VERTICAL HEAT TRANSFER FROM IMPRESSED TEMPERATURE FLUCTUATIONS

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[*Manuscript received November* 27, 1953]

Summary

The ability of buoyant elements to carry heat upwards through a stably stratified fluid depends on their rate of mixing and hence on their size. The largest and smallest elements are both relatively ineffective and there exists an optimum intermediate size yielding a maximum value of the buoyant heat flux for a given intensity of temperature disturbance.

For a layer of uniform unstable stratification the heat flux increases progressively with size of element and there is no theoretical upper limit apart from that set by the depth of the unstable layer.

The distribution, with respect to element size, of the intensity of temperature fluctuations impressed by external influences is modified by the effects of buoyancy and mixing, and relations are derived between the modified and unmodified distributions.

I. Introduction

It is proposed to examine theoretically the vertical flow of heat which results when a layer of fluid is subject to the continual creation of hot elements within it. Whenever turbulence occurs in a thermally stratified liquid, the action of pressure forces which are dissociated from the temperature fluctuations will bring about a state of affairs with elements at rest differing in temperature from their surroundings, but these differences have not normally been allowed for in constructing the equations of heat transfer. The problem therefore has quite general significance, but it becomes of special importance in meteorology where at least two further mechanisms for the creation of " hot spots " may be identified. The natural surface of the Earth is uneven in its physical properties, and so when heated or cooled is subject to local variations in surface temperature which are communicated to the overlying air ; the second mechanism occurs when convective clouds, generated in an unstable layer, penetrate a stable or less unstable layer above in which the heat flow requires to be studied.

The buoyant motions resulting from the creation of hot spots will account for a component of heat flux, F_H, whose sense will always be upwards but whose magnitude may depend on the thermal stratification of the fluid. The need is to express F_H in terms of the stratification, of the size of the heated elements, and of the intensity T_0' of the impressed† temperature fluctuations. This

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† The word is used to denote fluctuations arising from causes other than the motion or mixing of the elements themselves.

problem will here be treated for a layer in which the temperature gradient and the intensity T_0' and frequency of impressed fluctuations (i.e. the number appearing per unit height per unit time) are constant with height and time.

II. Basis of the Treatment

A formulation has been given earlier (Priestley and Swinbank 1947), but it is possible to treat in greater detail by invoking the model which may most suitably be described as the *open parcel*. This differs from the *closed parcel*, which has hitherto been used in studies of convection and turbulent transfer, in that as it moves it is subject to continuous mixing with its environment. The equations for the vertical motion w and excess temperature T' of an open parcel moving through an environment at rest at temperature T_e are (Priestley 1953)

$$\dot{w}=\frac{g}{T_e}T'-k_1w, \qquad \text{(1)}$$

$$\dot{T}'=-w\left(\frac{\partial T_e}{\partial z}+\Gamma\right)-k_2T', \qquad \text{(2)}$$

where g and Γ have their usual significance and k_1 and k_2 are the mixing rates. The latter are constant for an individual element but vary, in constant proportion to each other, from one element to another, taking relatively large values for the small and small for the large elements, and so are used in effect to identify the size of parcel under consideration.

T' may be eliminated from (**1**) and (**2**) and, assuming T'/T_e is small, the equation of motion of the individual parcel is derived as

$$\ddot{w}+(k_1+k_2)\dot{w}+\left[\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)+k_1k_2\right]w=0, \qquad \text{(3)}$$

an equation with constant coefficients whose solutions are readily obtained. We shall consider first a population of elements all of a given size (given k_1 and k_2) but starting from rest at different levels with temperature excess T_0', and derive expressions for the resulting heat flux F_H and r.m.s. temperature fluctuations σ_T at a fixed level in terms of k_1, k_2, and T_0'. The properties of these expressions will then be discussed, with particular reference to their dependence on k_1 and k_2. This will amount in essence to a discussion of the manner in which the size-distribution functions for F_H and σ_T are related to each other and to the corresponding function for the impressed temperature fluctuations T_0'.

III. Solution of the Problem

When

$$\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)+k_1k_2<0, \qquad \text{(4)}$$

that is, when the lapse rate is sufficiently unstable and the element sufficiently large, the motion is absolutely buoyant (Priestley 1953) and both the w and T'

of the individual element ultimately increase exponentially with time. This case will not be considered in this section, but it will be referred to in Section IV (*b*).

