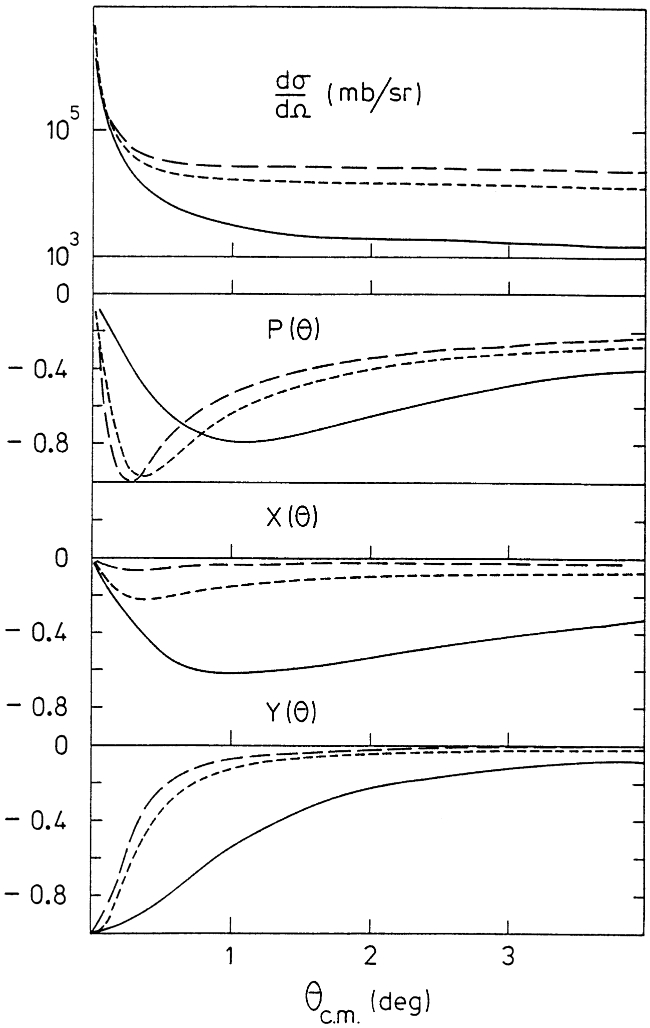

Fig. 4. Results of calculations (all with MS interaction included) at the three energies considered using the small-angle approximation. The solid, short-dashed and long-dashed curves display the results for the energies 0·5, 14·5 and 24·0 MeV respectively.

specific values of the expansion coefficients in these cases are listed in Table 1, as are the predictions of the polarisation peak values and the scattering angles at which they occur.

The small-angle ($<4°$) results at 0·5 MeV show that the cross sections rise rapidly, and over four orders of magnitude at very small angles, above the results obtained without the MS interaction. The polarisation (negative) peak occurs in this range too and is almost 80%. This strong negative peak characterises the MS potential, as was noted in the earliest studies (Schwinger 1948). The X variable result is very similar to that of the polarisation, while the MS effect upon the Y variable is very much a small-angle effect with the maximal value of -1 at $0°$.

Table 1. Small-angle expansion coefficients, maximum polarisation angles, maximum polarisation values and the momentum transfer values at those angles for neutron scattering from bismuth

The coefficient c_3 is a constant (0·2386) with energy

E (MeV)	0·5	14·5	24·0
a_1	−7·8029	−8·4646	−3·7493
a_2	5·1544	77·694	66·684
b_1	9·8538	36·877	50·810
b_2	−5·2973	−342·99	−636·87
c_1	−0·1323	−4·3054	−10·660
c_2	−1·9010	−39·796	−60·015
θ_m	1·08	0·36	0·27
$P(\theta_m)$	−0·783	−0·974	−0·997
$q_m(\mathrm{fm}^{-1})$	0·0075	0·005	0·005

The small-angle 14·5 MeV results show that it is the forward angle region that is the most affected and in this case the range of marked increase in the predicted cross section lies below 2°. The forward angle (negative) polarisation due to the MS potential is now almost total (100%). A characteristic of the MS effect is to give −100% for the Y variable at 0°, as is evident from the small-angle approximation formula. But the X variable is only slightly affected by MS corrections at small angles.