When

$$\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)+k_1k_2>0, \qquad (5)$$

the motion of the individual elements is always bounded, and there are two principal types of motion to consider. Writing

$$\mu^2=\left|\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)-\frac{(k_1-k_2)^2}{4}\right|, \qquad (6)$$

then, when the expression inside the modulus is positive, the solution of (3) which satisfies the initial conditions $w=0$, $T'=T_0'$ is of the oscillatory form

$$w=A\mathrm{e}^{-\frac{1}{2}(k_1+k_2)t}\sin\mu t, \qquad (7)$$

whence

$$\frac{T'}{T_e}=\frac{A}{g}\mathrm{e}^{-\frac{1}{2}(k_1+k_2)t}\left(\frac{k_1-k_2}{2}\sin\mu t+\mu\cos\mu t\right), \qquad (8)$$

where $A=(g/\mu)(T_0'/T_e)$. When the expression inside the modulus of (6) is negative, the solution satisfying the same initial conditions is

$$w=A\mathrm{e}^{-\frac{1}{2}(k_1+k_2)t}\sinh\mu t, \qquad (9)$$

$$\frac{T'}{T_e}=\frac{A}{g}\mathrm{e}^{-\frac{1}{2}(k_1+k_2)t}\left(\frac{k_1-k_2}{2}\sinh\mu t+\mu\cosh\mu t\right), \qquad (10)$$

with A as before. This is of the asymptotic form.

In deriving the statistical quantities F_H and σ_T^2 at a fixed reference level z_1, it is recognized that elements of a given size will all be subject to the same motion but will reach z_1 at different stages thereof through having started from different levels z_0. The contribution to the heat flux is

$$\rho c_p \times \text{average value of } wT',$$

the average being taken over all values of z_0, weighted according to their probability. The *a priori* probability of z_0 is uniform since conditions, including the frequency of impressed temperature fluctuations, are supposed constant with height, but the recognition that the element has reached the level z_1 affects the *a posteriori* probability of z_0; some values are excluded because of the bounded nature of the motion; others, when solutions (7) and (8) apply, must be counted n times where wT' is an n-valued function of z_1-z_0. Since, given the lapse and mixing rates, w and T' depend solely on the difference z_1-z_0, it is mathematically equivalent to keep z_0 fixed and vary z_1, so that the average wT' may be written

$$\int wT'\mathrm{d}s\bigg/\int\mathrm{d}s,$$

where the integrals are taken over the complete path of an individual element. In this way, and in a similar way for σ_T^2, is finally derived

$$\frac{F}{\rho c_p}=\int_0^\infty w\,|\,w\,|\,T'\mathrm{d}t\Big/\int_0^\infty |\,w\,|\,\mathrm{d}t, \qquad (11)$$

and

$$\sigma_T^2=\int_0^\infty |\,w\,|\,T'^2\mathrm{d}t\Big/\int_0^\infty |\,w\,|\,\mathrm{d}t, \qquad (12)$$

where w and T' are as in (7)–(10) and the integral is over the lifetime of the element.

The evaluation of the expressions (11) and (12) is straightforward though laborious, and the following results are obtained. In the case of asymptotic motion (solutions (9) and (10)),

$$\frac{F_H}{\rho c_p}=T_0'^2\frac{g}{T_e}\cdot\frac{\frac{2k_1}{3}}{\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)+k_1k_2+2(k_1+k_2)^2}, \qquad (13)$$

$$\sigma_T^2=\tfrac{1}{3}T_0'^2\cdot\frac{\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)+k_1k_2+2k_1^2}{\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)+k_1k_2+2(k_1+k_2)^2}, \qquad (14)$$

whence also

$$\frac{F_H}{\rho c_p}=\sigma_T^2\,\frac{g}{T_e}\cdot\frac{2k_1}{\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)+k_1k_2+2k_1^2}. \qquad (15)$$

In the case of oscillatory motion (solutions (7) and (8)), the expressions on the right of (13) and (14) must be multiplied by the factor

$$\coth\frac{3(k_1+k_2)\pi}{4\mu}\Big/\coth\frac{(k_1+k_2)\pi}{4\mu}$$

and the relation (15) continues to hold.

In discussing the results it will be convenient to define a function f_1 in the relation between heat flux F_H and impressed temperature fluctuations by

$$\frac{F_H}{\rho c_p}=\frac{g}{T_e}T_0'^2f_1,$$

where f_1 contains the dependence on lapse rate and, through k_1 and k_2, on element size ; similarly the relation between heat flux and r.m.s. temperature fluctuations at the same level may be described by the function f_2 defined by

$$\frac{F_H}{\rho c_p}=\frac{g}{T_e}\sigma_T^2f_2.$$

Of these two functions, the greater practical interest attaches to f_2, since it is possible to measure both σ_T^2 and the total heat flux, of which F_H is a component, whereas T_0' is not easily measurable. There is, however, some interest in the quantity $\sqrt{f_1/f_2}$ which is equal to σ_T/T_0' and so represents the factor by which the impressed temperature fluctuations are reduced when records of temperature are taken at a fixed level.