The marked effect of the MS interaction upon the differential cross section is even more forward peaked for the 24·0 MeV results. Otherwise the effects are essentially those observed at 14·5 MeV.

Finally, a study was made of the contributions to the results from the various components of the scattering amplitudes and of the low partial waves. Due to the relative size of the nuclear and MS potentials, one might expect there to be little difference between the phase shifts for $l < 20$ due to the MS potential. This is not the case because despite the major variation caused by the nuclear interaction to radial wavefunctions, the MS potential shifts solution shapes substantially over a large radial region outside the Saxon radius. Thus while the nuclear interaction creates a phase shift for each partial wave inside a small radial region, the MS potential makes its effect over large radial distances. This is the cause, predominantly, of the variation in scattering observables at larger scattering angles. When the scattering angle is small, however, the cot $\frac{1}{2}\theta$ term dominates.

6. Conclusions

This study of the Mott–Schwinger effect in low-energy neutron scattering from ^{209}Bi has shown that the differential cross section is influenced at larger scattering angles then previously thought (Hogan and Seyler 1969), although the major influence of the MS potential is for small angles (<5°). The polarisation is affected at all angles although markedly so for angles less than 50°. The results of these calculations in the 10–50° range are in disagreement with those obtained by Hogan and Seyler (1969) for the 14·5 and 24·0 MeV energies. We have also found that the effect of the MS potential in the small-angle cross sections tends to be larger for smaller energies, but the higher-energy polarisations are very much altered by the MS effect at all angles and for all energies. We have confirmed that the Born approximation can be used to specify phase shifts for $l > 50°$ and with only the MS interaction involved. The procedures used then give an accuracy to better than one part in 10^4. Finally, we have noted not only that the MS effect is very important in analyses of nucleon–nucleus elastic scattering, especially if one wishes to fit very low scattering angle data, but that it has a major effect upon the precise values of the specific, low angular momentum, scattering phase shifts that must be included in any realistic study of other nucleon–nucleus reactions.

One of the outcomes of this study is that we have developed a fast and accurate method to evaluate scattering observables and wavefunctions with the inclusion of the MS potential in the Schrödinger equation. The obvious next step is to compare results with actual measured data. However, in the region where the MS interaction has a very strong effect on cross sections, particularly $< 4°$, few if any data exist. But the significance of the MS effect should not be ignored in any accurate calculation of neutron scattering observables. Its inclusion in the analyses reported by Roberts *et al.* (1991), of extensive, high-accuracy $^{208}\mathrm{Pb(n,n)}$ data, is necessary for their optical model parameter search to be credible.

Acknowledgments

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Appendix A: Closure of the Born Series

An important characteristic of the MS potential is that the associated phase shifts converge as $1/l$, which gives rise to a summation in the $h(\theta)$ amplitude of the form

$$X(\theta) = \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} P_l^1(\cos\theta) . \tag{A1}$$

Numerically this summation (term by term) is difficult to do as one must generate associated Legendre polynomials to extremely large order and the convergence is very slow. In fact the task is not possible and so a closed expression is required. Such was found by Sample (1956) by using the integral representation of $P_l^1(\cos\theta)$,

$$P_l^1(\cos\theta) = \frac{i}{\pi}(l+1) \int_0^{\pi} (\cos\theta - i\sin\theta\cos\phi)^l \cos\phi \, d\phi , \tag{A2}$$

as then the summation $X(\theta)$ can be expressed by

$$X(\theta) = \frac{i}{\pi} \sum_{l=0}^{\infty} \frac{(2l+1)}{l} \int_0^{\pi} (\cos\theta - i\sin\theta\cos\phi)^l \cos\phi \, d\phi . \tag{A3}$$

Defining

$$z = \cos\theta - i\sin\theta\cos\phi , \tag{A4}$$

and using the series expansions

$$\frac{1}{1-z} = \sum_{l=0}^{\infty} z^l, \qquad \ln(1-z) = -\sum_{l=1}^{\infty} \frac{z^l}{l} , \tag{A5}$$

the summation becomes an integral form that is analytic, viz.

$$\begin{aligned} X(\theta) &= \frac{i}{\pi} \int_0^{\pi} \left(\frac{2}{1-z} - \ln(1-z) \right) \cos\phi \, d\phi , \\ &= \cot \tfrac{1}{2}\theta . \end{aligned} \tag{A6}$$

To establish that result, consider the form

$$\begin{aligned} 1 - z &= a + b\cos\phi \\ a &= 1 - \cos\theta, \qquad b = i\,\sin\theta , \end{aligned} \tag{A7}$$

and with which one finds the identities

$$\frac{2\ \cos\phi}{[a + b\cos\phi]} = \frac{2}{b} - \frac{2a}{b} \frac{1}{[a + b\cos\phi]} , \tag{A8}$$

$$\frac{b[1-\cos^2\phi]}{a+b\cos\phi} = \frac{a}{b} - \cos\phi + \frac{b^2-a^2}{b}\frac{1}{a+b\cos\phi}\,, \tag{A9}$$

$$-2a-(b^2-a^2) = -2[1-\cos\theta] + \sin^2\theta + [1-\cos\theta]^2 = 0\,. \tag{A10}$$

With the first of those three identities, the leading integral term becomes

$$\begin{aligned}\int_0^\pi \frac{2}{1-z}\cos\phi\,d\phi &= \left[\frac{2\phi}{b}\right]_0^\pi - \frac{2a}{b}\int_0^\pi \frac{1}{a+b\cos\phi}\,d\phi \\ &= -\frac{2\pi i}{\sin\theta} + (-2a)\frac{1}{b}\Im_\theta\,,\end{aligned} \tag{A11}$$

where the integral is also analytic (Gradshteyn and Ryzhik 1980), viz.

$$\Im_\theta = \frac{\pi}{\sqrt{2-2\cos\theta}}\,. \tag{A12}$$

The second integral in the expression is integrated by parts first to get

$$\begin{aligned}\int_0^\pi \ln[a+b\cos\phi]\cos\phi\,d\phi &= \{\ln[a+b\cos\phi]\,\sin\phi\}_0^\pi + b\int_0^\pi \frac{\sin^2\phi}{a+b\cos\phi}d\phi \\ &= \int_0^\pi \frac{b(1-\cos^2\phi)}{a+b\cos\phi}d\phi\,,\end{aligned} \tag{A13}$$

which, by using the second of the identities, can be reduced to

$$\begin{aligned}\int_0^\pi \ln[a+b\cos\phi]\cos\phi\,d\phi &= \frac{a}{b}\pi - \sin\phi|_0^\pi + \frac{b^2-a^2}{b}\Im_\theta \\ &= -\frac{(1-\cos\theta)\pi i}{\sin\theta} + (b^2-a^2)\frac{1}{b}\Im_\theta\,.\end{aligned} \tag{A14}$$

The final result is then found to be

$$\begin{aligned}\int_0^\pi \left(\frac{2}{1-z} - \ln(1-z)\right)\cos\phi\,d\phi = &-\frac{\pi i(1+\cos\theta)}{\sin\theta} \\ &+ \frac{1}{b}[-2a-(b^2-a^2)]\Im_\theta\,,\end{aligned} \tag{A15}$$

which, by virtue of the third identity, gives simply

$$\frac{i}{\pi}\int_0^\pi \left(\frac{2}{1-z} - \ln(1-z)\right)\cos\phi\,d\phi = \frac{\cos\theta+1}{\sin\theta} = \cot\tfrac{1}{2}\theta\,. \tag{A16}$$

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