The discussion will deal separately with stable and unstable stratifications, so that in writing

$$\lambda^2=\left|\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)\right|, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

there need be no confusion through use of the modulus. It will then be seen that both the criteria for the solutions and the values of λf_1 and λf_2 appertaining thereto depend only on the ratios k_1/k_2, k_1/λ, and k_2/λ. The results may therefore be presented in completeness by diagrams in which the dimensionless quantities λf_2 and $\sqrt{f_1/f_2}$ are displayed as functions of k_1/k_2 $(=\beta)$ and k_2/λ $(=\xi)$.

IV. Significance of the Results

(a) *Stably Stratified Medium*

Although either asymptotic or oscillatory motion can occur at stable lapse rates, the form of (15) and hence of f_2 is independent of the mode of motion occurring. With the notation adopted, (15) becomes

$$\lambda f_2=\frac{2\beta\xi}{1+\xi^2(2\beta^2+\beta)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

Isopleths of λf_2, as a function of β and ξ, are shown in Figure 1. The most significant feature is marked by the broken line, whose equation is

$$\xi^2=\frac{1}{2\beta^2+\beta}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

along which λf_2 attains its maximum values. As stated in Section II, the ratio β between the mixing rates for momentum and sensible heat is to be regarded as independent of the size of element, and as determined by factors not here under consideration. Special interest attaches to the case in which these mixing rates are equal, $\beta=1$, when from the above

$$k_1^2=k_2^2=\frac{\lambda^2}{3}=\frac{1}{3}\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right) \quad \ldots\ldots\ldots\ldots \quad (19)$$

is derived as the condition for maximum heat flux.

The interpretation of this result is that, at a given stable lapse rate, there exists an optimum size of element which is most efficient for the buoyant transport of heat, both very large and very small elements being relatively inefficient in this respect. The smallest elements are inefficient on account of their rapid rate of mixing and consequent short life; k_1 and k_2 are large and the term in $\exp\{-\frac{1}{2}(k_1+k_2)t\}$ dominates the solutions (7)–(10), whichever pair is applicable. The reason for the inefficiency of very large elements, to which the oscillatory solution applies, is that the damping becomes zero so that the motion approaches the simple harmonic with large amplitudes for both w and T', but the phase difference approaches $\pi/2$ and little heat flux results.

It is of interest to remark that, for solutions of the type under discussion, the vanishing of the mean product $\overline{wT'}$ while the motion remains finite requires *both* that the phase difference of the periodic part shall be $\pi/2$ *and* that the damping be negligible. For $k_1=k_2$ the first condition is satisfied for elements of all sizes, the second only for the largest ones.

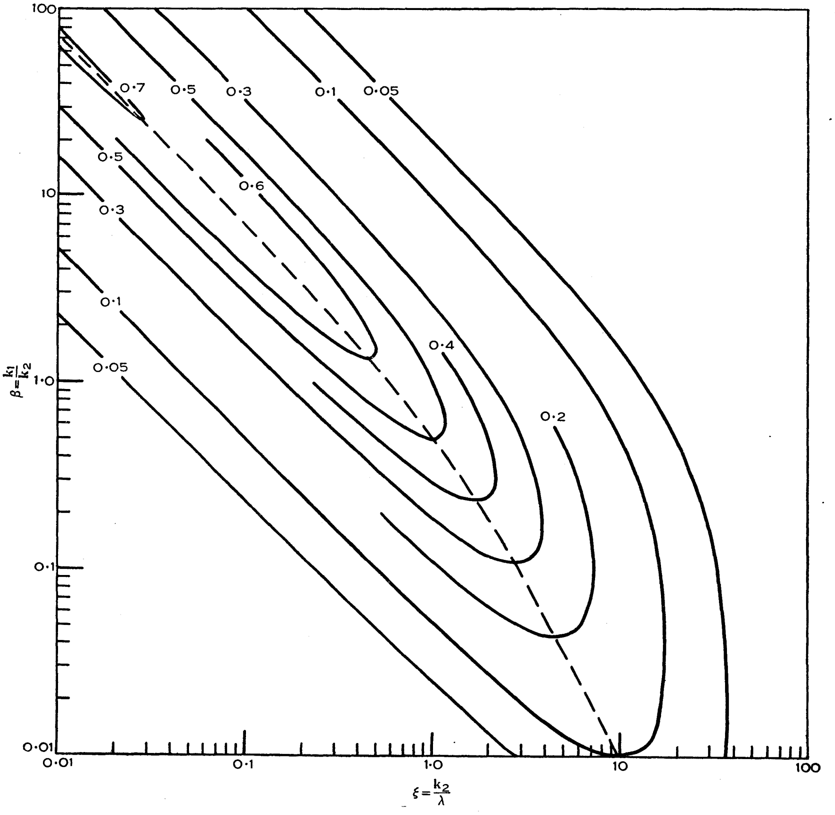

Fig. 1.—Isopleths of λf_2 in stable lapse rates.

For the optimum size of element, or mixing rate, given by (19), (15) takes the value

$$F_H=\frac{\rho c_p\sigma_T^2}{\sqrt{\frac{3T_e}{g}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)}}, \quad \ldots\ldots\ldots\ldots\ldots \quad (20)$$

which represents therefore an upper limit to the heat flux due to buoyancy, given σ_T and the temperature gradient. It may in practice prove a generous upper limit, since only a small fraction of elements may be close to the optimum size.

Reference to some actual magnitudes observed in the atmosphere is instructive. Fairly strong stable stratification at a height of 1 m above ground may be represented by $\partial T_e/\partial z+\Gamma=25\times10^{-4}$ °C/cm, for which the optimum k from (**19**) is given as 1/20 $\sec^{-1}$. Representative values of σ_T observed under these conditions over a uniform site at Edithvale, Victoria, are 0·2 °C, whence the upper limit to the heat flux due to buoyancy is about 1 mW/cm². This is of the order of magnitude of the *total downwards* heat flux under such conditions, with winds of 1–2 m/sec, as measured by Swinbank (1952).

For a layer above convection cloud, into which that cloud might penetrate, we might take $\partial T_e/\partial z+\Gamma=10^{-5}$ °C/cm, whence the optimum k would be 1/5 $\min^{-1}$. With σ_T of about 0·5 °C (Byers and Braham 1949), the upper limit of F_H would be about 100 times the above. There is therefore the possibility of a considerable upward flow of heat through statically stable layers in the free atmosphere.

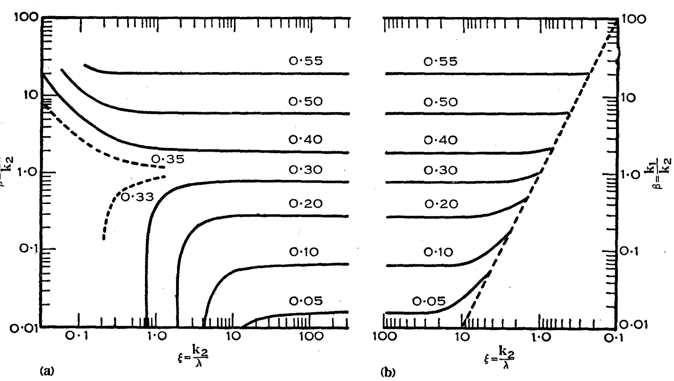

Fig. 2.—Isopleths of the reduction factor for temperature fluctuations σ_T/T_0'. (*a*) Stable lapse rates, (*b*) unstable lapse rates.

The values of the reduction factor $\sqrt{f_1/f_2}$ are shown in Figure 2 (*a*) as a function of β and ξ. When the mixing rates are equal ($\beta=1$) the factor is approximately $\frac{1}{3}$ for elements of all sizes. For the larger elements the factor remains $\frac{1}{3}$ except when the mixing rate for momentum greatly exceeds that for temperature, while for smaller elements the variation of $\sqrt{f_1/f_2}$ with k_1/k_2 is more regular, the two ratios increasing or decreasing in sympathy.

(*b*) *Unstably Stratified Medium*

When $(\partial T_e/\partial z+\Gamma)$ is negative, the reduction factor $\sqrt{f_1/f_2}$ takes the same value as before for the smaller elements, and there is little significant change in this value until the limit is reached at which (**14**) ceases to apply. This is given by $k_1k_2=-(g/T_e)(\partial T_e/\partial z+\Gamma)$. The function is shown in Figure 2 (*b*), the limit being indicated by the broken line.*

* This and the preceding result indicate that, under conditions of practical importance, the simple relation $\sigma_T=\frac{1}{3}T_0'$ will be widely valid.

The function f_2, which represents the variation of F_H/σ_T^2, is now

$$f_2=\frac{1}{\lambda}\cdot\frac{2\beta\xi}{-1+\xi^2(2\beta^2+\beta)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{21})$$

As with stable lapse rates the smallest elements (ξ large) can transport little heat, but f_2 increases progressively with size of element until $k_2=\lambda/\sqrt{\beta}$ at which the element becomes absolutely buoyant, when the relation ceases to apply. f_2 is still finite at this point. The elements will not, however, necessarily be confined to sizes below this critical value. Beyond it, the solutions for w and T' ultimately increase exponentially with time, and the present considerations provide no upper limit to the heat flux which might develop.

The heat flux will not in practice be unlimited since superadiabatic conditions occur only in layers of limited depth, with consequent restriction of a different type on the development of w and T'. In the layer close to a heated surface further restrictions reside in the motion-inhibiting presence of the boundary and consequent curvature of the temperature profile; the problem of heat transfer in this layer is discussed in a paper published concurrently with this (Priestley 1954).

V. References

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